# TAME DISTRIBUTIVE ALGEBRAS AND RELATED TOPICS

CHRISTOF GEISS

Contents



## Zusammenfassung


Diese Zusammenfassung ist im wesentlichen eine Übersetzung eines Teils des Abschnittes "Introduction" der vorliegenden Arbeit.

Sei k ein algebraisch abgeschlossener Körper. Diese Arbeit soll eine erste Studie über zahme (assoziative) Algebren mit distributivem Idealverband sein. Nach einem alten Resultat von Jans ist die Distributivität des Idealverbandes eine notwendige Bedingung für endlichen Darstellungstyp. "Zahm" bedeutet grob gesprochen, daß die unzerlegbaren Darstellungen in einer festen Dimension durch endlich viele 1-Parameter Familien beschrieben werden können. Das Theorem von Drozd besagt nun [Dr], daß die Darstellungstheorie einer Algebra, die nicht zahm ist, so kompliziert ist wie die Darstellungstheorie von $k\langle x, y\rangle$, der freien assoziativen Algebra in zwei Variablen; dies rechtfertigt auch die Bezeichnung "wild" für derartige Algebren. Nachdem die Theorie der darstellungsendlichen Algebren im wesentlichen


---


*Date*: .

1991 *Mathematics Subject Classification.* Primary 16G60; Secondary 14D25, 14L30.

*Key words and phrases.* tame algebras, distributive ideal lattice, cleaving, degenerations.

The resarch for this Thesis was supported by a grant from DAAD (Germany) and SRE (México) for a stay of the author at the Instituto de Matematicas, UNAM, México.

abgeschlossen ist, erscheint es in diesem Kontext natürlich, nun die zahmen Algebren in Angriff zu nehmen, siehe [Ga3]. Beim gegenwärtigen Stand der Forschung ist dies jedoch nur als Fernziel anzusehen, andererseits ist die Einschränkung auf den distributiven Fall in gewissem Sinne nicht zu groß (siehe Beispiel 3.3). Darüberhinaus hat dies den Vorteil, daß in dieser Situation die gut entwickelte Theorie der Strahlenkategorien zum Tragen kommt. An dieser Stelle sei noch darauf hingewiesen, daß wir aus technischen Gründen anstelle von Algebren mit lokal beschränkten Kategorien arbeiten, was aber bekanntlich keine Einschränkung ist.

Für unsere Aufgabe ist es von fundamantaler Wichtigkeit, den Darstellungstyp einer gegebenen (distributiven) Algebra entscheiden zu können. Es zeigte sich jedoch, daß die bekannten Techniken wie Überlagerungen im Zusammenspiel mit 1-Punkt-Erweiterungen sowie das Wissen über speziell zweireihige Algebren nicht immer ausreichend waren. Daher mußte die aus [BGRS] bekannte Cleaving-Technik auf den zahmen Fall übertragen werden um eine flexible Methode für "Wildheits Beweise" in die Hand zu bekommen (d.h. um gewisse Konstellationen für zahme Algebren ausschließen zu können), siehe Kapitel 3.

Andererseits konnten wir zeigen, daß, wenn $\Lambda_0$ eine Degeneration (im Sinne der algebraischen Geometrie) einer lokal beschränkten Kategorie $\Lambda_1$ ist, dann folgt aus "$\Lambda_0$ zahm" schon die Zahmheit von $\Lambda_1$. Dies ergiebt eine sehr wirkungsvolle Methode zur Bestimmung des Darstellungstyps; zum Beispiel ergeben sich damit sehr kurze Beweise fr die Zahmheit der lokalen Algebren in Ringels Liste [Ri1] – ein Problem, das von Ringel selbst in [Ri3] vorgeschlagen wurde, siehe Kapitel 2.

Als zentrales Ergebnis unserer Untersuchung über zahme distributive Algebren ist die Vervollständigung der von Hoshino und Miyachi [HoM] begonnenen Liste der zahmen distributiven Algebren mit zwei Isomorphieklassen von einfachen Moduln anzusehen. Als einfache Folgerung hieraus ergiebt sich eine abgeschwächte Version des Transit-Lemmas von Roiter [BGRS, 1.5], das uns erlaubt einige der Resultate aus dieser Arbeit auch für zahme distributive Algebren zu zeigen. Schließlich stellen wir noch einige Beispiele vor, die für den zahmen Fall typische Situationen repräsentieren, aber im darstellungsendlichen Fall nicht auftreten können. Diese Ergebnise werden in Kapitel 6 dargestellt.

## Introduction

Let k be an algebraically closed field. This work should be a first approach to the study of tame k-algebras with distributive lattice of ideals. Recall, that by an old result of Jans a representation finite algebra necessarily has a distributive lattice of ideals.

"Tame" means roughly speaking, that the indecomposable representations of a fixed dimension can be parametrized by 1-parameter families. By Drozd's theorem [Dr] the representation theory of an algebra that is not tame is as complicated as the representation theory of $k\langle x, y\rangle$, the free associative algebra in two variables. At the present state it seems hopeless to study tame algebras without any restrictions, on the other hand the restriction to the distributive case is in some sense not too restrictive for tame algebras (see example 3.3) moreover it has the advantage that in this case the well developed machinery of ray-categories may be used. In fact, for technical reasons we work with locally bounded categories instead of algebras; as it is well-known, this is no restriction.

For our task it is fundamental to be able to decide the representation type of a given (distributive) algebra. In fact it turned out, that the known methods, as covering-techniques together with 1-point extensions and the knowledge about special biserial algebras were not always adequate tools. Thus we had to adapt the cleaving technique from [BGRS] for the tame case in order to obtain an efficient device in "wildness proofs" (i.e. if we want to exclude certain constellations for tame algebras). On the other hand we could prove, that if a degeneration $\Lambda_0$ (in the sense of algebraic geometry) of a locally bounded categorie $\Lambda_1$ is tame, then follows already the tameness of $\Lambda_1$. This is a powerful tool for tameness-proofs. For example this provides new tameness proofs for the local algebras in Ringel's list [Ri1] – a problem proposed by Ringel himself in [Ri3].

As main result about tame distributive algebras we completed the list of tame distributive algebras with two isomorphism-classes of simple modules. A simple consequence of this is a weak version of Roiter's transit lemma [BGRS, 1.5.], which allows us to recover some of the results of that work for tame distributive algebras. Finally we provide several examples for the different behavior of tame distributive algebras in comparison with representation finite categories. These results are presented in chapter 6.

The work is organized as follows: In chapter 1 we set up some basic material such as quivers, locally bounded categories and their representations. Then we present special biserial and clannish categories, which are important examples of tame representation type. Also we introduce the classical technique of one-point extensions; an example how this method may be used will help us later on to prove the tameness of the algebra (4) in Ringel's list [Ri1] – the only one we can't treat with the above mentioned degenerations.

In chapter 2 we present first some important facts from algebraic geometry, including a perhaps well-known characterization of degenerations, which we needed in a previous version of our result. Then we introduce the variety $\mathrm{alg}(\underline{d})$, the algebra structures with fixed Cartan-matrix. Next we introduce the varieties $\mathrm{mod}_\Lambda(\underline{z}, t)$, which describes the $\Lambda$-module structures with dimension-vector $\underline{z}$ and dimension of the endomorphism ring $\geq t$; we will see, that $\Lambda \mapsto \dim \mathrm{mod}_\Lambda(\underline{z}, t)$ is a upper-semicontinuous function on $\mathrm{alg}(\underline{d})$. We present also a characterization of "tame" from [P2], since from this follows easily that weakly tame already implies tame (weakly tame means, that for each dimension $\underline{z}$ there is a finite set of 1-parameter families of modules, such that each indecomposable with dimension $\underline{z}$ is direct summand of one of these modules); this is crucial for our adaptation of the cleaving technique to the tame case. Next we give a new characterization of "tame" in terms of the dimensions of the varieties $\mathrm{mod}_\Lambda(\underline{z}, t)$. With the above semicontinuity statement follows our result about degenerations. We include also a new proof of Gabriel's result, that finite representation type is open.

In chapter 3 we introduce the already anounced cleaving technique, and prove that if $F\colon \Gamma \to \Lambda$ is a cleaving functor between locally bounded categories, then $\Gamma$ wild implies $\Lambda$ wild. As an example we present a necessary condition for tame representation type which corresponds to the distributivity condition for finite representation type. Also we give a new proof that the algebra (c) of Ringel's list is wild.

In chapter 4 we collect some results about Galois-coverings. As example we prove the tameness of (4) of Ringel's list. Also we provide a (new) example of a Galois covering $\tilde{\Lambda} \to \Lambda$ with $\tilde{\Lambda}$ tame and $\Lambda$ wild. This example occured in our study of the tame distributive 2-point algebras.

In chapter 5 we recall the relations between distributive categories and ray categories given in [BGRS].

Finally, in chapter 6 we present our results about tame distributive algebras as mentioned above. The last two chapters contain the completeness- and tameness proofs for our list. There we use all the material presented in the previous chapters.

**Remarks.** The tameness proof of the algebra (4) of Ringel's list is essentially taken from [GeP]. The fact, that weakly tame implies tame was first stated in [Ge1], however we give her a different proof. The result about degenerations is new. The proof for our result about cleaving functors is taken (with small changes) from [Ge1], while the examples in the corresponding chapter are new. Our example of a tame covering of a wild algebra is from [GeP]. The list of tame distributive 2-point algebras is copied from [Ge3]; on the other hand the remaining results and examples in chapter 6 are presented here the first time. Finally, the proofs in the chapters 7 and 8 are a compilation from [Ge3] and [GeP].

**Acknowledgement.** It is a pleasure to thank J.A. de la Peña for many fruitful discussions on the material presented here as well as for detailed suggestions on the final form of this thesis. Also his continous encouragement and selfdenying help during my stay in Mexico was of unfathomable value.

Also I'm indebted to the Instituto de Matematicas, UNAM (México) represented by its director R. Bautista for warm hospitality and for providing quite perfect working conditions.

Finally my thanks are directed to W. Müller, who made possible in a generous way this work and who gave me all the support I needed in Bayreuth.

## 1. Locally bounded categories

1.1. Let Q be a quiver. The path category $W_0Q$ has as objects the vertices of Q; given two vertices $x$ and $y$, the morphisms from $x$ to $y$ are given by the directed paths from $x$ to $y$ in Q; in furthermore we add formally the zero path ${}_x0_y$. The composition is given in the obvious way. Thus $W_0Q$ is a category with zeros.

An equivalence relation $\sim$ on the morphisms of a small category P is called *stable* if i) $\varphi_1 \sim \varphi_2$ implies that start- and endpoint of $\varphi_1$ and $\varphi_2$ are equal; ii) $\varphi_1 \sim \varphi_2$ implies $\mu\varphi_1\nu \sim \mu\varphi_2\nu$ whenever this makes sense. If $\sim$ is a stable equivalence relation on P we can construct the quotient $P/\sim$, it has the same objects as P and as morphisms the equivalence classes of morphisms of P. By construction the composition using representants is well defined.

The *linearization* k(P) of a category P with zeros has the same objects as P and $\mathrm{k(P)(x,y)} := \oplus_{\mu \in \mathrm{P(x,y)}} \mathrm{k}\mu/\mathrm{k}_\mathrm{x}0_\mathrm{y}$ with the composition as induced by P.

1.2. **Definition.** A category $\Lambda$ is called a *locally bounded* k*-category* if it satisfies the following axioms a)–d).

a) The morphism spaces $\Lambda(x,y)$ are k-vectorspaces and the composition is k-linear.
b) Different Objects are not isomorphic.
c) $\sum_{y \in \mathrm{Obj}\,\Lambda}[\Lambda(y,x) : \mathrm{k}]$ and $\sum_{y \in \mathrm{Obj}\,\Lambda}[\Lambda(x,y) : \mathrm{k}]$ are finite.
d) The rings $\Lambda(x,x)$ are local for all $x \in \mathrm{Obj}\,\Lambda$.

*Remark .* Locally bounded categories with a finite number of objects will be also called *bounded categories.* Recall, that there is a one to one correspondence between bounded k-categories and finitedimensional basic k-algebras which sends $\Lambda$ to $\oplus\Lambda := \oplus_{i,j \in \mathrm{Obj}\,\Lambda}\Lambda(i,j)$ the matrix-algebra of $\Lambda$ with the obvious multiplication.

1.3. The *radical* $\mathcal{R}\Lambda$ of a locally bounded k-category $\Lambda$ is the ideal given by the non-invertible morphisms. A morphism $\mu$ has depth $\mathrm{d\!I}\,(\mu) = n$ if it belongs to $\mathcal{R}^n\Lambda$ (= the n-th power of $\mathcal{R}\Lambda$) but not to $\mathcal{R}^{n+1}\Lambda$; a zero morphism has by definition depth $\infty$. If in turn we want to refer to the (Jacobson) radical of the $\Lambda(x,x)$-$\Lambda(y,y)$-bimodule $\Lambda(x,y)$ we will write $\mathrm{rad}\Lambda(x,y)$.

With each locally bounded category $\Lambda$ we associate a locally finite *quiver* $\mathrm{Q} = \mathrm{Q}_\Lambda$ whose vertices coincide with $\mathrm{Obj}\,\Lambda$; if $d = [\mathcal{R}\Lambda(x,y) : \mathrm{k}]$, $Q$ is endowed with a sequence $\mu_1, \ldots, \mu_d$ of arrows from $x$ to $y$. By definition, a *presentation* $\pi$ of $\Lambda$ maps the arrows $\mu_1, \ldots, \mu_d$ onto morphisms $\mu_1^\pi, \ldots, \mu_d^\pi \in \mathcal{R}\Lambda(x,y)$ whose classes modulo $\mathcal{R}^2(x,y)$ form a basis of $(\mathcal{R}\Lambda/\mathcal{R}^2\Lambda)(x,y)$. Each presentation $\pi$ gives rise to a surjective k-linear functor $\phi^\pi : \mathrm{k(W_0Q)} \longrightarrow \Lambda$ in the obvious way. The kernel $I^\pi$ of $\phi^\pi$ is an (admissible) ideal, and $\phi^\pi$ induces an isomorphism $\mathrm{k(W_0Q)/I^\pi} \xrightarrow{\sim} \Lambda$. *If $w$ is a path of* $W_0Q$ *we often write $\bar{w}$ in place of* $\phi^\pi(w)$ implicitly assuming $\pi$ to be given. In case of $\Lambda = \mathrm{k(W_0Q)/I}$ we take the canonical projection as presentation.

1.4. Let $A$ be a finitely generated k-algebra. The category of $\Lambda$-$A$-*bimodules* is given by the contravariant k-linear functors $M : \Lambda \longrightarrow \text{mod-}A$, where mod-$A$ denotes the category of right $A$-modules. $M$ is a finitely generated (resp. free) right $A$-module, if $\oplus_{x \in \text{Obj}\,\Lambda} M(x)$ is finitely generated (resp. free) right A-module. Note that $\oplus_{x \in \text{Obj}\,\Lambda} M(x)$ has a natural $\oplus\Lambda$-$A$ bimodule structure in the classical sense if $\Lambda$ is bounded. Similarily, if $\Gamma$ is an other locally bounded k-category the $\Lambda$-$\Gamma$-bimodules are k-linear functors $\Lambda^{\text{op}} \times \Gamma \longrightarrow \text{k-mod}$.

Of particular interest is the case $A = \text{k}$ where we obtain the left $\Lambda$-*modules.* We write $\Lambda$-mod for the category of finitely generated modules and $\Lambda$-ind denotes the full subcategory of indecomposable modules; $\Lambda$-Mod stands for the category of $\Lambda$-modules $M$ with $[M(x) : \text{k}] < \infty$ for all $x \in \text{Obj}\,\Lambda$; finally $\Lambda$-MOD is the category of all $\Lambda$-modules. For $M \in \Lambda\text{-Mod}$ we set $\underline{\dim} M := ([M(x) : \text{k}])_{\text{x} \in \text{Obj}\,\Lambda}$ the *dimension vector* of $M$ and $\operatorname{supp} M := \{x \in \text{Obj}\,\Lambda \mid M(x) \neq 0\}$, the support of $M$.

First examples are the representatives for the isoclasses of indecomposable projective modules $\Lambda(-, x)$ (resp. indecomposable injective modules $D\Lambda(x, -)$) for all $x \in \text{Obj}\,\Lambda$. Here and furher on $D$ *will denote the duality* $\text{Hom}_{\text{k}}(-, \text{k})$.

1.5. $\Lambda$ is said to be *representation finite* if there are only finitely many isomorphism classes of indecomposable $\Lambda$-modules; it is called *locally representation finite,* if for every $x \in \text{Obj}\,\Lambda$ there are only finitely many isomorphism classes of indecomposable $\Lambda$-modules $M$ with $M(x) \neq 0$;

*tame* if for every dimension vector $\underline{z} \in \mathbb{N}^{\text{Obj}\,\Lambda}$ there is a finite number of $\Lambda$-k[x] bimodules $M_1, \dots, M_t$ which are finitely generated free right k[x] modules such that for every indecomposable $\Lambda$-module $M$ with $\underline{\dim} M = \underline{z}$ we have $M \cong M_i \otimes_{\text{k[x]}} S$ for some $i$ and some simple k[x]-module S.

*wild*, if there is a $\Lambda$-k$\langle$x, y$\rangle$ bimodule $M$ being a finitely generated free right k$\langle$x, y$\rangle$ module such that the functor $M \otimes_{\text{k}\langle\text{x,y}\rangle} - : \text{k}\langle\text{x, y}\rangle\text{-mod} \longrightarrow \Lambda\text{-mod}$ preserves isoclasses and indecomposability.

For $x \in \text{Obj}\,\Lambda$ we denote by $\Lambda_x$ the full subcategory of $\Lambda$ which has as objects the set $\{y \in \text{Obj}\,\Lambda \mid y \in \operatorname{supp} M \text{ for some } M \in \Lambda\text{-ind with } M(x) \neq 0\}$; we call $\Lambda$ *locally support-finite,* if $\Lambda_x$ is finite for all $x \in \text{Obj}\,\Lambda$.

See [BGRS, 1],[DoSk1, 1] for more details. We will use without further reference Drozd's deep result that every bounded category is either tame or wild, see e.g. [CB1], [Dr].

*Remark .* We call a locally bounded category $\Lambda$ *weakly tame* if for every dimension vector $\underline{z}$ there are finitely many $\Lambda$-k[T] bimodules $M_1, \dots, M_{\mu(\underline{z})}$ which are finitely generated free k[T]-modules and have the property that for any indecomposable $\Lambda$-module $X$ with $\underline{\dim} X = \underline{z}$ there is $i \in \{1, \dots, \mu(\underline{z})\}$ and a simple k[T]-module $S$ such that $X$ is a direct summand of $M_i \otimes_{\text{k[T]}} S$. Of course this is weaker than tame, however in [Ge2] it was shown, that weakly tame implies tame; in 2.4 we will give an independent short proof, using the type of arguments in [P2]. This characterization is crucial for our result on cleaving diagrams.

1.6. A locally bounded category $\Lambda$ is called *special biserial* if

1) The number of arrows starting or stopping at any vertex of $\mathrm{Q} = \mathrm{Q}_\Lambda$ is bounded by two;
2) There is a presentation $\pi$ of $\Lambda$ such that for any arrow $\beta$ of Q there is at most one arrow $\alpha$ and at most one arrow $\gamma$ such that $\bar{\alpha}\bar{\beta} \neq 0$ and $\bar{\beta}\bar{\gamma} \neq 0$.

It was shown in [WW] that all these categories are tame; another proof using Galois-coverings is given in [DoSk3]. In both cases one obtains an explicit description of the indecomposable modules.

Let $\Gamma = \mathrm{k}(\mathrm{W}_0\mathrm{Q})/\mathrm{I}$ and take a subset Sp of the loops of Q. Call the arrows in Sp *special loops* and the remaining arrows *ordinary arrows.* $\Gamma$ is called *clannish* if:

0) $I$ is generated by $Z \cup \{q_\beta(\beta) \mid \beta \in \mathrm{Sp}\}$ where the $q_\beta$ are quadratic polynomials and $Z$ a set of paths in $Q$ which do not start or end with a special loop neither involve the square of a special loop.
1) The number of arrows starting or stopping at any vertex of Q is bounded by two;
2) For any *ordinary* arrow $\beta$ of Q there is at most one arrow $\alpha$ and at most one arrow $\gamma$ such that $\bar{\alpha}\bar{\beta} \neq 0$ and $\bar{\beta}\bar{\gamma} \neq 0$.
3) The polynomials $q_\beta$ are *non singular* that is $q_\beta(0) \neq 0$, and *separably reducible* that is they have distinct zeros lying in k for all $\beta \in \mathrm{Sp}$.

Note that in general these categories are not locally bounded. However, in [CB2] there is shown the tameness and given an explicit description of the indecomposable modules using the technique of functorial filtrations.

**Example .** Let $\mathrm{Q} = \begin{tikzcd} \mathrm{a} \arrow[loop left, "\sigma"] \arrow[r, shift left, "\nu"] & \mathrm{b} \arrow[l, shift left, "\gamma"] \arrow[loop right, "\rho"] \end{tikzcd}$ and $\Lambda := \mathrm{k}(\mathrm{W}_0\mathrm{Q})/\mathrm{I}$ where $I$ is generated by $\sigma^2 - \nu\gamma$, $\rho^2 - \gamma\nu$, $\rho\gamma - \gamma\sigma$ and $\sigma\nu - \nu\rho$; this is not locally bounded, but all the categories $\Lambda 9_{p,q,r,s,1,1}$ of table T are quotients of $\Lambda$. We will prove their tameness by giving an equivalence $\psi : \Gamma\text{-mod} \xrightarrow{\sim} \Lambda\text{-mod}$, where $\Gamma$ is the *clannish category* given by the quiver with relations

$$\begin{tikzcd} * \arrow[loop left, "\epsilon"] \arrow[loop right, "\delta"] \end{tikzcd} \qquad \bar{\delta}^2 = 0, \quad q(\bar{\epsilon}) = (\bar{\epsilon} - K_1)(\bar{\epsilon} - K_2) = 0$$

with $K_1, K_2$ different elements of $\mathrm{k}^*$.

Let $Y \in \Gamma$-mod and consider the decomposition $Y(*) = Y_1 \oplus Y_2$ of the k-vectorspace $Y(*)$ into the eigenspaces of $Y(\epsilon)$, thus

$$Y(\bar{\delta}) = \begin{pmatrix} Y_{1,1}(\bar{\delta}) & Y_{1,2}(\bar{\delta}) \\ Y_{2,1}(\bar{\delta}) & Y_{2,2}(\bar{\delta}) \end{pmatrix}, \qquad \text{where} \quad Y_{i,j}(\bar{\delta}) \colon Y_j \longrightarrow Y_i$$

We set:

$$\begin{array}{ll} \psi(Y)(a) = Y_1, & \psi(Y)(b) = Y_2, \\ \psi(Y)(\bar{\sigma}) = Y_{1,1}(\bar{\delta}), & \psi(Y)(\bar{\nu}) = Y_{1,2}(\bar{\delta}), \\ \psi(Y)(\bar{\gamma}) = -Y_{2,1}(\bar{\delta}), & \psi(Y)(\bar{\sigma}) = -Y_{2,2}(\bar{\delta}) \end{array}$$

Since $Y(\bar{\delta})^2 = 0$, then $\psi(Y)$ is a well defined $\Lambda$-module. If $f \in \mathrm{Hom}_\Gamma(Y, X)$ we may decompose also $X(*) = X_1 \oplus X_2$ using the endomorphism $X(\bar{\epsilon})$. Since $K_1 \neq K_2$ in $\mathrm{k}^*$, then

$$f(Y_i) \subseteq X_i, \quad \text{for} \quad i = 1, 2.$$

We may therefore define the morphism $\psi(f) \colon \psi(Y) \longrightarrow \psi(X)$ in the obvious way. It is very easy to verify that $\psi$ is a functor admitting an inverse $\psi^{-1}$.

1.7. We make explicit the relations between subspace categories and one-point (co-) extensions of locally bounded categories. For more details on the representation theory of vectorspace categories we refer to [Ri2]

1.7.1. Let K be a Krull-Schmidt category, and $|-| \colon \mathrm{K} \to \mathrm{k}\text{-mod}$ an additive functor. The pair $(\mathrm{K}, |-|)$ is called a *vectorspace category.* By $\check{\mathcal{U}}(\mathrm{K}, |-|)$ we denote the category of triples $V = (V_0, V_\omega, \gamma_V)$, where $V_0$ is an object of K, $V_\omega$ is a k-vectorspace and $\gamma_V : V_\omega \to |V_0|$ a k-linear map. Given two such triples $V$ and $W$, a map $f \colon V \to W$ is given by a pair $(f_0, f_\omega)$, where $f_0 \in \mathrm{K}(\mathrm{V}_0, \mathrm{W}_0)$, $f_\omega \in \mathrm{Hom}_\mathrm{k}(V_\omega, W_\omega)$ and $\gamma_W f_\omega = |f_0|\gamma_V$.

If we have a contravariant functor $||-|| : \mathrm{K} \longrightarrow \mathrm{k}\text{-mod}$ we denote by $(\mathrm{K}, ||-||)\check{\mathcal{U}}$ the category of triples $V = (V_0, V_\alpha, \gamma_V : DV_\alpha \to ||V_0||)$, in this case a morphism $f : V \to W$ between two such triples is given by a pair $(f_0, f_\alpha)$ with $f_0 \in \mathrm{K}(\mathrm{V}_0, \mathrm{W}_0)$ and $f_\alpha \in \mathrm{Hom}_\mathrm{k}(V_\alpha, W_\alpha)$ such, that $\gamma_V D f_\alpha = ||f_0||\gamma_W$.

1.7.2. Given $(\mathrm{K}, |-|)$ as before, we let $\mathrm{K}(|-|)$ be the category obtained from K by factoring out the ideal in K given by all maps $f$ with $|f| = 0$. Then we may consider $|-|$ as being defined on $\mathrm{K}(|-|)$ as faithful functor. This induces canonical functors $(\mathrm{K}, |-|) \to (\mathrm{K}(|-|), |-|)$ and $\check{\mathcal{U}}(\mathrm{K}, |-|) \to \check{\mathcal{U}}(\mathrm{K}(|-|), |-|)$. *This last functor maps the objects in* $\check{\mathcal{U}}(\mathrm{K}, |-|)$ *which are of the form* $(X, 0, o)$ *with* $|X| = 0$ *to zero, and it gives a bijection on the remaining indecomposable objects.* (See [Ri2, 2.6(5)].)

1.7.3. Let $\Lambda$ be a locally bounded category and $M \in \Lambda$-mod, then $\Lambda$ may be considered as full subcategory of the one-point extension $\Lambda[M]$, where $\mathrm{Obj}\,\Lambda[M] = \mathrm{Obj}\,\Lambda \cup \{\omega\}$, $\Lambda[M](\omega, \omega) := \mathrm{k}$ and $\Lambda[M](x, \omega) := M(x)$, $\Lambda[M](\omega, x) := 0$ for all $x \in \mathrm{Obj}\,\Lambda$, while the composition is the obvious one. We have $\Lambda[M]\text{-mod} \cong \check{\mathcal{U}}(\Lambda\text{-mod}, \mathrm{Hom}_\Lambda(M, -))$; indeed $T \in \Lambda[M]$-mod can be interpreted as $(T|_\Lambda, T(\omega), \gamma_T)$, where

$$T(x) \xleftarrow{T(\mu)} T(\omega), \; t \mapsto (\gamma_T(t))_x(\mu) \qquad (\mu \in \Lambda[M](x, \omega) = M(x)).$$

By the foregoing remark we can reduce from $\Lambda[M]$-mod to $\Lambda$-mod and the subspace-category $\check{\mathcal{U}}(\Lambda\text{-mod}(\mathrm{Hom}_\Lambda(M, -)), \mathrm{Hom}_\Lambda(M, -))$

Dually $\Lambda$ may be considered as full subcategory of the one-point co-extension $[M]\Lambda$, where $\mathrm{Obj}\,[M]\Lambda = \{\alpha\} \cup \mathrm{Obj}\,\Lambda$, $[M]\Lambda(\alpha, \alpha) := \mathrm{k}$ and

$[M]\Lambda(\alpha,x) := DM(x)$, $[M]\Lambda(x,\alpha) := 0$ for all $x \in \operatorname{Obj}\Lambda$, while the composition is the obvious one. Here $[M]\Lambda\text{-mod} \cong (\Lambda\text{-mod}, \operatorname{Hom}_\Lambda(-,M))\check{\mathcal{U}}$; indeed, $S \in [M]\Lambda\text{-mod}$ can be interpreted as $(S\mid_\Lambda, S(\alpha), \delta_S)$ where

$$DDS(\alpha) = S(\alpha) \xleftarrow{S(\mu^*)} S(x),\ s_x \mapsto \mu^*((\delta_S(-))_x(s_x)) \quad (\mu^* \in [M]\Lambda(\alpha,x) = DM(x))$$

1.7.4. *Example.* Let $D \in \mathrm{k}\setminus\{0,1\}$ and consider the category $\tilde\Lambda_D$ which we describe by its quiver and relations which generate $I^\pi$

$$\cdots -1 \underset{y_{-1}}{\overset{x_{-1}}{\rightrightarrows}} 0 \underset{y_0}{\overset{x_0}{\rightrightarrows}} 1 \underset{y_1}{\overset{x_1}{\rightrightarrows}} 2 \cdots \qquad \left.\begin{array}{l} x_iy_{i+1} - Dy_iy_{i+1} \\ y_ix_{i+1} - \ \ x_ix_{i+1}\end{array}\right\}.$$

Let $\tilde\Lambda_D^{(i,j)}$ be the full subcategory with the vertices $i, i+1, \dots, j, j+1$ and $\tilde\Lambda_D^{(i)} := \tilde\Lambda_D^{(i,i)}$, which is the Kronecker-category. For $i \in \mathbb{Z}$ and $C \in \mathrm{k}\cup\{\infty\}$, consider the module $M_C^{(i)}$ given by

$$M_C^{(i)}(j) = \begin{cases} \mathrm{k}, & \text{if } j \in \{i, i+1\}; \\ 0, & \text{else.}\end{cases}$$

$$M_C^{(i)}(\bar x_i) = \begin{cases} 1, & \text{if } C \in \mathrm{k}; \\ 0, & \text{if } C = \infty.\end{cases}$$

$$M_C^{(i)}(\bar y_i) = \begin{cases} C, & \text{if } C \in \mathrm{k}; \\ 1, & \text{if } C = \infty.\end{cases}$$

Since $\mathcal{R}^3\tilde\Lambda_D = 0$, then

$$\tilde\Lambda_D^{(i,j+1)} \cong \tilde\Lambda_D^{(i,j)}[M_1^{(j)} \oplus M_{D^{-1}}^{(j)}] \cong [M_0^{(i+1)} \oplus M_\infty^{(i+1)}]\tilde\Lambda_D^{(i+1,j+1)}$$

We will show, that $\tilde\Lambda_D$ is locally support finite. Let $X$ be an indecomposable $\tilde\Lambda_D$-module, we will show that $\operatorname{supp} X$ is contained in some $\tilde\Lambda_D^{i,i+1}$. By the structure of $\tilde\Lambda_D^{i,j}$ it is sufficient to prove the following:
If $\operatorname{supp} X$ is contained in $\tilde\Lambda_D^{i,i+2}$ and $X(i+3) \neq 0$, then $X(i) = 0$. Indeed, the restriction of $X$ to $\tilde\Lambda_D^{i,i+1}$ decomposes as $\oplus Y_j$, where each $Y_j$ is indecomposable and by 1.7.2

$$0 \neq \operatorname{Hom}_{\tilde\Lambda_D^{(i,i+1)}}(M_1^{(i+1)} \oplus M_{D^{-1}}^{(i+1)}, Y_j) \cong \operatorname{Hom}_{\tilde\Lambda_D^{(i+1)}}(M_1^{(i+1)} \oplus M_{D^{-1}}^{(i+1)}, Y_j|_{\tilde\Lambda_D^{(i+1)}}).$$

If some $Y_j$ has $Y_j(i) \neq 0$, then the restriction $Y_j|_{\tilde\Lambda_D^{(i+1)}}$ decomposes as $\oplus Z_t$ where each $Z_t$ is indecomposable and

$$0 \neq \operatorname{Hom}_{\tilde\Lambda_D^{(i+1)}}(Z_t, M_0^{(i+1)} \oplus M_\infty^{(i+1)}).$$

Therefore, $Z_t$ is a preprojective or a regular $\tilde\Lambda_D^{(i+1)}$-module in the tube of $M_0^{(i+1)}$ or $M_\infty^{(i+1)}$. Since $D \neq 1$, we conclude that $\operatorname{Hom}_{\tilde\Lambda_D^{i+1}}(M_1^{(i+1)} \oplus M_{D^{-1}}^{(i+1)}, Z_t) = 0$ which yields a contradiction. Hence, $X(i) = 0$.

## 2. Varieties and Representations

2.1. We recall some important facts from algebraic geometry.

**Lemma 2.1.1.** *Let $\pi : X \longrightarrow Y$ be a morphism between (not necessarily irreducible) varieties. Suppose, that $\sigma$ is a section of $\pi$ that meets each irreducible component of each fibre of $\pi$. Then the function $d\colon Y \longrightarrow \mathbb{N}_0,\ y \mapsto \dim \pi^{-1}(y)$ is upper semicontinuous.*

*Proof.* Let $X = \cup X_i$ be the decomposition of $X$ into irreducible components and $\pi_i$ the restriction of $\pi$ to $X_i$. Now Chevalley's theorem says that $d_i'\colon X_i \longrightarrow \mathbb{N}_0, x \mapsto \dim \pi_i^{-1}(\pi_i(x))$ is upper semicontinuous. By hypothesis $\sigma(Y) \cap X_i$ may be identified with the (closed) subset $\operatorname{Im} \pi_i \subset Y$, thus the map $d_i\colon Y \longrightarrow \mathbb{N}_0, y \mapsto \dim \pi^{-1}(y)$ is upper semicontinuous. Since $d(y) = \max_{i=1,\dots n} d_i(y)$, we are finished. $\square$

This is not new, but since we know no adequate reference we included the proof. Note however, that without further assumptions on $\pi$ the function $d$ of the lemma may happen to be *not* upper semicontinuous as the following example shows: Set $\pi\colon \mathrm{k}^3 \longrightarrow \mathrm{k}^3,\ (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \mapsto (\mathrm{x}_1, \mathrm{x}_2(\mathrm{x}_1\mathrm{x}_2 - 1), \mathrm{x}_3(\mathrm{x}_1\mathrm{x}_2 - 1))$. (Observe, that $\pi$ is surjective, but $\pi^{-1}(c, 0, 0) = \{(c, 0, 0)\} \cup \{(c, 1/c, t) \mid t \in \mathrm{k}\}$ if $c \neq 0$, while $\pi^{-1}(0, 0, 0) = \{(0, 0, 0)\}$.

**Lemma 2.1.2.** [Ga1] *Let $Z$ be a variety over a not countable (algebraically closed) field, $C_1 \subset C_2 \subset \dots$ an increasing sequence of constructible subsets and $S_1 \supset S_2 \supset \dots$ a decreasing sequence of open subsets with*

$$\bigcup_{i=1}^{\infty} C_i = \bigcap_{j=1}^{\infty} S_j$$

*Then we have $S_r = C_t$ for some $r, t$.*

**Lemma 2.1.3.** [P2, 1.4] *Let $V$ be an algebraic variety and $G$ be an algebraic group acting on $V$. Let $U_1$, $U_2$ be two constructible subsets of $V$ satisfying:*

$$U_1^G = V, \text{ and } U_2 \text{ intersects each orbit of } G \text{ in at most one point}$$

*Then* $\dim U_2 \leq \dim U_1$.

If $G$ is an algebraic group acting on a variety $V$, by definition $v_0 \in V$ is a *degeneration* of $v_1 \in V$ iff $v_0$ is contained in the (Zariski-) closure of the orbit $v_1^G$. There is the following useful characterization of degenerations. Again, by the lack of references we include a proof.

**Lemma 2.1.4.** *Let $v_0$ be a degeneration of $v_1$, then we can find a nonsingular, irreducible curve $C, c_0 \in C$ as well as regular maps $\lambda\colon C \longrightarrow V$ and $\gamma\colon C^* := C \setminus \{c_0\} \longrightarrow G$ such that $\lambda(c_0) = v_0$ and $v_1^{\gamma(c)} = \lambda(c)$ for all $c \in C^*$.*

*Proof.* (Compare [K, III.2.3 Lemma 1]) We may assume $V = \overline{v_1^G}$. Set $\mu\colon G \longrightarrow V, g \mapsto v_1^g$, the orbit map. First we get a curve $C' \subset G$ such that

$v_0 \in \overline{\mu(C')}$. With $R = \mathcal{O}(\overline{\mu(C')})$ and $S := \mathcal{O}(C')$ and $L := \operatorname{quot} S$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(G) \arrow[r] & S \arrow[r, "\subset"] & \tilde{S} \arrow[r, "\subset"] & L \\
\mathcal{O}(V) \arrow[r] \arrow[u, "\mu^*"'] & R \arrow[r, "\subset"] \arrow[u, "\cup"'] & \tilde{R} \arrow[u, "\cup"'] &
\end{tikzcd}$$

where $\tilde{R}$ (resp. $\tilde{S}$) is the integral closure of $R$ (resp. $S$) in $L$. Note that $\tilde{R}$ is a finitely generated $R$ module and $\operatorname{quot} \tilde{R} = L$ (see e.g. [ZS, V.Th.7]). Thus $\operatorname{Spec} \tilde{S}$ and $\operatorname{Spec} \tilde{R}$ are birationally equivalent nonsingular curves and it is easy to see that we can take $C = \operatorname{Spec} \tilde{R}$ (after removing if necessary a finite number of points), $\lambda$ and $\gamma$ as induced by the above diagram. $\square$

2.2. We fix $n \in \mathbb{N}$ and $\underline{d} \in \mathbb{N}_0^{n\times n}$ and will study the bounded k-categories $\Lambda$ with $\mathrm{Obj}(\Lambda) = \{1, \dots, n\}$ and Cartan - matrix $(\underline{d}(i,j) + \delta_{i,j})_{i,j \in \mathrm{Obj}\,\Lambda}$. Then $\mathrm{alg}(\underline{d})$ is the closed subset of the affine space

$$\prod_{i,j,k=1}^{n} \{\lambda_{i,j,k} \colon \mathrm{k}^{\underline{\mathrm{d}}(\mathrm{i,j})} \times \mathrm{k}^{\underline{\mathrm{d}}(\mathrm{j,k})} \longrightarrow \mathrm{k}^{\underline{\mathrm{d}}(\mathrm{i,k})} \mid \lambda_{\mathrm{i,j,k}}\mathrm{bilinear}\}$$

formed by tuples $(\lambda_{i,j,k})_{i,j,k=1,\dots,n}$ satisfying the equations (of trilinear maps) for associativity

$$\lambda_{i,k,l}((\lambda_{i,j,k}(-,-)),-) = \lambda_{i,j,l}(-,(\lambda_{j,k,l}(-,-))) \text{ for all } i,j,k,l = 1,\dots,n$$

as well as the equations of ($\underline{d}(i,i)$-linear maps) for nilpotency of the radical

$$\underbrace{\lambda_{i,i,i}(\lambda_{i,i,i}(\dots\lambda_{i,i,i}}_{\underline{d}(i,i)\text{-times}}(-,-),-),\dots,-),-) = 0 \text{ for } i = 1,\dots,n$$

Any $(\lambda_{i,j,k}) \in \mathrm{alg}(\underline{d})$ defines canonically a bounded k-category $\Lambda$: Set $\mathcal{R}\Lambda(i,j) := \mathrm{k}^{\mathrm{d(i,j)}}$ and define the composition of morphisms $i \xrightarrow{\mu} j \xrightarrow{\nu} k$ in $\mathcal{R}\Lambda$ as $\lambda_{i,j,k}(\mu,\nu) \in \mathcal{R}\Lambda(i,k)$; note that we have to determine the composition only on $\mathcal{R}\Lambda$. Thus we identify sometimes $\Lambda$ with $(\lambda_{i,j,k})$. On $\mathrm{alg}(\underline{d})$ operates the group $\mathrm{Gl}(\underline{d}) := \prod_{i,j\in\{1,\dots,n\}} \mathrm{Gl}_{\underline{d}(i,j)}(\mathrm{k})$ via transport of structure: Let $g = (g(i,j))_{i,j\in\{1,\dots,n\}}$ and $\lambda = (\lambda_{i,j,k,})_{i,j,k\in\{1,\dots,n\}}$ then

$$\lambda^g = \Big(g^{-1}(i,k) \cdot \lambda_{i,j,k}\,(g(i,j)\cdot -, g(j,k)\cdot -)\Big)_{i,j,k\in\{1,\dots,n\}}$$

Thus the orbits of $\mathrm{Gl}(\underline{d})$ on $\mathrm{alg}(\underline{d})$ correspond to the isoclasses of locally bounded k-categories $\Lambda$ with $\mathrm{Obj}\,\Lambda = \{1,\dots,n\}$ and Cartan-Matrix $C = (\underline{d}(i,j) + \delta_{i,j})_{i,j\in\{1,\dots,n\}}$ (if we admit only isomorphisms which fix the points).

Observe that $\mathrm{alg}(\underline{d})$ is connected since every structure on it degenerates into the structure $\overline{\Lambda} = (\overline{\lambda}_{i,j,k})$, where $\overline{\lambda}_{i,j,k} = 0$ for all indices. The quiver of $\overline{\Lambda}$ has $\underline{d}(i,j)$ arrows from $i$ to $j$ and $\mathcal{R}^2\overline{\Lambda} = 0$.

2.3. Let $\Lambda$ be as in 2.2 and $\underline{z} \in \mathbb{N}_0^n$ be a dimension vector. The variety of $\Lambda$-modules of dimension $\underline{z}$ is the closed subset $\mathrm{mod}_\Lambda(\underline{z})$ of the affine space

$$\prod_{i,j\in\{1,\dots,n\}} \{M_{i,j} : \mathrm{k}^{\underline{\mathrm{d}}(\mathrm{i,j})} \longrightarrow \mathrm{Hom_k}(\mathrm{k}^{\underline{\mathrm{z}}(\mathrm{j})}, \mathrm{k}^{\underline{\mathrm{z}}(\mathrm{i})}) \mid \mathrm{M_{i,j}}\ \text{k-linear}\}$$

formed by tuples $(M_{i,j})_{i,j\in\{1,\dots,n\}}$ which fulfill the equations (of bilinear maps) in order to respect the composition of morphisms in $\Lambda$

$$M_{i,j}(-)\cdot M_{j,k}(-) = M_{i,k}(\lambda_{i,j,k}(-,-)) \text{ for all } i,j,k \in \{1,\dots,n\}.$$

Any $(M_{i,j}) \in \mathrm{alg}(\underline{d})$ defines in a canonical way a $\Lambda$-module $M$: Set $M(i) := \mathrm{k}^{\underline{\mathrm{z}}(\mathrm{i})}$ and $M(\nu) := M_{i,j}(\nu)$ if $\nu \in \mathcal{R}\Lambda(i,j)$.

On $\mathrm{mod}_\Lambda(\underline{z})$ operates the group $\mathrm{G}(\underline{z}) = \prod_{i=1}^n \mathrm{Gl}_{\underline{z}(i)}(\mathrm{k})$ by conjugation (thus the orbits of $\mathrm{G}(\underline{z})$ on $\mathrm{mod}_\Lambda(\underline{z})$ correspond to the isoclasses of $\Lambda$-modules $M$ with $\underline{\dim} M = \underline{z}$). Clearly, $\dim \mathrm{G}(\underline{z}) = \underline{z}^2 := \sum_{i=1}^n \underline{z}(i)^2$.

The variety $\mathrm{mod}_\Lambda(\underline{z})$ contains the constructible subset $\mathrm{ind}_\Lambda(\underline{z})$ of indecomposable structures In addition we define the following $\mathrm{G}(\underline{z})$-invariant and closed subsets of $\mathrm{mod}_\Lambda(\underline{z})$:

$$\mathrm{mod}_\Lambda(\underline{z},t) := \{M \in \mathrm{mod}_\Lambda(\underline{z}) \mid [\mathrm{End}_\Lambda(M) : \mathrm{k}] \geq \mathrm{t}\}; \quad \mathrm{t} \in \{1,\dots,\underline{\mathrm{z}}^2\}$$

*Remark .* (1) $\mathrm{mod}_\Lambda(\underline{z},t)$ is the union of all $\mathrm{G}(\underline{z})$ orbits with (orbit-) dimension smaller then $\underline{z}^2 - t$. Furthermore any $M \in \mathrm{mod}_\Lambda(\Lambda)(\underline{z},t)$ degenerates into the semi-simple structure $S$ where $S_{i,j}$ is the zero-map for all $i,j$; thus $\mathrm{mod}_\Lambda(\underline{z},t)$ is connected. See also [Bo3, 2.1].

(2) We define $\mathrm{mod}_{\mathrm{k[x]}}(n) := \mathrm{End_k}(\mathrm{k^n})$ resp. $\mathrm{mod}_{\mathrm{k\langle x,y\rangle}}(n) := \mathrm{End_k}(\mathrm{k^n}) \times \mathrm{End_k}(\mathrm{k^n})$, on which operates $\mathrm{Gl}_n(\mathrm{k})$ by conjugation, resp. simultaneos conjugation. Thus the orbits correspond to the isoclasses of $n$-dimensional modules of the respective algebras. If $M$ is a $\Lambda$-$A$-bimodule with $A = \mathrm{k[x]}$, (resp. $A = \mathrm{k\langle x,y\rangle}$) as in the definition of tame (resp. wild), then the functor $M\otimes_A -$ induces regular maps $\mathrm{mod}_A(n) \longrightarrow \mathrm{mod}_\Lambda(n\underline{z})$ if $\underline{z}(i)$ is the rank of the free $A$-module $M(i)$, see [DoSk1, 1].

**Proposition .** *Fix $\underline{z} \in \mathbb{N}_0^n$ and $t \leq \underline{z}^2$. Then the function $\Lambda \mapsto \dim \mathrm{mod}_\Lambda(\underline{z},t)$ is upper semicontinuous on* $\mathrm{alg}(\underline{d})$

*Proof.* We define the variety $\mathrm{algmod}(\underline{d},\underline{z},t)$ which consists of pairs $(\Lambda, M)$ with $\Lambda \in \mathrm{alg}(\underline{d})$ and $M \in \mathrm{mod}_\Lambda(\underline{z},t)$. Then we have the map $\pi : \mathrm{algmod}(\underline{d},\underline{z},t) \to \mathrm{alg}\,\underline{d}$, $(\Lambda, M) \mapsto \Lambda$ and the section $\sigma : \mathrm{alg}\,\underline{d} \to \mathrm{algmod}(\underline{d},\underline{z},t)$, $\Lambda \mapsto (\Lambda, S)$, where $S$ is the semisimple structure in $\mathrm{mod}_\Lambda(\underline{z},t)$. By the above remark $\sigma$ fulfils the requirements of Lemma 2.1.1. □

2.4. We recall the following characterization of tame representation type from [P2]:

**Proposition .** *Let $\Lambda$ a bounded* k*-category as above, then the following are equivalent:*

a) *$\Lambda$ is tame.*
b) *For every $\underline{z} \in \mathbb{N}_0^n$ there is a constructible subset $C \subset \operatorname{ind}_\Lambda(\underline{z})$ such that $\mathrm{G}(\underline{z}) \cdot C = \operatorname{ind}_\Lambda(\underline{z})$ and $\dim C \leq 1$.*
c) *For every $\underline{z} \in \mathbb{N}_0^n$, if $C$ is a constructible subset of $\operatorname{ind}_\Lambda(\underline{z})$ which intersects each orbit of $\mathrm{G}(\underline{z})$ in at most one point, then $\dim C \leq 1$.*

*Proof.* a)$\Longrightarrow$ b) is clear (the $\Lambda$-k[t] bimodules induce regular maps $\mathrm{k} \longrightarrow \operatorname{mod}_\Lambda(\underline{z})$); b)$\Longrightarrow$ c) is Lemma 2.1.3; c)$\Longrightarrow$ a) follows, since for wild categories it is very easy to construct two-parameter families of indecomposable, pairwise nonisomorphic modules. $\square$

**Corollary .** *A locally bounded category is tame if and only if it is weakly tame.*

*Proof.* Let $\Lambda$ be a locally bounded category as before and fix a dimension vector $\underline{z} \in \mathbb{N}_0^{\operatorname{Obj}\Lambda}$ and suppose that there are $\Lambda$-k[x] bimodules $L_1, \dots, L_m$ as in the definition of weakly tame. Then set

$$\mathcal{F}^{\rangle}(\underline{\ddagger}) = \{Y \in \operatorname{mod}_\Lambda(\underline{z}) \mid Y \text{ is a direct summand of } M_i \otimes_{\mathrm{k[t]}} S \text{ for some simple k[t]-module S}\}$$

By hypothesis we have $\operatorname{ind}_\Lambda(\underline{z}) = \bigcup_{i=1}^m (\mathcal{F}^{\rangle}(\underline{\ddagger}) \cap \operatorname{ind}_\Lambda(\underline{\ddagger}))$. Taking into account Proposition c), we have to show that the sets $\mathcal{F}^{\rangle}(\underline{\ddagger})$ are constructible and if $C$ is a constructible subset of $\mathcal{F}^{\rangle}(\underline{\ddagger})$ which intersects each $\mathrm{G}(\underline{z})$ orbit in at most one point, then $\dim C \leq 1$.

Indeed, the proof for this is literally the same as the second part of the proof of [P2, 3.1]. Let $\underline{d}_i \in \mathbb{N}_0^{\operatorname{Obj}\Lambda}$ be such that $\underline{d}_i(x)$ is the rank of the free k[t]-module $L_i(x)$ and $f_i \colon \operatorname{mod}_{\mathrm{k[t]}}(1) \longrightarrow \operatorname{mod}_\Lambda(\underline{d}_i)$ the regular map induced by $L_i \otimes_{\mathrm{k[t]}} -$. Then $Z_i := \operatorname{Im} f_i$ is a constructible subset of $\operatorname{mod}_\Lambda(\underline{d}_i)$ with $\dim Z_i \leq 1$. Let $K_i'$ be the subset of

$$Z_i \times \left( \prod_{x \in \operatorname{Obj}\Lambda} \operatorname{End}_{\mathrm{k}}(\mathrm{k}^{\underline{\mathrm{d}}_{\mathrm{i}}(\mathrm{x})}) \right) \times \operatorname{mod}_\Lambda(\underline{z}) \times \left( \prod_{x \in \operatorname{Obj}\Lambda} \operatorname{Hom}_{\mathrm{k}}(\mathrm{k}^{\underline{\mathrm{z}}(\mathrm{x})}, \mathrm{k}^{\underline{\mathrm{d}}_{\mathrm{i}}(\mathrm{x})}) \right)$$

formed by tuples $(X, f, Y, j)$ satisfying:

$$f \in \operatorname{End}_\Lambda(X), \quad f^2 = f \quad \text{and} \quad 0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{f} X \text{ exact}$$

Clearly $K_i'$ is constructible. Therefore, the subset $K_i$ of $Z_i \times \operatorname{mod}_\Lambda(\underline{z})$ formed by pairs $(X, Y)$ such that $Y$ is a direct summand of $X$ is constructible. Consider the canonical projections

$$\begin{tikzcd} K_i \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Z_i \\ \mathcal{F}^{\rangle}(\underline{\ddagger}) & \end{tikzcd}$$

Then $\mathcal{F}^{\rangle}(\underline{\ddagger}) = \operatorname{Im}\pi_{\in}$ is constructible. Let $C$ be a constructible subset of $\mathcal{F}^{\rangle}(\underline{\ddagger})$ intersecting each orbit of $G(\underline{z})$ in at most one point. By the Krull-Schmidt theorem the induced regular map $\operatorname{res}\pi_1 : \pi_2^{-1}(C) \longrightarrow Z$ has only finite fibres. This implies that $\dim C \leq \dim \pi_2^{-1}(C) \leq \dim Z_i = 1$. □

2.5. The following description of tame representation type turns out to be crucial for our result on degenerations of categories.

**Theorem .** *Let $\Lambda$ be a locally bounded category as above, then the following are equivalent:*

a) *$\Lambda$ is tame.*
b) *For every $\underline{z} \in \mathbb{N}_0^n$ there is a constructible subset $C \subset \operatorname{mod}_\Lambda(\underline{z})$ such that $\mathrm{G}(\underline{z}) \cdot C = \operatorname{mod}_\Lambda(\underline{z})$ and $\dim C \leq |\,\underline{z}\,| := \sum_{i=1}^n \underline{z}(i)$.*
c) *For every $\underline{z} \in \mathbb{N}_0^n$ and every $t \in \{1, \dots, \underline{z}^2\}$ we have $\dim \operatorname{mod}_\Lambda(\underline{z}, t) \leq |\,\underline{z}\,| + (\underline{z}^2 - t)$.*

*Proof.* a)$\Longrightarrow$ b) Let $\underline{z} = \underline{e}_1 + \cdots + \underline{e}_m$ for some $0 \neq \underline{e}_i \in \mathbb{N}_0^n$ and take $C_i \subset \operatorname{ind}_\Lambda(\underline{e}_1)$ as in proposition 2.4 b). Then we can interpret $C_{\underline{e}_1,\dots,\underline{e}_m} := C_1 \times \cdots \times C_m$ naturally as constructible subset of $\operatorname{mod}_\Lambda \underline{z}$ by the Krull-Schmidt theorem every $M \in \operatorname{mod}_\Lambda \underline{z}$ is contained in some $C_{\underline{e}_1,\dots,\underline{e}_m}^{\mathrm{G}(\underline{z})}$, thus we can take for $C$ the (finite) union over all possible $C_{\underline{e}_1,\dots,\underline{e}_m}$, which has clearly dimension $\leq |\,\underline{z}\,|$. For a)$\Longleftrightarrow$ b) see also [DrGr, 2]).

b)$\Longrightarrow$ c) Let $C$ as in b) and set $C_t := C \cap \operatorname{mod}_\Lambda(\underline{z}, t)$, then $\mathrm{G}(\underline{z}) \cdot C_t = \operatorname{mod}_\Lambda(\underline{z}, t)$; since every orbit in $\operatorname{mod}_\Lambda(\underline{z}, t)$ has dimension at most $\underline{z}^2 - t$ we conclude $\dim \operatorname{mod}_\Lambda(\underline{z}, t) \leq |\,\underline{z}\,| + (\underline{z}^2 - t)$.

c)$\Longrightarrow$ a) This is a variation on the proof of Proposition 1.5. in [P1]. Assume $\Lambda$ not to be tame, thus $\Lambda$ is wild and let $M$ be a $\Lambda - \mathrm{k}\langle \mathrm{x}, \mathrm{y}\rangle$ bimodule as in the definition of wildness. Then let $\underline{z}(i)$ be the rank of the free $\mathrm{k}\langle \mathrm{x}, \mathrm{y}\rangle$-module $M(i)$. Consider the regular maps

$$f^{(n)} \colon \operatorname{mod}_{\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle}(n) \longrightarrow \operatorname{mod}_\Lambda(n\underline{z}) \quad \text{induced by} \quad M \otimes_{\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle} -$$

In $\operatorname{mod}_{\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle}(n)$ there is a open dense subset $U_n$ of module structures with stabilizer $\mathrm{k}^*$ (under the action of $\mathrm{Gl}_n(\mathrm{k})$); thus $\dim U_n = 2n^2$. Consider now the regular map

$$\hat{f}^{(n)} \colon \mathrm{G}(n\underline{z}) \times U_n \longrightarrow \operatorname{mod}_\Lambda(n\underline{z}), \quad (g, Y) \mapsto \left(f^{(n)}(Y)\right)^g.$$

Let $X \in \operatorname{Im} \hat{f}^{(n)}$, then we find as in [P1, 1.5]

$$[\mathrm{End}_\Lambda(X) : \mathrm{k}] + \mathrm{n}^2 - 1 \geq \dim(\hat{\mathrm{f}}^{(\mathrm{n})})^{-1}(\mathrm{X}) \geq \mathrm{n}^2\underline{\mathrm{z}}^2 + 2\mathrm{n}^2 - \dim \operatorname{Im} \hat{\mathrm{f}}^{(\mathrm{n})}.$$

Now choose $X_0 \in \operatorname{Im} \hat{f}^{(n)}$ with $[\mathrm{End}_\Lambda(X_0) : \mathrm{k}] = \mathrm{t}_0$ minimal, then $\operatorname{Im} \hat{f}^{(n)} \subset \operatorname{mod}_\Lambda(n\underline{z}, t_0)$ and $\dim \operatorname{mod}_\Lambda(n\underline{z}, t_0) \geq \dim \operatorname{Im} \hat{f}^{(n)} \geq n^2\underline{z}^2 + n^2 + 1 - t_0$; this contradicts condition d) for $n \geq |\,\underline{z}\,|$. □

**Example .** Let $\Lambda := \mathrm{k}[\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3]/(\mathrm{a}_\mathrm{i}\mathrm{a}_\mathrm{j} \mid 1 \leq \mathrm{i} \leq \mathrm{j} \leq 3)$.
$\Lambda$ is well known to be wild, however in [P1, 2] it is shown that $\dim \operatorname{mod}_\Lambda(2n) = 4n^2$ and $\dim \operatorname{mod}_\Lambda(2n+1) = 4n^2 + 4n$.

Consider the $\Lambda - \mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle$ bimodule $M$ with $M(\cdot) = (\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle)^2$ and

$$M(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad M(a_2) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}; \quad M(a_3) = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$$

It is trivial to check that $M$ is a bimodule as required in the definition of wild. The functor $M \otimes_{\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle} -$ induces regular maps

$$\begin{aligned} f^{(n)} \colon \mathrm{mod}_{\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle}(n) &\longrightarrow \mathrm{mod}_{\Lambda}(2n), \\ (N(x), N(y)) &\mapsto \left( \left(\begin{smallmatrix} 0 & 0 \\ I_n & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ N(x) & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ N(y) & 0 \end{smallmatrix}\right) \right). \end{aligned}$$

We study $\mathrm{Im}\, \hat{f}^{(n)}$ as in the proof above and find

$$\dim \mathrm{Im}\, \hat{f}^{(n)} \geq 4n^2 + n^2 + 1 - t_0$$

Now for $Y \in U_n$ we have

$$\mathrm{End}_{\Lambda}(\hat{f}^{(n)}(\mathrm{I}_n, Y)) = \left\{ \left(\begin{smallmatrix} f, & 0 \\ g, & f \end{smallmatrix}\right) \mid f \in \mathrm{End}_{\mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle}(Y), g \in \mathrm{End}_{\mathrm{k}}(\mathrm{k}^{\mathrm{n}}) \right\}$$

Since all $Y \in U_n$ have trivial endomorphism rings we conclude that $\dim \mathrm{End}_{\Lambda}(X) = n^2 + 1$ for all $X \in \mathrm{Im}\, \hat{f}^{(n)}$, thus $\mathrm{Im}\, \hat{f}^{(n)} \subset \mathrm{mod}_{\Lambda}(2n, n^2+1)$ and

$$4n^2 \leq \dim \mathrm{mod}_{\Lambda}(2n, n^2+1) \leq \dim \mathrm{mod}_{\Lambda}(2n) = 4n^2$$

For a tame local algebra $\Lambda'$ our proposition allows however only the inequality $\dim \mathrm{mod}_{\Lambda'}(2n, n^2+1) \leq 3n^2 + 2n - 1$.

2.6. **Degenerations of algebras.** A direct consequence of Proposition 2.3 and Theorem 2.5 c) is:

**Theorem .** *Let $\Lambda_0$ be a degeneration of $\Lambda_1$ in $\mathrm{alg}\,\underline{d}$, then $\Lambda_0$ tame implies $\Lambda_1$ tame.*

*Proof.* $\Lambda_0$ lies in the closure of $\Lambda_1^{G(\underline{d})}$, thus $\dim \mathrm{mod}_{\Lambda_1}(\underline{z}, t) \leq \dim \Lambda_0\text{-}\mathrm{mod}(\underline{z}, t)$ for all $\underline{z} \in \mathbb{N}_0^n$ and $1 \leq t \leq \underline{z}^2$ by 2.3. Now 2.5 c) implies, that from $\Lambda_0$ tame follows $\Lambda_1$ tame. $\square$

**Example .** Usually a bounded k-category $\Lambda$ is given by its quiver $Q_\Lambda$ and relations, and not by its structure constants. For example consider in alg(6), the variety of 7-dimensional local algebras and the algebras $\Lambda_T = \mathrm{k}(\mathrm{W_0 Q})/\mathrm{I}^{\pi(\mathrm{T})}$, where $\mathrm{Q} = \alpha \circlearrowright \cdot \circlearrowleft \beta$ and $I^{\pi(T)}$ is generated by $\alpha^2 - T\beta\alpha\beta$, $\beta^2 - T\alpha\beta\alpha$ and $(\alpha\beta)^2$, $(\beta\alpha)^2$; now it is a trivial matter to calculate the structure constants of $\Lambda_T$ with respect to the base $\mathcal{B} = \{\bar{\alpha}, \bar{\beta}, \bar{\alpha}\bar{\beta}, \bar{\beta}\bar{\alpha}, \bar{\alpha}\bar{\beta}\bar{\alpha}, \bar{\beta}\bar{\alpha}\bar{\beta}\}$ of $\mathcal{R}\Lambda_T$. Then we set

$$g(T) = \begin{pmatrix} T^{-1} & & & & & \\ & T^{-1} & & & 0 & \\ & & T^{-2} & & & \\ & & & T^{-2} & & \\ & 0 & & & T^{-3} & \\ & & & & & T^{-3} \end{pmatrix} \in \mathrm{Gl}_6(\mathrm{k})$$

and find $\Lambda_T = \Lambda_1^{g(T)}$ for all $T \in \mathrm{k}^*$. Thus the special biserial algebra $\Lambda_0$ is a degeneration of $\Lambda_1$ and our theorem gives (as far as we know) the first proof for the tameness of $\Lambda_1$ in the case of $\operatorname{char} \mathrm{k} \neq 2$.

Similarily we find in Ringel's list of tame local algebras [Ri1] degenerations of No. (3),(2) into (1) and of No. (5),(6),(7),(8) into (9), where (1) and (9) as special biserial algebras are well known to be tame. Sadly enough our method provides no description of the modules. Note that in 4.3 we will give a proof of the tameness of No. (4).

Our observation from the example above generalizes a little bit: Suppose, we have a family of locally bounded categories $\Lambda_T$, $(T \in \mathrm{k})$ in $\mathrm{alg}(\underline{d})$ given by a quiver $Q$ and relations which depend algebraically on $T$, and suppose furthermore that there are for all $i, j \in \{1, \ldots, n\}$ paths $\mathcal{P}(i,j) := \{w_1, \ldots, w_{\underline{d}(i,j)}\}$ in $Q$ such that the $\bar{w}_i$ form a base $\mathcal{B}(i,j)$ of $\mathcal{R}\Lambda_T(i,j)$ for every $T \in \mathrm{k}$. Then to see that $\Lambda_0$ is a degeneration of $\Lambda_1$ it is often sufficient to check that the generators of $I^{\pi(T)}$, $T \in \mathrm{k}^*$ are obtained from the generators of $I^{\pi(1)}$ by a 'change of presentation' $\alpha \mapsto \hat{\alpha} = T^{i(\alpha)}\bar{\alpha}$ (with $i(\alpha) \in \mathbb{Z}$ for the arrows $\alpha$ of $Q$).

**Example .** See [E, p. 295/303]. Let $Q : \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd}$ and $\Lambda_T := Q/I^{\pi(T)}$ with generators $\gamma\beta - T^5\eta^2$, $\beta\eta - T^5(\alpha\beta\gamma)\alpha\beta$, $\eta\gamma - T^5(\gamma\alpha\beta)\gamma\alpha$, $\alpha^2 - T^3(\beta\gamma\alpha)\beta\gamma + T^6(\beta\gamma\alpha)^2$, $(\alpha\beta\gamma)^2 - (\beta\gamma\alpha)^2$, $(\gamma\alpha\beta) - \eta^3$ for $I^{\pi(T)}$. We find that the following paths will give bases of $\Lambda_T(i,j)$:

$$\begin{aligned}
\mathcal{P}(1,1) &= \{\alpha, \beta\gamma, \alpha\,\beta\gamma, \beta\gamma\,\alpha, \alpha\,\beta\gamma\,\alpha, \beta\gamma\,\alpha\,\beta\gamma, (\alpha\,\beta\gamma)^2\};\\
\mathcal{P}(2,2) &= \{\eta, \eta^2, \eta^3, \gamma\alpha\beta\};\\
\mathcal{P}(1,2) &= \{\beta, \alpha\,\beta, \beta\gamma\,\alpha\,\beta, \alpha\,\beta\gamma\,\alpha\,\beta\};\\
\mathcal{P}(2,1) &= \{\gamma, \gamma\,\alpha, \gamma\,\alpha\,\beta\gamma, \gamma\,\alpha\,\beta\gamma\,\alpha\}.
\end{aligned}$$

Study now the 'change of presentation'

$$\hat{\gamma} = T^{-3}\bar{\gamma}, \hat{\eta} = T^{-4}\bar{\eta}, \hat{\alpha} = T^{-3}\bar{\alpha}, \hat{\beta} = \bar{\beta};$$

then we obtain for $T \in \mathrm{k}^*$ just the geenrators of $I^{\pi(T)}$. Thus again $\Lambda_0$ is a degeneration of $\Lambda_1$. Similarly we find for every algebra of the family $Q(2B)_1$ (in the notation of [E]) a degeneration into one of the form $D(2B)$ which is special biserial, thus tame. There should be a theoretical reason why many of the algebras of quaternion and semidihedral type admit degenerations into algebras of dihedral type.

Similarly we may prove that $\Lambda 12_m$ (table T) is tame. Thus study the following (algebraic) family of categories $\Lambda_T$ with quiver $\begin{tikzcd} a \arrow[loop left, "\sigma"] \arrow[r, shift left, "\nu"] & b \arrow[l, shift left, "\gamma"] \arrow[loop right, "\rho"] \end{tikzcd}$ and presentation $\pi(T)$ which we determine by the following generators of $I^{\pi(T)}$: $\bar{\gamma}\bar{\nu} - T\bar{\rho}^m, \bar{\gamma}\bar{\sigma}\bar{\nu}, \bar{\nu}\bar{\rho}, \bar{\rho}\bar{\gamma}, \bar{\sigma}^2 - T\bar{\nu}\bar{\gamma}, \bar{\rho}^{m+1}, \bar{\nu}\bar{\gamma}\bar{\nu}, \bar{\gamma}\bar{\nu}\bar{\gamma}$. We find the following base: $\mathcal{B}(\rangle, |)$ for $\mathcal{R}\Lambda_T(i,j)$ (independent of $T$): $\mathcal{B}(\dashv, \dashv) = \{\bar{\sigma}, \bar{\nu}\bar{\gamma}\}$, $\mathcal{B}(a,b) = \{\bar{\nu}, \bar{\sigma}\bar{\nu}\}$, $\mathcal{B}(\lfloor, \dashv) = \{\bar{\gamma}, \bar{\gamma}\bar{\sigma}\}$, $\mathcal{B}(\lfloor, \lfloor) = \{\bar{\rho}, \bar{\rho}^{\in}, \ldots, \bar{\rho}^{\Updownarrow}\}$. Furthermore

$\Lambda_T \cong \Lambda 12_m$ for all $T \in k^*$ and $\Lambda_0$ is special biserial, thus $\Lambda 12_m$ degenerates into the tame category $\Lambda_0$

2.7. We use the oportunity to discuss within our context some aspects of Gabriel's result "Finite representation type is open". Write $\mathrm{alg}_{\mathrm{fin}}(\underline{d})$ for the structures with finite representation type in $\mathrm{alg}(\underline{d})$.

**Proposition .** *We have*

$$\mathrm{alg}_{\mathrm{fin}}(\underline{d}) = \bigcap_{\substack{\underline{z}\in\mathbb{N}_0^n,\\ 1\le t\le \underline{z}^2}} \{\Lambda \in \mathrm{alg}(\underline{d}) \mid \dim \mathrm{mod}_\Lambda(\underline{z},t) \le \underline{z}^2 - t\}$$

*Proof.* The inclusion $\subseteq$ is clear, since for representation-finite $\Lambda$ the varieties $\mathrm{mod}_\Lambda(\underline{z},t)$ only consist of a fionite number of orbits. The other inclusion is exactly Brauer-Thrall II (see e.g. [NRo]), that is for representation infinite $\Lambda$ there exists in some dimension an infinite family of (indecomposable) modules. $\square$

Now there are two possibilities to give a new proof of "$\mathrm{alg}_{\mathrm{fin}}(\underline{d})$ *is open in* $\mathrm{alg}(\underline{d})$":

(1) We can repeat Gabriel's proof (using Auslander's construction [A], compare also [M]) to see that $\mathrm{alg}_{\mathrm{fin}}(\underline{d})$ is a countable union of constructible subsets. Then follows, if k is not countable from lemma 2.1.2, the above propsition and proposition 2.3, that $\mathrm{alg}_{\mathrm{fin}}$ is open.

(2) On the other hand there is a *numerical version* of Brauer-Thrall II, essentialy based on the deep results of [BGRS] (see remark below) that is, there is some number $D = D(\underline{d})$ such that if $\Lambda$ in $\mathrm{alg}(\underline{d})$ is representation infinite, then there is for some $\underline{z}_0 \in \mathbb{N}_0^n$ with $|\underline{z}_0| \le D(\underline{d})$ a (algebraic) 1-parameter family in $\mathrm{mod}_\Lambda(\underline{z}_0)$ that intersects each orbit in at most one point. This implies, that $\dim \mathrm{mod}_\Lambda(\underline{z}_0,t_0) > \underline{z}_0^2 - t_0$ for some $t_0 \in \{1,\dots \underline{z}_0^2\}$. Thus we obtain even a sharper version of the above proposition:

$$\mathrm{alg}_{\mathrm{fin}}(\underline{d}) = \bigcap_{\substack{\underline{z}\in\mathbb{N}_0^n,\ |\underline{z}|\le D\\ 1\le t\le \underline{z}^2}} \{\Lambda \in \mathrm{alg}(\underline{d}) \mid \dim \mathrm{mod}_\Lambda(\underline{z},t) \le \underline{z}^2 - t\}$$

and it follows immediatly from proposition 2.3 that this is open. Note that here we can avoid the hypothesis that k should be not countable.

*Remark .* To prove this numerical version of Brauer-Trall II one can reduce to minimal representation infinite categories $\Lambda$. (i.e. every proper quotient is representation finite). If such $\Lambda$ is not distributive, the required family can be constructed as quotients of some indecomposable projective module, see 3.3.

If $\Lambda$ is distributive it follows from [BGRS] and [Bo2] that it is of the form k(P) for some ray category P. (It is not too hard to see, that the hypothesis about chains made in these works can be dropped, see for example [Ge1].) Now, if this P contains no infinite chains, it follows from [F1] (see also [F2]) and [Bo1], that the required family exists in some $\mathrm{mod}_\Lambda(\underline{z})$ with $|\underline{z}| \le$

$4(\sum(\underline{d}(i,j)+\delta_{i,j}))+30$. In this case the modules in the family are even pairwise nonisomorphic.

If P contains infinite chains, the family can be constructed as in [BGRS, 3.7], but there is only a finite number of isoclasses of such $\Lambda = \mathrm{k}(\mathrm{P})$ for fixed $\underline{d}$.

A finer analysis of the possible chains in the last case should show, that anyway $D(\underline{d})$ can be chosen as $4(\sum(\underline{d}(i,j)+\delta_{i,j}))+30$.

## 3. Cleaving functors

3.1. Associated with each k-linear functor $F : \Gamma \longrightarrow \Lambda$ between locally bounded k-categories we have the *restriction* or *pull-up* functor $F. : \Lambda\text{-MOD} \longrightarrow \Gamma\text{-MOD}$ which is given by $F.M := M \circ F$ for $M \in \Lambda\text{-MOD}$. The restriction functor admits a left-adjoint, the *extension* or *push-down* functor $F_\lambda : \Gamma\text{-MOD} \longrightarrow \Lambda\text{-MOD}$ which is uniquely determined (up to isomorphy) by the requirement, that it commutes with direct limits (i.e. with cokernels and direct summs) and that $F_\lambda\Gamma(*,x) = \Lambda(-,Fx)$. The composition $F.F_\lambda$ is connected to the identity in $\Gamma$-MOD by the canonical transformation $\Phi_F : \mathrm{id}_{\Gamma\text{-MOD}} \longrightarrow F.F_\lambda$ which is uniquely determined by the requirement,that $\Phi_F(\Gamma(*,x))$ coincides for all $x \in \operatorname{Obj}\Gamma$ with the map $\Gamma(*,x) \longrightarrow \Lambda(F*,Fx)$ associated with $F$.

**Definition .** The functor above is called *cleaving* if the canonical transformation $\Phi_F$ admits a retraction, i.e. if there is a transformation $\Psi : F.F_\lambda \longrightarrow \mathrm{id}_{\Gamma\text{-MOD}}$ such that $\Psi(M)\Phi_F(M) = \mathrm{id}_M$ for each $M \in \Gamma\text{-MOD}$

*Remark .* The above definitions are almost literally taken from [BGRS, 3]. There it was shown, that if $F$ is cleaving, then $\Gamma$ not locally representation finite implies that $\Lambda$ is not locally representation finite.

Furthermore there the authors give a more accessible characterization: $F : \Gamma \longrightarrow \Lambda$ is cleaving if and only if the natural transformation

$$\varphi_F : \Gamma(*_1,*_2) \longrightarrow \Lambda(F*_1,F*_2) \text{ of functors } \Gamma^{\mathrm{op}} \times \Gamma \longrightarrow \text{k-mod}$$

admits a retraction, i.e. if there is a transformation

$$\epsilon : \Lambda(F*_1,F*_2) \longrightarrow \Gamma(*_1,*_2) \text{ of functors } \Gamma^{\mathrm{op}} \times \Gamma \longrightarrow \text{k-mod}$$

such that $\epsilon(x,y)\varphi_F(x,y) = \mathrm{id}_{\Gamma(x,y)}$ for all $x,y \in \Gamma$. In other words we will have to give a decomposition $S \oplus \Gamma = \Lambda(D*_1,D*_2)$ of $\Gamma$-$\Gamma$ bimodules if we want to prove that $F$ is cleaving; in this case $S$ will be called a *cleavage* for $F$.

3.2. The significance of cleaving functors for the study of tame categories is based on the following:

**Proposition .** *Suppose that $F : \Gamma \longrightarrow \Lambda$ is cleaving. Then $\Gamma$ wild implies $\Lambda$ wild.*

*Proof.* By the tame-wild Theorem of Drozd (see e.g [CB]) and 2.4 it will suffice to prove that we have in this situation $\Lambda$ tame $\Longrightarrow \Gamma$ weakly tame. Because of [DoSk1, 2] we may suppose, that $\Gamma$ is finite (the full inclusion functor is cleaving, and the composition of cleaving-functors is cleaving). Let $\underline{e} = (\underline{e}(y))_{y \in \mathrm{Obj}\,\Gamma}$ be a dimension vector.

We will prove that there are only finitely many dimension vectors $\underline{z} = (\underline{z}(x))_{x \in \mathrm{Obj}\,\Lambda}$ with $\underline{z} = \underline{\dim} F_\lambda(N)$ for some $\Gamma$-module $N$ with $\underline{\dim} N = \underline{e}$. Indeed we have canonical isomorphisms

$$(F_\lambda N)(x) \cong \mathrm{Hom}_\Lambda(F_\lambda N, D\Lambda(x,-)) \cong \mathrm{Hom}_\Gamma(N, D\Lambda(x, F*))$$

from which we see:

$$\dim(F_\lambda N)(x) \le \sum_{y \in \mathrm{Obj}\,\Gamma} \underline{e}(y) \cdot \dim \Lambda(x, Fy)$$

Thus we conclude, since $\mathrm{Obj}\,\Gamma$ is finite and $\Lambda$ locally finite, that $\mathrm{supp}(\underline{z})$ is finite and that $\underline{z}(x) \le C|\underline{e}|$.

Since $\Lambda$ is tame, there is a finite number of $\Lambda$-k[t]-bimodules $M_i$ which are finitely generated free right-k[t] modules and parametrize all $\mathrm{ind}_\Lambda(\underline{z})$, where $\underline{z}$ fullfils the above estimation. From the cleaving property we conclude, that any indecomposable $N \in \Gamma$-mod with $\underline{\dim} N = \underline{e}$ is isomorphic to a direct summand of $F.(M_i) \otimes_{\mathrm{k[T]}} S$ for some simple k[t]-module $S$. In other words $\Gamma$ is weakly tame. □

3.3. In practice, if we want to produce a cleavage for a given functor $F\colon \Gamma \longrightarrow \Lambda$, we will work only with functors, where $\Gamma$ may be identified with a subbimodule of $\Lambda(D*_1, D*_2)$. Furthermore we will consider as sources $\Gamma$ always categories where the endomorphism rings $\Gamma(x,x)$ all are trivial. Thus the natural choice for $S(x,x)$ is $\mathcal{R}\Lambda(Fx,Fx)$, which in turn impose on $S$ the condition $S(x,y) \supset \mathcal{R}\Lambda(x,x)\Gamma(x,y) + \Gamma(x,y)\mathcal{R}\Lambda(y,y)$. In particular, if $\Gamma(x,y)$ generates the bimodule ${}_{\Lambda(Fx,Fx)}\Lambda(Fx,Fy)_{\Lambda(Fy,Fy)}$ the inclusion becomes an equality; also, if $\Gamma(x,y) = 0$ we have to take $S(x,y) = \Lambda(Fx,Fy)$. We shall not specify $S(x,y)$ in such cases. For the remaining cases we have to give a decomposition of vectorspaces $\Lambda(Dx,Dy) = \Gamma(x,y) \oplus S(x,y)$ such, that $S(x, s_\beta)F\bar{\beta} \subset S(x,y)$ and $F\bar{\alpha}S({}_\alpha s, y) \subset S(x,y)$ for all arrows $x \xrightarrow{\alpha} {}_\alpha s$ starting in $x$ and all arrows $s_\beta \xrightarrow{\beta} y$ ending in $y$.

We describe a cleaving functor $F$ always by the quiver $\mathrm{Q}_\Gamma$ where we write over every arrow $\varphi \in \mathrm{Q}_\Gamma$ the image $F(\bar{\varphi})$. This determines $F$ uniquely, thus we will call this data a *cleaving diagram*.

**Example .** (1) $\Lambda$ tame $\Longrightarrow [\mathrm{rad}^i\Lambda(x,y)/\mathrm{rad}^{i+1}\Lambda(x,y) : \mathrm{k}] \le 2$ for all $x, y \in \mathrm{Obj}\,\Lambda$, $i \in \mathbb{N}$.

*Proof.* We may suppose $[\mathrm{rad}^i\Lambda(x,y) : \mathrm{k}] = 3$ and $\mathrm{rad}^{i+1}\Lambda(x,y) = 0$. Thus let $\alpha, \beta, \gamma$ be a base of $\mathrm{rad}^i\Lambda(x,y)$. Then we have the following diagram:

$$1 \;\substack{\xrightarrow{\alpha} \\ \xrightarrow{\beta} \\ \xrightarrow{\gamma}}\; 2$$

Here we can take as $S(1,2)$ any complement of $\mathrm{rad}^i\Lambda(x,y)$ in $\Lambda(x,y)$, while the remaining spaces are clear by our convention. The bimodule stucture of $S$ is trivial, since the $\alpha, \beta, \gamma$ are in the socle of the $\Lambda(x,x)$-$\Lambda(y,y)$-bimodule. $\square$

(2) By the same type of argument we see, that a locally representation finite category is distributive, see 5.4. Indeed, if $\Lambda$ is not distributive we find $[\mathrm{rad}^i\Lambda(x,y)/\mathrm{rad}^{i+1}\Lambda(x,y) : \mathrm{k}] \geq 2$ for some $x, y \in \mathrm{Obj}\,\Lambda$, $i \in \mathbb{N}$ and we obtain a cleaving diagram with the Kronecker-category $\Gamma : 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$. Moreover, $M_C \in \Gamma$-mod with $M_C(1) = M_C(2) = \mathrm{k}$, $M_C(\alpha) = \mathrm{id}_{\mathrm{k}}$, $M_C(\beta) = C \cdot \mathrm{id}_{\mathrm{k}}$ (and $C \in \mathrm{k}$) is a 1-parameter family of pairwise nonisomorphic indecomposable modules. We have an exact sequence

$$0 \longrightarrow \Gamma(*,1) \xrightarrow{f_C} \Gamma(*,2) \xrightarrow{p_C} M_C \longrightarrow 0$$

and thus an exact sequence

$$\Lambda(-,F1) \xrightarrow{F_\lambda f_C} \Lambda(-F2) \xrightarrow{F_\lambda p_C} F_\lambda M_C \longrightarrow 0$$

Since $M_C$ is a direct summand of $F.F_\lambda M_C$ we conclude by the Krull-Schmidt-theorem, that for a fixed $C_0$ there are only finitely many $C_i \in \mathrm{k}$ such that $F_\lambda M_{C_i} \cong F_\lambda M_{C_0}$.

(3) Next we show, that the local algebra $\Lambda := \mathrm{k}\langle \mathrm{x}, \mathrm{y}\rangle/(\mathrm{x}^2 - \mathrm{y}^2, \mathrm{yx})$ is wild, this is (c) in [Ri1]; we interpret $\Lambda$ in the obvious way as locally bounded category with one object. Thus consider the following diagram which is tilted of type $\tilde{\tilde{E}}_8$:

$$\begin{tikzcd}
 & & & & 9 \arrow[r, "\bar{y}"] \arrow[d, "\bar{x}"] & 10 \\
 & & & 7 \arrow[r, "\bar{y}"] \arrow[d, "\bar{x}"] & 8 & \\
\Gamma & 4 \arrow[r, "\bar{x}"] \arrow[d, "\bar{y}"'] & 5 \arrow[r, "\bar{y}"] \arrow[d, "x"'] & 6 & & \\
1 \arrow[r, "\bar{x}"] & 2 \arrow[r, "\bar{y}"] & 3 & & &
\end{tikzcd}$$

Observe that the morphisms of $\Gamma(i,j)$ is always mapped to one of the summands of

$$\Lambda = \Lambda(1,1) = \mathrm{k}1_\Lambda \oplus \mathrm{k}\bar{\mathrm{x}} \oplus \mathrm{k}\bar{\mathrm{y}} \oplus \mathrm{k}\bar{\mathrm{x}}\bar{\mathrm{y}} \oplus \mathrm{k}\bar{\mathrm{x}}^2,$$

thus we can take as $S(i,j)$ the sum of the four complementary summands (in accordance with our convention!). Furthermore $\bar{x} \notin \Lambda\bar{y} + \bar{y}\Lambda$, $\bar{y} \notin \Lambda\bar{x} + \bar{x}\Lambda$ and $\bar{x}\bar{y} \notin \Lambda\bar{x} + \bar{y}\Lambda$ from which directly follow the inclusions which show that $S$ is a bimodule.

## 4. Galois Coverings

We will use covering techniques mainly for tameness proofs. On the other hand we will see that this method is not as powerful as in the representation finite case.

4.1. Let $\Lambda$ be a locally bounded category. Let $G$ be a group of k-linear automorphisms of $\Lambda$; we say that $G$ is *acting freely* on $\Lambda$, if $g(x) = x$ for some $x \in \operatorname{Obj}\Lambda$ implies $g = 1_G$.

The *quotient category* $\Lambda/G$ has as objects the $G$-orbits of $G$ in $\operatorname{Obj}\Lambda$; a morphism $a \xrightarrow{\mu} b$ in $\Lambda/G$ is a family $\mu = ({}_y\mu_x) \in \prod_{x\in a, y\in b} \Lambda(x,y)$ such, that $g_y\mu_x = {}_{gy}\mu_{gx}$ for all $g \in G$. The composition of $a \xrightarrow{\mu} b \xrightarrow{\nu} c$ in $\Lambda/G$ is defined by ${}_x\mu\nu_z = \sum_{y\in b} {}_x\mu_y\, {}_y\mu_z$; this sum is well defined since $\Lambda$ is locally bounded. The canonical projection $F\colon \Lambda \to \Lambda/G$ maps $x \in \operatorname{Obj}\Lambda$ onto its orbit $Gx$; for $\xi \in \Lambda(x,y)$ we have ${}_{hx}F\xi_{gy} = g\xi$ or 0 according as $h = g$ or $h \neq g$. Thus $F$ is a covering functor, that is, the induced maps

$$\oplus_{Fy=a}\Lambda(x,y) \to \Lambda/G(Fx,a) \quad \text{and} \quad \oplus_{Fy=a}\Lambda(y,x) \to \Lambda/G(a,Fx)$$

are bijective for all $x \in \operatorname{Obj}\tilde{\Lambda}$ and $a \in \operatorname{Obj}\Lambda$. This motivates the following:

**Definition .** A k-linear functor $F\colon \tilde{\Lambda} \to \Lambda$ is a *Galois covering* defined by the action of the group $G$ if the following requirements are satisfied:

a) $G$ is a group of k-linear automorphisms acting freely on $\tilde{\Lambda}$.
b) $F \circ g = F$ for every $g \in G$.
c) $F$ is onto on objects and $G$ acts transitively on $F^{-1}(a)$ for every $a \in \operatorname{Obj}\Lambda$.
d) $F$ is a covering functor, that is, the induced maps

$$\oplus_{Fy=a}\tilde{\Lambda}(x,y) \to \Lambda(Fx,a) \quad \text{and} \quad \oplus_{Fy=a}\tilde{\Lambda}(y,x) \to \Lambda(a,Fx)$$

are bijective for all $x \in \operatorname{Obj}\tilde{\Lambda}$ and $a \in \operatorname{Obj}\Lambda$.

*Remark .* Note that covering functors are cleaving.

4.2. Let $F\colon \Lambda \to \Lambda/G$ be a Galois covering. Then $G$ acts on $\Lambda$-MOD by translation: We define $M^g := M \circ g$. We write $G_M := \{g \in G \mid M^g \cong M\}$ for the *stabilizer* of $M$.

Recall from 3.1 that $F$ induces a pull-up $F.\colon \Lambda/G\text{-MOD} \to \Lambda\text{-MOD}$; note that $(F.N)^g = F.N$ for all $N \in \Lambda$-MOD. This in turn induces an equivalence $\Lambda/G\text{-MOD} \cong \Lambda\text{-MOD}^G$, where the last one is the category of $G$-invariant $\Lambda$-modules and $G$-equivariant $\Lambda$-homomrphisms.

The push down $F_\lambda\colon \Lambda\text{-MOD} \longrightarrow \Lambda/G\text{-MOD}$ admits a explicit description: For $M \in \Lambda\text{-MOD}$ we can set

$$(F_\lambda M)(a) = \bigoplus_{x\in F^{-1}(a)} M(x) \quad \text{for all } a \in \operatorname{Obj}\Lambda/G;$$

if $a \xrightarrow{\xi} b$ is a morphism in $\Lambda/G$, we will have

$$(F_\lambda M)(a) \xleftarrow{(F_\lambda M)(\xi)} (F_\lambda M)(b), \quad (m_y)_{y\in F^{-1}(b)} \mapsto \left(\sum_y (M({}_x\xi_y))(m_y)\right)_{x\in F^{-1}(a)}$$

Furthermore we have $F_\lambda M^g \cong F_\lambda M$ and $\oplus_{h\in G} M^g \cong F.F_\lambda M$ for each $M \in \Lambda\text{-MOD}$. Thus, if $M$ is indecomposable and $M^g \cong M$ implies $g = 1$ we infer, that $F_\lambda M$ is indecomposable.

4.3. A particularly nice situation occurs, when $G$ acts freely on the isomorphism classes of indecomposable $\Lambda$-modules. This hypothesis is satisfied for example if $G$ is torsion-free.

**Theorem .** *Let $\Lambda \longrightarrow \Lambda/G$ be a Galois covering. Suppose, that $G$ acts freely on $(\Lambda\text{-ind})/\cong$. Then we have:*

- *a) $F_\lambda$ induces an injection from the set $((\Lambda\text{-ind})/\cong)/G$ to $(\Lambda/G\text{-ind})/\cong$.*
- *b) If $\Lambda$ is furthermore locally support-finite then this injection is even a bijection.*
- *c) The push-down of an Auslander-Reiten sequence of $\Lambda$ is an Auslander-Reiten sequence of $\Lambda/G$.*

*Remark .* Assertion a) and c) of the theorem are from Theorem 3.6 of [Ga2], while assertion b) is from [DoSk3].

Since for a locally representation finite $\Lambda$ any k-linear automorphism moves the isoclasses of indecomposables (see [MP1]) we conclude that $\Lambda$ is locally representation finite iff $\Lambda/G$ so is. (Recall that $F$ is cleaving.)

If $G$ operates freely on $\Lambda$-ind and $\Lambda$ is locally support finite we conclude that $\Lambda$ tame implies $\Lambda/G$ tame.

**Example .** Let $D \in \mathrm{k}\setminus\{0,1\}$, then $\Lambda_D = \mathrm{k}\langle \mathrm{x},\mathrm{y}\rangle/(\mathrm{yx}-\mathrm{x}^2, \mathrm{xy}-\mathrm{Dy}^2)$ is tame. This algebra is marked as (4) in Ringel's list of tame local algebras [Ri1]. Note however, that the following proof is independent from the proof given there.

*Proof.* Consider the Galois covering $F\colon \tilde{\Lambda}_D \longrightarrow \Lambda_D$ defined by the action of $\mathbb{Z}$, where $\tilde{\Lambda}_D = \mathrm{k}(\mathrm{W}_0\mathrm{Q})/\mathrm{I}$ with the following quiver Q and generators of $I$:

$$\mathrm{Q}\colon \cdots -1 \;\underset{y_{-1}}{\overset{x_{-1}}{\rightrightarrows}}\; 0 \;\underset{y_0}{\overset{x_0}{\rightrightarrows}}\; 1 \;\underset{y_1}{\overset{x_1}{\rightrightarrows}}\; 2 \cdots \qquad \left.\begin{matrix} x_i y_{i+1} - D y_i y_{i+1} \\ y_i x_{i+1} - \; x_i x_{i+1} \end{matrix}\right\}$$

Let $\tilde{\Lambda}_D^{(i,j)}$ be the full subcategory with the vertices $i, i+1, \dots, j, j+1$. As we have seen in 1.7.4, $\tilde{\Lambda}_D$ is locally support finite and every indecomposable has its support in some $\tilde{\Lambda}_D^{(i,i+1)}$.

Furthermore, $\tilde{\Lambda}_D^{(0,1)}$ is isomorphic to a special biserial algebra. Indeed, just consider the change of presentation $\bar{x}_0' := \bar{x}_0 - D\bar{y}_0$ and $\bar{y}_0' := \bar{y}_0 - \bar{x}_0$. Hence $\tilde{\Lambda}_D^{(0,1)}$ is tame. The pushdown functor $F_\lambda : \tilde{\Lambda}_D^{(0,1)}\text{-mod} \longrightarrow \Lambda_D\text{-mod}$ gives a description of the indecomposable $\Lambda_D$-modules. $\square$

4.4. Given $p \in \mathbb{N}_0$ a group $G$ is called *p-residually finite* if for each finite subset $S \subset G \setminus \{1_G\}$ there is a normal subgroup $H \lhd G$ of finite index such that $S \cap H = \emptyset$ and $p \nmid |G/H|$. Especially if $G$ is finite, and $p$ a prime that does not divide the order of $G$ then $G$ is $p$ residually finite; also if $G$ is a free group it is $p$-residually finite for any $p \in \mathbb{N}_0$.

**Theorem** ([P2]). *Let $F : \tilde{\Lambda} \longrightarrow \Lambda$ be a Galois covering given by the action of a p-residually finite Group $G$, where $p = \operatorname{char} \mathrm{k}$. Assume, that $\Lambda$ is locally support finite, then $\Lambda$ is tame if and only if $\tilde{\Lambda}$ is tame.*

*Proof.* (*We indicate only the main steps.*) Without loss of generality we may assume that $\operatorname{Obj} \Lambda$ is finite.

(1) As in [DoSk3] we see that any $Y \in \Lambda$-mod is direct summand of $F_\lambda F.Y$. This is only true since $G$ is $p$-residually finite. Essentially one reduces to the case of finite $G$ in order to use the argument of [Ga2, 3.4].

(2) Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a sequence of finite full subcategories of $\tilde{\Lambda}$ such that $\cup \operatorname{Obj} \Gamma_i = \operatorname{Obj} \tilde{\Lambda}$ and $\tilde{\Lambda}(x, \operatorname{Obj} \Gamma_i) \neq 0$ or $\tilde{\Lambda}(\operatorname{Obj} \Gamma_i, x) \neq 0$ for some $i$, then $x \in \operatorname{Obj} \Gamma_{i+1}$; denote by $\epsilon^n : \Gamma_n \longrightarrow \tilde{\Lambda}$ the inclusion functor. The restriction Functor $\epsilon.$ admits a left adjoint $\epsilon_\lambda$ such that $\epsilon^n \circ \epsilon^n_\lambda = \mathrm{id}_{\Gamma_n\text{-mod}}$. Thus $F_n := F_\lambda \epsilon_\lambda$ is right exact.

(3) There is some $n \in \mathbb{N}$ and a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that for every $Y \in \Lambda$-ind with dimension d there is some $X \in \Gamma_n$-ind with dimension $\leq f(d)$ such that $Y$ is isomorphic to a direct summand of $F_n X$. This is consequence of (1) and the hypothesis that $\tilde{\Lambda}$ is locally supportfinite.

(4) Since $F_n$ is right exact we conclude from (3) that $\Gamma_n$ tame implies $\Lambda$ weakly tame, thus $\Lambda$ is tame by 2.4 $\square$

*Remark .* (1) The hypothesis on $G$ is essential as we shall see by an example. The theorem avoids the restriction of the foregoing theorem that $G$ should move the isoclasses of indecomposables, which is a strong restriction.

(2) In [P2] the theorem was stated under the additional hypothesis that k is not countable. As we observed it is easy to avoid this restriction.

**Example** ([GeP]). Set $\Lambda_A := \mathrm{k}(\mathrm{W}_0\mathrm{Q})/\mathrm{I}_\mathrm{A}$, $A \in \mathrm{k}^*$ with $Q = \sigma \circlearrowright a \underset{\nu}{\overset{\gamma}{\rightleftarrows}} b \circlearrowleft \rho$ and $I_A$ generated by $\sigma\nu - A^{-1}\nu\rho$, $\rho\gamma - \gamma\sigma$, $\nu\gamma - A\sigma^2$, $\gamma\nu - \rho^2$ and all paths of length 3; this is $\Lambda 9_{2,2,3,3,A^{-1},A}$ in Table T of 6.1. The algebra $\Lambda_A$ admits a k-linear automorphism $g$ of order 2. Indeed, consider

$$\begin{aligned} &g\colon \Lambda_A \longrightarrow \Lambda_A \\ &g(a) = b,\ g(b) = a, \\ &g(\bar\rho) = \sqrt{A}\,\bar\sigma,\ g(\bar\sigma) = \sqrt{A}^{-1}\bar\rho,\ g(\bar\nu) = \sqrt{A}\,\bar\gamma,\ g(\bar\gamma) = \sqrt{A}^{-1}\bar\nu \end{aligned}$$

where $\sqrt{A} \in \mathrm{k}$ is such that $(\sqrt{A})^2 = A$.

Consider the local quotient $\Lambda_A^0 = \Lambda_A/\langle g\rangle \cong \mathrm{k}\langle\alpha,\beta\rangle/(\beta^2 - \sqrt{\mathrm{A}}\,\alpha^2, \beta\alpha - \sqrt{\mathrm{A}}\,\alpha\beta)$ and the Galois covering $F : \Lambda_A \longrightarrow \Lambda_A^0$ defined by the action of $\langle g\rangle = \mathbb{Z}_2$ sending

$$F(\bar\rho) = F(\sqrt{A}\,\bar\sigma) = \bar\alpha, \quad F(\bar\nu) = F(\sqrt{A}\,\bar\gamma) = \bar\beta.$$

We give new presentations to the algebras $\Lambda_A^0$ in order to identify the corresponding algebras in [Ri1].

i) Assume $\mathrm{char}\,\mathrm{k} \neq 2$ and $A = 1$. For simplicity $\sqrt{A} = 1$.
Set $\bar x := \bar\alpha - \bar\beta$ and $\bar y := \bar\alpha + \bar\beta$. Thus $\Lambda_1^0$ can be presented as $\mathrm{k}\langle\mathrm{x},\mathrm{y}\rangle/(\mathrm{xy},\mathrm{yx})$ – number (1) in [Ri1] (this is special biserial, thus tame).

ii) Assume $\mathrm{char}\,\mathrm{k} \neq 2$ and $A \neq 1$. Hence $\sqrt{A} \neq 1$.
Set $\bar x := \bar\alpha - \bar\beta$ and $\bar y := S(\bar\alpha + \bar\beta)$, where $S := (1+\sqrt{A})/(1-\sqrt{A})$. Thus $\Lambda_A^0$ can be presented as $\mathrm{k}\langle\mathrm{x},\mathrm{y}\rangle/(\mathrm{yx}-\mathrm{x}^2, \mathrm{xy}-\mathrm{S}^2\mathrm{y}^2)$ – number (4) in [Ri1] (compare example in 4.3!).

iii) Assume $\mathrm{char}\,\mathrm{k} = 2$ and $A \neq 1$.
Set $\bar x := \sqrt[4]{A}\,(\bar\alpha + \bar\beta)$ and $\bar y := \bar\beta + \sqrt{A}\,\bar\alpha$. Thus $\Lambda_A^0$ can be presented as $\mathrm{k}\langle\mathrm{x},\mathrm{y}\rangle/(\mathrm{x}^2 - \mathrm{y}^2, \mathrm{yx})$ – number (d) in [Ri1] (and we have seen in Example 3.3, that this is wild).

iv) Assume $\mathrm{char}\,\mathrm{k} = 2$ and $A = 1$.
Set $\bar x := \bar\alpha + \bar\beta$ and $\bar y := \bar\beta$. Thus $\Lambda_1^0$ can be presented as the algebra $\mathrm{k}\langle\mathrm{x},\mathrm{y}\rangle/(\mathrm{x}^2, \mathrm{xy}+\mathrm{yx})$ which has (c) in [Ri1] as a quotient (note, that this is a degeneration of the wild algebra from the foregoing case, thus it is wild).

We conclude already from the fact, that covering functors are cleaving, that if $\mathrm{char}\,\mathrm{k} \neq 2$ then $\Lambda_A$ is tame. On the other hand we have already seen in 1.6, that $\Lambda_1$ is tame independently of the characteristic of the field while $\Lambda_1^0$ is wild if $\mathrm{char}\,\mathrm{k} = 2$. This illustrates the importance of the hypothesis on the group in the theorem – compare with the situation in the representation finite case (remark 4.3). Obviously the push-down functor $F_\lambda\colon \Lambda_1\text{-mod} \longrightarrow \Lambda_1^0\text{-mod}$ and the pull-up functor $F_\cdot : \Lambda_1^0\text{-mod} \longrightarrow \Lambda_{1,1}\text{-mod}$ cannot behave as nicely as in the case $\mathrm{char}\,\mathrm{k} \neq 2$ (i.e. if $\langle g\rangle$ is $\mathrm{char}\,\mathrm{k}-$residually finite). We give some example:

Consider the indecomposable $\Lambda_1^0$-module $M_t$, $t \in \mathrm{k}$, given by:

$$M_t(*) = \mathrm{k}^3, \quad \mathrm{M_t}(\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{M_t}(\beta) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ t & 1 & 0 \end{pmatrix}.$$

It is easy to see that $M_t \cong M_{t'}$ iff $t = t'$ However, for any $t \in \mathrm{k}$ we get an isomorphism (use $\operatorname{char}\mathrm{k} = 2$):

$$f_t = (f_t(a), f_t(b)) = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \right) : F.M_t \longrightarrow F.M_0.$$

We conclude that $M_t$ can not be a direct summand of $F_\lambda F.M_t \cong F_\lambda F.M_0$. In fact a simple calculation shows that $F_\lambda F.M_t$ is indecomposable.

Compare with the situation $\operatorname{char}\mathrm{k} \neq 2$, see step (1) of the proof above.

Anyway we will prove in 8.1.2 the tameness of $\Lambda_A$ by covering techniques.

## 5. The ray category of a distributive category

5.1. **Definition.** A *ray category* is a category P which satisfies the following axioms a)–f):

a) P is a category with zeros.
b) Different objects are not isomorphic.
c) For each $x \in \operatorname{Obj}\mathrm{P}$, there are only finitely many non-zero morphisms starting or stopping in $x$.
d) An endomorphism $\rho \in \mathrm{P(x,x)}$ is nilpotent iff $\rho \neq 1_x$.
e) Each morphism set $\mathrm{P(x,y)}$ is linearly ordered by the prescription $\mu \leq \nu \Longleftrightarrow \nu = \sigma\mu\rho$ for some $\sigma \in \mathrm{P(x,x)}, \rho \in \mathrm{P(y,y)}$.
f) $0 \neq \mu\nu = \mu\rho\nu$ implies that $\rho$ is an identity.

*Remark* . Axioms d) and e) are equivalent to

d') $\mathrm{P(x,x)}$ is for all $x \in \operatorname{Obj}\mathrm{P}$ a cyclic half-group with nilpotent generator.
e') $\mathrm{P(x,y)}$ is for all $x, y \in \operatorname{Obj}\mathrm{P}$ cyclic over $\mathrm{P(x,x)}$ or $\mathrm{P(y,y)}$.

We say, that a morphism $x \xrightarrow{\nu} y$ *permits transit*, if for every $\sigma \in \mathrm{P(x,x)}$ there is a $\rho \in \mathrm{P(y,y)}$ with $\sigma\nu = \nu\rho$ and we say $\nu$ *prohibits transit* if it does not permit transit; in the dual situation we say that $\nu$ *permits cotransit* (resp. *prohibits cotransit*); by e') in a ray category every a morphism which prohibits transit will permit cotransit.

For ray categories we have the following *cancellation law*: $0 \neq \mu\nu_1 = \mu\nu_2 \Longrightarrow \nu_1 = \nu_2$ and $0 \neq \mu_1\nu = \mu_2\nu \Longrightarrow \mu_1 = \mu_2$.

5.2. Similary to the locally bounded categories the radical $\mathcal{R}\mathrm{P}$ of a ray category P is the ideal of all non-invertible morphisms (or here equivalently all morphisms which are no identities). A morphism $\mu$ has *depth* $\mathrm{d\!I}(\mu) = n$ if it belongs to $\mathcal{R}^n\mathrm{P} := (\mathcal{R}\mathrm{P})^{\mathrm{n}}$ but not to $\mathcal{R}^{n+1}\mathrm{P}$ (the multiplication of ideals is the obvious one). The quiver $\mathrm{Q_P}$ of a ray category P has as vertices the objects of P and as arrows the irreducible morphisms of P. We have a canonical functor $\overrightarrow{\phantom{x}} : \mathrm{W_0Q_P} \longrightarrow \mathrm{P}$ or in other words $\mathrm{P} \cong \mathrm{W_0Q_P}/ \overset{\mathrm{P}}{\sim}$ for some stable equivalence relation $\overset{\mathrm{P}}{\sim}$. Note that because of axiom e) the quiver $\mathrm{Q_P}$ has no multiple arrows.

**Definition .** Let $w\colon x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \cdots x_{n-1} \xrightarrow{\alpha_n} x_n$ be a path in $\mathrm{Q_P}$, then $w$ is a
*stable path,* if $\vec{w} := \vec{\alpha}_1\vec{\alpha}_2\cdots\vec{\alpha}_n \neq 0$;
*singular path*, if $\vec{w} = 0 \neq \mathrm{P(x_0,x_1)\cdot P(x_1,x_2)\cdots P(x_{n-1},x_n)}$;
*critical path,* if $w$ is singular and $\vec{\alpha}_1\vec{\alpha}_2\cdots\vec{\alpha}_{n-1} \neq 0 \neq \vec{\alpha}_2\vec{\alpha}_2\cdots\vec{\alpha}_n$;
*zero path,* if $\mathrm{P(x_0,X_1)\cdot P(x_1,x_2)\cdots P(x_{n-1},x_n)} = 0$;
*minimal zero path*, if it is a zero path but $\mathrm{P(x_1,x_2)\cdots P(x_{n-1},x_n)} \neq 0 \neq \mathrm{P(x_0,x_1)\cdots P(x_{n-2},x_{n-1})}$.

For every pair $(x,y) \in \mathrm{Obj\,P} \times \mathrm{Obj\,P}$ we choose a (stable) path $\omega(x,y)$ in Q such that $\vec{\omega}(x,y)$ *generates* $\mathcal{R}\mathrm{P(x,y)}$; if $\mathcal{R}\mathrm{P(x,y)} = 0$ we agree that $\omega(x,y)$ is the corresponding zero path.

We say that two paths are *parallel* if they have the same start- and the same endpoint. A pair $(v,w)$ of parallel non zero paths is called *interlaced*, if there is a finite sequence $v = v_1, v_2, \ldots, v_n = w$ of parallel paths such that $\vec{v}_1 = \vec{v}_2 = \cdots = \vec{v}_n$ and $v_i$ and $v_{i+1}$ share their first or last arrow. The pair $(v,w)$ is called *contour* if $\vec{v} = \vec{w} \neq 0$, a contour is called *essential* if the two paths are not interlaced.

*Remark .* (1) Clearly any zero path contains a minimal zero path and any singular path contains a critical path.

(2) If we have a ray category $\mathrm{P} \cong \mathrm{W_0Q_P}/\sim$ as above, then $\sim$ consists of the contours and the pairs $(w,0),(0,w),(w,w)$ where $w$ runs over all zero paths and singular paths. $\sim$ is already generated by the essential contours, the minimal zero paths and the critical parths.

(3) Let $w$ be a path as in the definition above. It is not hard to see that there is some $i \in \{0,1,\ldots,n\}$ such that $\mathrm{P(x_0,x_1)\cdots P(x_{n-1},x_n)} = \vec{\alpha}_0\cdots\vec{\alpha}_{\mathrm{i}}\mathrm{P(x_i,x_i)}\vec{\alpha}_{\mathrm{i+1}}\cdots\vec{\alpha}_{\mathrm{n}}$ (compare [BGRS, 2.3]).

**Lemma .** *Let* $w\colon x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \cdots x_{n-1} \xrightarrow{\alpha_n} x_n$ *be a path in the quiver* $\mathrm{Q_P}$ *of some ray category* P*, suppose furthermore that* $\mathrm{P(x_0,x_n)}$ *is cyclic over* $\mathrm{P(x_n,x_n)}$*. Then we have:*

a) *$w$ is singular iff $\vec{w} = 0 \neq \vec{\alpha}_0\cdots\vec{\alpha}_i\mathrm{P(x_i,x_i)}\vec{\alpha}_{\mathrm{i+1}}\cdots\vec{\alpha}_{\mathrm{n}}$ for some $i \in \{1,\ldots,n-1\}$.*

b) *$w$ is stable iff $\mathrm{P(x_0,x_1)\cdots P(x_{n-1},x_n)} = \vec{\mathrm{w}}\mathrm{P(x_n,x_n)} \neq 0$.*

This is an immediate consequence of the above remark (3).

5.3. **Example.** We describe the ray categories P with quiver Q : $\begin{tikzcd} a \arrow[loop left, "\sigma"] \arrow[r, shift left, "\nu"] & b \arrow[l, shift left, "\gamma"] \arrow[loop right, "\rho"] \end{tikzcd}$ by its essential contours and minimal zero paths.

i) $(\sigma\nu, \nu\rho^i)$, $\quad i \in \mathbb{N}^\infty \cup \{1/n \mid n \in \mathbb{N}^\infty\}$, where $\mathbb{N}^\infty := \mathbb{N} \cup \{\infty\}$

ii) $(\rho\gamma, \gamma\sigma^j)$, $\quad j \in \mathbb{N}^\infty \cup \{1/n \mid n \in \mathbb{N}^\infty\}$

iii) $(\sigma^k, \nu\rho^l\gamma)$, $\quad 0 \leq l < \min\{i, 1/j, m\}, k \geq 2$

iv) $(\rho^m, \gamma\sigma^n\nu)$, $\quad 0 \leq n < \min\{j, 1/i, k\}, m \geq 2$

*Remark* 5.3.1. We write $i = 1/n$ to indicate the contour $(\sigma^n\nu, \nu\rho)$; with $i = \infty$ we mean $(\sigma\nu, 0)$ and with $i = 1/\infty$ we mean $(0, \nu\rho)$. Indeed we are always in one of these situations, since $\mathrm{P(a, b)}$ is cyclic over $\mathrm{P(a, a)}$ or over $\mathrm{P(b, b)}$ (clearly $(\sigma\nu, 0)$ does not exclude $(0, \nu\rho)$). Analogous observations hold for ii).

*Remark* 5.3.2. We write $k = \infty$ to indicate $(0, \nu\rho^l\gamma)$ for all $l \in \mathbb{N}_0$; in other words there is no essential contour from $a$ to $a$. If $\nu\rho^l\gamma \neq 0$, we must have a contour $(\sigma^{k(l)}, \nu\rho^l\gamma)$; furthermore there is at most one $l \in \{0, 1, \ldots, \min\{i, 1/j\} - 1\}$ with $\nu\rho^l\gamma \neq 0$ and consequently there is at most one essential contour from $a$ to $a$. Indeed, suppose $\nu\rho^{l_1}\gamma \neq 0 \neq \nu^{l_2}\gamma$ for different $l_1, l_2 \in \{0, 1, \ldots, \min\{i, 1/j\} - 1\}$. This implies $\min\{i, 1/j\} \geq 2$ and because of axiom e) we can suppose $k(l_1) < k(l_2)$. Thus we calculate

$$0 \neq \nu\rho^{l_2}\gamma = \sigma^{k(l_2)-k(l_1)}\nu\rho^{l_1}\gamma = \begin{cases} 0, & \text{if } i = \infty; \\ \nu\rho^{i(k(l_2)-k(l_1))+l_1}\gamma, & \text{otherwise.} \end{cases}$$

This is anyway a contradiction (in the second case we obtain $l_2 \geq i$). In case of a essential contour $(\sigma^k, \nu\rho^l\gamma)$ with $l \geq 1$ the paths $\nu\rho^i\gamma$ are *singular* for $i = 0, 1, \ldots, l - 1$.

*Remark* 5.3.3. We have $\sigma\nu \neq 0 \Longrightarrow m > i$, $\nu\rho \neq 0 \Longrightarrow k > 1/i$ etc. Suppose for example $2 \leq m \leq i$ then $0 \neq \sigma\nu = \nu\rho^i = \nu\rho^{i-m}\gamma\sigma^n\nu$ which is impossible because of the cancellation law. Clearly one can find even more restrictions but we will not need them.

5.4. **Definition.** A locally bounded category $\Lambda$ (see 1.2) is *distributive* if it satisfies in addition the following axiom:

e) $\Lambda(x, y)$ is uniserial as $\Lambda(x, x)$-$\Lambda(y, y)$ bimodule for all $x, y \in \mathrm{Obj}\,\Lambda$.

*Remark* . Indeed, the lattice of Ideals of a locally bounded category $\Lambda$ is distributive if and only if it satisfies axiom e). From e) we conclude $[\mathrm{rad}^i\Lambda/\mathrm{rad}^{i+1}\Lambda(x, y) : \mathrm{k}] \leq 1$ always, especially for $\mu, \nu \in \Lambda(x, y)$ with $\mathrm{d\!I}\,(\mu) = \mathrm{d\!I}\,(\nu)$ we can find a unique $c \in \mathrm{k}$ with $\mathrm{d\!I}\,(\mu + c\nu) > \mathrm{d\!I}\,(\mu)$.

Furthermore, axiom d) and e) are equivalent to the following:

d') $\Lambda(x, x) \cong \mathrm{k[T]/(T^{d_x})}$ for all $x \in \mathrm{Obj}\,\Lambda$.

e') For all $x, y \in \mathrm{Obj}\,\Lambda$ the bimodule $\Lambda(x, y)$ is cyclic as $\Lambda(x, x)$ or as $\Lambda(y, y)$ module.

Compare with the definiton of ray categories. From e') we conclude that a morphism that prohibits cotransit will allow transit.

The following result is fundamental for the study of distributive categories:

**Lemma .** *Let $\Lambda$ be a distributive category and given two morphisms $a \xrightarrow{\mu} c \xrightarrow{\nu} b$ in $\Lambda$ one of the following mutually exclusive situations occcurs:*

i) $\mathrm{d\!I}\,(\mu\gamma\nu) = \mathrm{d\!I}\,(\mu\nu)$ *for all invertible $\gamma \in \Lambda(c, c)$. This is the case iff $\mu\mathcal{R}\Lambda(c, c)\nu \subseteq \mathcal{R}\Lambda(a, a)\mu\nu + \mu\nu\mathcal{R}\Lambda(b, b)$.*

ii) *The morphisms $\mu\gamma\nu$, where $\gamma$ ranges over the automorphisms of $c$, form a nonzero subbimodule of $\Lambda(a,b)$. This is the case iff $\mu\nu \in \mu\mathcal{R}\Lambda(c,c)\nu$*

5.5. **Construction.** Let $\Lambda$ be a distributive category. For $\mu \in \Lambda(x,y)$ we set $\vec{\mu} := \{\rho\mu\sigma \mid \rho \in \Lambda(x,x), \sigma \in \Lambda(y,y) \text{ invertible}\}$, the equivalence class of $\mu$ with respect to the relation induced by $\mathrm{d\!I}$. Set $\operatorname{Obj} \overrightarrow{\Lambda} := \operatorname{Obj}\Lambda$, and take $\overrightarrow{\Lambda}(x,y) := \{\vec{\mu} \mid \mu \in \Lambda(x,y)\}$ for all $x,y \in \operatorname{Obj}\Lambda$ and define for $\mu \in \overrightarrow{\Lambda}(x,y)$, $\nu \in \overrightarrow{\Lambda}(y,z)$ the composition $\vec{\mu}\cdot\vec{\nu}$ as $\overrightarrow{\mu\nu}$ if this is independent of the representants and ${}_x\vec{0}_z$ else. Then $\overrightarrow{\Lambda}$ is the *ray category* of $\Lambda$ and satisfies the axioms a)–f) of 5.1.

*Remark* . (1) If $\Lambda$ is distributive we find for morphisms $\mu,\nu$ in $\Lambda$, that $\mu$ factors through $\nu$ iff $\vec{\mu}$ factors through $\vec{\nu}$ in $\overrightarrow{\Lambda}$ especially the quivers $\mathrm{Q}_\Lambda$ and $\mathrm{Q}_{\overrightarrow{\Lambda}}$ may be identified. Also $\Lambda(x,y)$ is cyclic over $\Lambda(x,x)$ (resp. over $\Lambda(y,y)$) iff $\overrightarrow{\Lambda}(x,y)$ is cyclic over $\overrightarrow{\Lambda}(x,x)$ (resp. over $\overrightarrow{\Lambda}(y,y)$). Furthermore the lattices of ideals of $\Lambda$ and $\overrightarrow{\Lambda}$ are canonically isomorphic.

(2) For every ray category P the linearization k(P) is a distributive category, and one calculates $\overrightarrow{\mathrm{k(P)}} \cong \mathrm{P}$. Finally $\mathrm{k}(\overrightarrow{\Lambda})$ is called the *standard form* of $\Lambda$ resp. $\overrightarrow{\Lambda}$; if $\Lambda \cong \mathrm{k}(\overrightarrow{\Lambda})$ then it is called *standard.* This is the definition of [BGRS], note however that the definition of "standard" in [Bo1] is more restrictive.

5.6. From lemma 5.2, and the above remarks we obtain:

**Proposition .** *Let $\Lambda$ be a distributive* k*-category, $\phi^\pi : \mathrm{k}(\mathrm{W_0Q}) \longrightarrow \Lambda \cong \mathrm{k}(\mathrm{W_0Q})/\mathrm{I}^\pi$ induced by a presentation $\pi$ of $\Lambda$ and $\overrightarrow{\Lambda} = \mathrm{P} = \mathrm{W_0Q}/\overset{\mathrm{P}}{\sim}$ with $\overset{\mathrm{P}}{\sim}$ a stable equivalence relation, then we have:*

a) *If $w$ is a stable path in* Q *then $\mathrm{d\!I}\,(\bar{w}) = \mathrm{d\!I}\,(\vec{w})$ and furthermore $\bar{\omega}(x,y)$ generates $\mathcal{R}\Lambda(x,y)$.*

b) *If $(v,w)$ is a contour in* P *from $x$ to $y$ and* $\mathrm{P(x,y)}$ *is cyclic over say* $\mathrm{P(y,y)}$ *we find $v - wP_{(v,w)}(\omega(y,y)) \in I^\pi$ for some $P_{(v,w)} \in$* $\mathrm{k[T]}, \mathrm{P_{(v,w)}}(0) \neq 0$.

c) *If $s$ is a singular path from $x$ to $y$ and* $\mathrm{P(x,y)}$ *is cyclic say over* $\mathrm{P(x,x)}$ *then $s - Q_s(\omega(x,x))\omega(x,y) \in I^\pi$ for some $Q_s \in$* $\mathrm{k[T]}$.

d) *If $z$ is a zero path, then $z \in I^\pi$.*

e) *Running over all essential contours, singular paths and minimal zero paths the mentioned elements generate $I^\pi$.*

5.7. **Definition.** Let P be a ray category and $(Z,\cdot)$ an abelian group. A $Z$-valued *contour-function* on P is a map which asigns a value $c(p,q)$ to each contour $(p,q)$ of P and satisfies $c(p,q)\cdot c(q,r) = c(p,r)$ and $c(sp,sq) = c(p,q) = c(pt,qt)$ whenever this makes sense. The contour-function is *exact* if there is a $Z$-valued function $a$ on the arrows of P such, that $c(\alpha_1\cdots\alpha_s, \beta_1\cdots\beta_t) = \delta a := \prod_i a(\alpha_i)/\prod_j a(\beta_j)$ for all contours.

This definition is motivated by the following observation: The values $P_{(v,w)}(0)$ (proposition 5.6 b)) determine a $\mathrm{k}^*$-valued contour-function; if and only if this is exact, we can find a new presentation $\pi'$ such that $P'_{(v,w)}(0) = 1$ for all contours $(v, w)$, see [BGRS, 8.2]. In other words, the class of $P_{(v,w)}$ modulo the exact contour-functions is an invariant of $\Lambda$.

Let $K$ be the group of contour-functions with values in $Z$ and $E$ the subgroup of exact contour-functions of P. It is remarkable that $Z/E \cong \mathrm{H}^2(\mathrm{P}, \mathrm{Z})$; see [BGRS, 1.10] for the definition of the cohomology groups $\mathrm{H}^i(\mathrm{P}, \mathrm{Z})$ with values in $Z$ and [BGRS, 8.2] for the isomorphism.

**Example .** Let P be one of the ray categories of example 5.3: $i = 1$, $j = \infty$, $k = m = 2$, $l = n = 0$; this implies already that $\gamma\sigma^2$, $\sigma^3$ and $\rho^3$ are (minimal) zero-paths (thus $\mathrm{k}(\mathrm{P}) = \Lambda 10$ of Table T in 6.1).

Clearly a contour-function is already determined by its values on the essential contours; one should note however, that sometimes the values even on the essential contours may not be independent as we will see:

First consider the exact functions, so let $a(\sigma) = a_1$, $a(\nu) = a_2$, $a(\gamma) = a_3$ and $a(\rho) = a_4$, thus $\delta(\sigma\nu, \nu\rho) = a_1/a_4$, $\delta a(\sigma^2, \nu\gamma) = a_1^2/(a_2a_3)$, $\delta a(\rho^2, \gamma\nu) = a_4^2/(a_2a_3)$. From this it is clear, that every contour-function is modulo the exact functions of the form $c_A(\sigma\nu, \nu\rho) = A$, $c_A(\sigma^2, \nu\gamma) = 1$, $c_A(\rho^2, \gamma\nu) = 1$ and moreover $c_A = c_{-A}$ modulo the exact functions. But we have the following restriction: $\vec{\nu}\vec{\rho}^2 \neq 0$ thus

$$A^2 = c_A(\sigma^2\nu, \nu\rho^2) = c_A(\sigma^2\nu, \nu\gamma\nu) = c_A(\sigma^2\nu, \sigma^2\nu) = 1$$

and we conclude $\mathrm{H}^2(\mathrm{P}, \mathrm{k}^*)$ vanishes. This last restriction disappears in $\mathrm{P}/\langle\vec{\nu}\vec{\rho}^2\rangle$, while the description of the contour-functions and the exact contour-functions are the same, and we conclude $\mathrm{H}^2(\mathrm{P}/\langle\vec{\nu}\vec{\rho}^2\rangle, \mathrm{k}^2) = \mathrm{k}^*/\langle -1\rangle$, where $\langle -1\rangle$ is the (multiplicative) subgroup of $\mathrm{k}^*$ generated by $-1$.

## 6. Results on tame distributive categories

6.1. **Theorem.** **(A)** *Let $\Lambda$ be a bounded, distributive* k*-category with two objects. If $\Lambda$ is tame, representation infinite, then it is (up to duality) isomorphic to a quotient of one of the categories in Table T which are given by their quiver with relations.*

**(B)** *Every category in Table T with the possible exception of $\Lambda 9'_{m,A}$, $\Lambda 9''_{m,A}$, $\Lambda 10$, $\Lambda 10'_1$ is tame.*

*Remark .* The algebras $\Lambda 9'_{m,A}$ and $\Lambda 9''_{m,A}$ degenerate into no. 20 of Bekkert's list of tame algebras [Be]; $\Lambda 10$ and $\Lambda 10'_1$ correspond to the case $\gamma = 1$ of no. 19 in the same list. However we have no acces to a proof of the main theorem there.

$\Lambda 1)$ $\begin{tikzcd} a \arrow[r, "\nu"] & b \arrow[loop right, "\rho"] \end{tikzcd}$ with $\bar{\nu}\bar{\rho}^2 = \bar{\rho}^6 = 0$

$\Lambda 2)$ $\begin{tikzcd} a \arrow[loop left, "\sigma"] \arrow[r, "\nu"] & b \arrow[loop right, "\rho"] \end{tikzcd}$ with $\bar{\sigma}\bar{\nu} = \bar{\nu}\bar{\rho}, \bar{\rho}^6 = 0, \bar{\sigma}^3 = 0$

$\Lambda 3)$ with $\bar\sigma\bar\nu = \bar\nu\bar\rho, \bar\rho^4 = 0, \bar\sigma^4 = 0$

$\Lambda 4)$ $\quad a \underset{\gamma}{\overset{\nu}{\rightleftarrows}} b \circlearrowleft \rho \quad$ with $\bar\gamma\bar\nu = \bar\rho^2, \bar\nu\bar\rho\bar\gamma = 0$

$\Lambda 4)^{\mathrm{bis}}_{\text{if char k=3}}$ with $\bar\gamma\bar\nu = \bar\rho^2, \bar\nu\bar\rho\bar\gamma = \bar\nu\bar\rho^2\bar\gamma, \bar\rho^5 = 0$

$\Lambda 5)$ with $\bar\gamma\bar\nu = \bar\rho^3, \bar\nu\bar\gamma = 0, \bar\nu\bar\rho = 0$

$\Lambda 6)$ with $\bar\gamma\bar\nu = \bar\rho^3, \bar\nu\bar\gamma = 0, \bar\nu\bar\rho^2 = 0, \bar\rho^2\bar\gamma = 0$

$\Lambda 6)^{\mathrm{bis}}_{\text{if char k=3}}$ with $\bar\gamma\bar\nu = \bar\rho^3, \bar\nu\bar\gamma = \bar\nu\bar\rho\bar\gamma, \bar\nu\bar\rho^2 = 0, \bar\rho^2\bar\gamma = 0, \bar\rho^5 = 0$

$\Lambda 7)$ with $\bar\gamma\bar\nu = \bar\rho^3, \bar\nu\bar\gamma = 0 = \bar\nu\bar\rho\bar\gamma$

$\Lambda 8)$ with $\bar\gamma\bar\nu = \bar\rho^4, \bar\nu\bar\rho = 0, \bar\rho^2\bar\gamma = 0$

$$\sigma \circlearrowright a \underset{\gamma}{\overset{\nu}{\rightleftarrows}} b \circlearrowleft \rho$$

$\Lambda 9_{p,q,r,s,P,Q})$ with $\begin{cases} \bar\sigma\bar\nu = \bar\nu\bar\rho P(\bar\rho), \bar\rho\bar\gamma = \bar\gamma\bar\sigma, \bar\nu\bar\gamma = \bar\sigma^2 Q(\bar\sigma), \bar\gamma\bar\nu = \bar\rho^2 \\ \bar\sigma^p\bar\nu = 0, \bar\rho^q\bar\gamma = 0, \bar\sigma^s = 0, \bar\rho^r = 0 \\ P, Q \in \mathrm{k[T]}, \mathrm{P(0)} \neq 0 \neq \mathrm{Q(0)} \end{cases}$

$\Lambda 9'_{m,A})$ with $\begin{cases} \bar\sigma\bar\nu = \bar\nu\bar\rho, \bar\rho\bar\gamma = \bar\gamma\bar\sigma, \bar\nu\bar\gamma = \bar\sigma^2, \bar\gamma\bar\nu = \bar\rho^m(1_b + A\bar\rho) \\ \bar\sigma^2\bar\nu = 0, \bar\rho^2\bar\gamma = 0, \bar\sigma^4 = 0, \bar\rho^{m+2} = 0 \\ m \geq 3, A \in \mathrm{k} \end{cases}$

$\Lambda 9''_{m,A})$ with $\begin{cases} \bar\sigma\bar\nu = A\bar\nu\bar\rho, \bar\rho\bar\gamma = \bar\gamma\bar\sigma, \bar\nu\bar\gamma = \bar\sigma^2, \bar\gamma\bar\nu = \bar\rho^m \\ \bar\sigma^2\bar\nu = 0, \bar\rho^2\bar\gamma = 0, \bar\sigma^3 = 0, \bar\rho^{m+1} = 0 \\ m \geq 3, A \in \mathrm{k}^* \end{cases}$

$\Lambda 10)$ with $\begin{cases} \bar\sigma\bar\nu = \bar\nu\bar\rho, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = \bar\sigma^2, \bar\gamma\bar\nu = \bar\rho^2 \\ \bar\gamma\bar\sigma^2 = 0, \bar\sigma^3 = 0, \bar\rho^3 = 0 \end{cases}$

$\Lambda 10'_A)$ with $\begin{cases} \bar\sigma\bar\nu = A\bar\nu\bar\rho, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = \bar\sigma^2, \bar\gamma\bar\nu = \bar\rho^2 \\ \bar\sigma^2\bar\nu = 0, \bar\gamma\bar\sigma^2 = 0, \bar\sigma^3 = 0, \bar\rho^3 = 0 \\ A \in \mathrm{k}^*, \end{cases}$ $\quad \Lambda 10'_A \cong \Lambda 10'_{-A}$

$\Lambda 10''_m)$ with $\begin{cases} \bar\sigma\bar\nu = \bar\nu\bar\rho, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = \bar\sigma^2, \bar\gamma\bar\nu = \bar\rho^m \\ \bar\sigma^2\bar\nu = 0, \bar\gamma\bar\sigma^2 = 0, \bar\sigma^3 = 0, \bar\rho^{m+1} = 0 \\ m \geq 3 \end{cases}$

$\Lambda 11)$ with $\bar\gamma\bar\nu = 0 = \bar\rho^2, \bar\sigma\bar\nu = 0, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = 0 = \bar\sigma^2$

$\Lambda 12_m)$ with $\bar\gamma\bar\nu = \rho^m, \bar\gamma\bar\sigma\bar\nu = 0, \bar\nu\bar\rho = 0, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = \bar\sigma^2$

$\Lambda 13_m)$ with $\bar\gamma\bar\sigma\bar\nu = \rho^m, \bar\gamma\bar\nu = 0, \bar\nu\bar\rho = 0, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = \bar\sigma^2$

$\Lambda 13_m)^{\mathrm{bis}}_{\text{if char k=2}}$ with $\bar\gamma\bar\sigma\bar\nu = \rho^m, \bar\gamma\bar\nu = \bar\gamma\bar\sigma\bar\nu, \bar\nu\bar\rho = 0, \bar\rho\bar\gamma = 0, \bar\nu\bar\gamma = \bar\sigma^2$
$\bar\sigma^4 = 0, \bar\sigma^2\nu = 0, \bar\nu\rho^2 = 0$

$\Lambda 14_{m,n})$ with $\bar\gamma\bar\nu = \bar\rho^m, \bar\sigma\bar\nu = 0 = \bar\nu\bar\rho, \bar\rho\bar\gamma = 0 = \bar\gamma\bar\sigma, \bar\nu\bar\gamma = \bar\sigma^n$

**Table T**

For the bounded distributive k-categories (short: distributive categories) with quiver different from $\sigma \circlearrowright \cdot \underset{\gamma}{\overset{\nu}{\rightleftarrows}} \cdot \circlearrowleft \rho$ the Theorem was already proved in [HoM], so we have to deal only with this last case. The proofs for this case we will give in section 7 and 8

6.2. **Weak Transit Lemma.** *Let $\Lambda$ be a tame distributive category.*

a) *If $x \xrightarrow{\nu} y$ is a morphism in $\Lambda$ that prohibits cotransit, then $\nu(\mathcal{R}\Lambda(y,y))^3 = 0$.*

b) *If $x \xrightarrow{\vec{\nu}} y \xrightarrow{\vec{\gamma}} z$ is a sequence of morphisms in $\overrightarrow{\Lambda}$ such that $\vec{\nu}$ prohibits cotransit and $\vec{\gamma}$ prohibits transit, then $\vec{\nu}\overrightarrow{\Lambda}\vec{\gamma} \setminus \{{}_x0_z\}$ is exactly one of the following sets: $\emptyset$ (1), $\{\vec{\nu}\vec{\gamma}\}$ (2), $\{\vec{\nu}\vec{\gamma}, \vec{\gamma}\vec{\rho}^2\vec{\nu}\}$ (3), $\{\vec{\nu}\vec{\rho}^2\vec{\gamma}\}$ (4), $\{\vec{\nu}\vec{\rho}\vec{\gamma}\}$ (5), where $\vec{\rho}$ denotes the generator of $\mathcal{R}\overrightarrow{\Lambda}(y,y)$.*

*Proof.* a) We can suppose $\Lambda = \Lambda\{x,y\}$; now the assertion may be verified by inspection of table T.

b) By a) it is clear that $\vec{\nu}\overrightarrow{\Lambda}(y,y)\vec{\gamma}\setminus\{{}_x0_z\} \subseteq \{\vec{\nu}\vec{\gamma}, \vec{\nu}\vec{\rho}\vec{\gamma}, \vec{\nu}\vec{\rho}^2\gamma\}$. Suppose now for example $\vec{\nu}\vec{\gamma} \neq 0 \neq \vec{\rho}\vec{\gamma}$, then by property f) of ray-categories $\vec{\nu}\vec{\gamma} \neq \vec{\nu}\vec{\rho}\vec{\gamma}$; thus, if $\overrightarrow{\Lambda}(x,z)$ is cyclic over, say $\overrightarrow{\Lambda}(z,z)$, we must have either $\vec{\nu}\vec{\rho}\vec{\gamma} = \vec{\nu}\vec{\gamma}\vec{\pi}$ or $\vec{\nu}\vec{\gamma} = \vec{\nu}\vec{\rho}\vec{\gamma}\vec{\pi}$ for some $\vec{\pi} \in \overrightarrow{\Lambda}(z,z)$. In the first case we get $\vec{\nu}\vec{\gamma}\vec{\pi} = \vec{\nu}\vec{\rho}^k\vec{\gamma}$ with some $k \geq 2$ since $\vec{\gamma}$ prohibits transit – in this case a contradiction to property f) of a ray-category. The other case is also impossible by similar arguments. Also the assumption $\vec{\nu}\vec{\rho}\vec{\gamma} \neq 0 \neq \vec{\nu}\vec{\rho}^2\vec{\gamma}$ leads to a similar contradiction. $\square$

*Remark .* With the notation of Table T above $\Lambda 4$ is a example for Situation (3), and $\Lambda 7$ for situation (4). The other situations appear already in the representation-finite case.

6.3. With the help of the weak transit lemma we can recover some basic results that were obtained in [BGRS] for the representation-finite (mild) case with Roiter's transit lemma, see [BGRS, 2.5, 2.6].

By definition a path $w : x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n$ in $Q_\Lambda = Q_{\vec{\Lambda}}$ (with $\Lambda$ distributive) is *deep*, if $\omega(x_0,x_0)w$ and $w\omega(x_n,x_n)$ are zero-paths. Thus $w$ is deep iff $\varphi^\pi(w)$ lies in the socle of the bimodule $\Lambda(x_0,x_0)$ for any presentation $\pi$ of $\Lambda$. Recall, that $w$ is called *simple* if $n \geq 1$ and if $0 < i < n$ and $i \neq j$ imply $x_i \neq x_j$. Thus a simple path may be closed (i.e. $x_0 = x_n$) or not.

**Proposition .** *Let $\Lambda$ be a tame distributive category and $w$ a path as before.*

a) *If $\vec{w}\vec{\omega}(x_n,x_n) = 0$ (resp.$\vec{\omega}(x_0,x_0)\vec{w} = 0$), then this is a (stable) zero-path. Especially $w$ is already deep if $\vec{w}\vec{\omega}(x_n,x_n) = 0 = \vec{\omega}(x_0,x_0)\vec{w}$.*

b) *If $w$ is neither simple nor deep, then $w$ is interlaced with $\omega(x_0,x_0)^r\omega(x_0,x_n)$ or with $\omega(x_0,x_n)\omega(x_n,x_n)^r$ for some $r \geq 1$, especially $\omega(x_0,x_n)$ is simple if it is not deep.*

*Proof.* a) We suppose that $w\omega(x_n,x_n)$ is singular, then if $\omega(x_n,x_n)$ is given by $x_n \xrightarrow{\alpha_{n+1}} x_{n+1} \xrightarrow{\alpha_{n+2}} \cdots x_{m-1} \xrightarrow{\alpha_m} x_n$. we find by 5.2 some $i \in \{1,\dots m-1\}$ such that $\vec{\alpha_0}\cdots\vec{\alpha_i}\vec{\omega}(x_i,x_i)^r\vec{\alpha_{i+1}}\cdots\vec{\alpha_n} \neq 0$ where $r \geq 1$. Clearly we may suppose $1 \leq i \leq n-1$, thus $\vec{\nu} := \vec{\alpha}_1\cdots\vec{\alpha}_i$ prohibits cotransit and $\vec{\gamma} := \vec{\alpha}_{i+1}\cdots\vec{\alpha}_n$ prohibits transit. Now we have $0 \neq \vec{\nu}\vec{\omega}(x_i,x_i)^r\vec{\gamma}\vec{\omega}(x_n,x_n) =$

$\vec{\nu}\vec{\omega}(x_ix_i)^{r+k}\vec{\gamma}$ with $k \geq 2$ since $\vec{\gamma}$ prohibits transit – this is impossible because of the weak transit lemma.

b) By assumption, $w$ admits a decomposition $x_0 - \xrightarrow{u} z - \xrightarrow{s} z - \xrightarrow{v} x_n$, where $\{s\} \neq \{1_z\} \neq \{u, v\}$. As in [BGRS, 2.6] we see, that in the case $1_z \in \{u, v\}$ our assertion is true without any hypothesis about $\overrightarrow{\Lambda}$.

It remains the case $u \neq 1_z \neq v$. Since $usv$ is not deep $\vec{v}$ allows transit or $\vec{u}$ allows cotransit by a). By duality we may restrict to the first case. We then have $\vec{s}\vec{v} = \vec{\omega}(x_i, x_n)\vec{\omega}(x_n, x_n)^b$ for some $b \geq 1$, hence $w = usv \sim u\omega(x_i, x_n)\omega(x_n, x_n)^b$ and we are reduced to the first case in the proof. $\square$

*Remark* . As a special case of a) we see, that in $\Lambda$ singular paths are deep.

Recall that $w$ is called *critical* if $\alpha_0 \cdots \alpha_{n-1}$ and $\alpha_1 \cdots \alpha_n$ are both stable, and $w$ is singular. With Roiter's transit lemma follows, that if $w$ is critical, $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n-1}$ allow bitransit. Note however, that the proof of this statement, given in [BGRS, 4.2], does not work with our weak transit lemma.

6.4. **Examples.** We provide some examples in order to illustrate the different behavior of tame distributive categories in comparison with representation finite (resp. mild) categories.

In $\Lambda 4$ note, that $\nu\rho\gamma$ is a critical path of length 3 – compare with the structure theorem for singular paths [BGRS, 2.4].

In $\Lambda 6$ we observe, that the contour $(\gamma\nu, \rho^3)$ is not deep, but it is none of the three types (dumb-bell, penny-farthing, diamond) which can appear in the mild case – see the structure theorem for non-deep essential contours [BGRS, 2.7].

In $\Lambda 9_{pqrs,P,Q}$ (with $(p, q, r, s) \geq (3, 3, 4, 4)$ any of the four non deep essential contours share at least one arrow – compare the disjointness theorem for non-deep essential contours [BGRS, 2.9]. Moreover this family provides many examples for tame distributive algebras without multiplicative or semi-multiplicative basis.

Now consider the following category $\Lambda = \mathrm{k}(\mathrm{W}_0\mathrm{Q})/\mathrm{I}$ which we describe by the following quiver Q and generators for $I$:

$$\begin{tikzcd} a_1 \arrow[r, shift left, "\nu_1"] & b \arrow[l, shift left, "\gamma_1"] \arrow[loop above, "\rho"] \arrow[r, shift left, "\gamma_2"] & b_2 \arrow[l, shift left, "\nu_2"] \end{tikzcd} \qquad \left.\begin{array}{ll} \nu_1\gamma_1, & \gamma_1\nu_1 - \rho_0^2 \\ \nu_2\gamma_2, & \gamma_2\nu_2 - \rho_0^2 \\ \nu_1\gamma_2 & \\ \nu_2\gamma_1 & \end{array}\right\}$$

First we want to see that $\Lambda$ is tame; thus note that $\Lambda(-, b) \cong D\Lambda(b, -)$ and the socle of this projective-injective module is generated by $\bar{\rho}^3$, hence $\Lambda$ is tame iff $\Lambda_1 := \Lambda/(\bar{\rho}^3)$ is tame. Next we observe, that $\Lambda_1$ degenerates into $\Lambda_0$ with the same quiver but relations

$$\left.\begin{array}{ll} \nu_1\gamma_1, & \gamma_1\nu_1 - \gamma_2\nu_2 \\ \nu_2\gamma_2, & \rho_0^2 \\ \nu_1\gamma_2, & \gamma_1\nu_1\rho_0 \\ \nu_2\gamma_1, & \rho_0\gamma_2\nu_2 \end{array}\right\}$$

By our result on degenerations it is enough, to show the tameness of $\Lambda_0$. Now, $\Lambda_0$ is not distributive, but we will produce an equivalence $\phi\colon \Lambda_0\text{-mod} \longrightarrow \Gamma\text{-mod}$ where $\Gamma$ is the following clannish category:

$$\begin{tikzcd} a \arrow[loop left, "\epsilon"] \arrow[r, shift left, "\nu"] & b_0 \arrow[l, shift left, "\gamma"] \arrow[loop right, "\rho_0"] \end{tikzcd} \qquad \left.\begin{array}{c} (\bar\epsilon - K_1)(\bar\epsilon - K_2) = 0 = \bar\rho^2 \\ \bar\nu\bar\gamma = 0 = \bar\gamma\bar\nu \\ \bar\gamma\bar\epsilon\bar\nu\bar\rho = 0 = \bar\rho\bar\gamma\bar\epsilon\bar\nu \end{array}\right\}$$

indeed, for $M \in \Lambda$-mod we can set $\psi(M)(b_0) = M(b)$, $\psi(M)(a) = M(a_1) \oplus M(a_2)$ and

$$\psi(M)(\bar\epsilon) := \begin{pmatrix} K_1\mathrm{id}_{M(a_1)} & 0 \\ 0 & K_2\mathrm{id}_{M(a_2)} \end{pmatrix},\ \psi(M)(\bar\nu) := \begin{pmatrix} M(\bar\nu_1) \\ M(\bar\nu_2) \end{pmatrix}, \tag{1}$$

$$\psi(M)(\bar\gamma) := \left(M(\bar\gamma_1)M(\bar\gamma_2)\right), \qquad \psi(M)(\bar\rho_0) := M(\bar\rho). \tag{2}$$

To construct the inverse of $\psi$ decompose simply for $N \in \Gamma$-mod the space $N(a)$ into the eigenspaces of $N(\bar\epsilon)$ (compare example in 1.6).

We observe, that the three essential contours $(\gamma_1\nu_1, \rho^2)$, $(\rho^2, \gamma_2\nu_2)$ and $(\gamma_2\nu_2, \gamma_1\nu_1)$ are not deep in $\Lambda$ and each pair of them shares one path. Furthermore the critical paths $\nu_1\gamma_1$, $\nu_1\gamma_2$, $\nu_2\gamma_2$ and $\nu_2\gamma_1$ of length 2 share pairwise one arrow – compare the disjointness theorem for singular paths [BGRS, 2.4]. Moreover consider $\Lambda' = \mathrm{k}(\mathrm{W_0Q})/\mathrm{I}'$ with the same quiver as $\Lambda$ and the following generators of $I'$:

$$\nu_1\gamma_1 - \nu_1\rho_0\gamma_1,\ \gamma_1\nu_1 - \rho_0^2,\ \gamma_2\nu_2 - \rho_0^2, \rho^3,\ \nu_2\gamma_2,\ \nu_1\gamma_2,\ \nu_2\gamma_1$$

Thus $\overrightarrow{\Lambda'} = \overrightarrow{\Lambda}/\langle\vec\rho^3\rangle$, but there is no presentation of $\Lambda'$ which annihilates each singular path. On the other hand $\Lambda'$ degenerates obviously into $\Lambda/\langle\bar\rho^3\rangle$, hence it is tame.

## 7. Completeness of Table T

Recall, that we have to study only the cases with quiver Q : $\begin{tikzcd} \mathrm{a} \arrow[loop left, "\sigma"] \arrow[r, shift left, "\nu"] & \mathrm{b} \arrow[l, shift left, "\gamma"] \arrow[loop right, "\rho"] \end{tikzcd}$. In 5.3 we have studied already the ray categories with this quiver, and we will use further on the notations introduced there.

7.1. The following lemmas exclude certain "bad" situations in the ray category of a tame distributive category with the above quiver. The proofs use the cleaving technique as presented in 3.2. As sources for cleaving functors we will use wild hereditary and concealed algebras.

**Lemma 7.1.1.** *Let* P *be a ray category as in 5.3. If* $\mathrm{P} = \overrightarrow{\Lambda}$ *for a tame distributive category* $\Lambda$*, then we have* $i, j \in \{1, \infty, 1/\infty\}$.

*Proof.* Let $\phi^\pi : \mathrm{k}(\mathrm{W_0Q}) \longrightarrow \Lambda$ be induced by a fixed presentation $\pi$ of a form $\Lambda$ of P; for typographical reasons we write $\bar v := \phi^\pi(v)$ if $v$ is a path in Q.

Now suppose for example $\infty \neq i \geq 2$. We claim that in this case the following diagram is cleaving:

$$
\Gamma\colon\quad
\begin{tikzcd}
2 & 1 \arrow[l, "\bar\gamma"'] \arrow[d, "\bar\rho"] \\
4 \arrow[u, "\bar\sigma"] \arrow[r, "\bar\nu"] & 3 \arrow[d, "\bar\rho"] \\
 & 5
\end{tikzcd}
$$

Clearly $\overrightarrow{\bar\nu\bar\rho} = \vec\nu\vec\rho \neq 0 \neq \vec\rho^2 = \overrightarrow{\bar\rho\bar\rho}$. We give a cleavage $S$ following [BGRS, 3.4], thus set $S(1,3) = S(3,5) = \mathrm{k}1_{\mathrm{b}} + \bar\rho\mathcal{R}\Lambda(\mathrm{b},\mathrm{b})$, $S(1,5) = \mathrm{k}1_{\mathrm{b}} + \mathrm{k}\bar\rho + \bar\rho^2\mathcal{R}\Lambda(\mathrm{b},\mathrm{b})$, $S(4,5) = \mathrm{k}\bar\nu + \bar\nu\bar\rho\mathcal{R}\Lambda(\mathrm{b},\mathrm{b})$. The only relations we have to check are:

$\bar\gamma S(2,5) = \bar\gamma\Lambda(a,b) = \bar\gamma\bar\nu\Lambda(b,b) \overset{(*)}{\subseteq} \bar\rho^3\Lambda(b,b) \subseteq S(1,5)$. The inclusion $(*)$ holds, since $\infty \neq i \geq 2$ implies that $\vec\rho^2$ does not factor through the point $a$, but this is then also true for $\bar\rho^2$.

$\bar\sigma S(2,5) = \bar\sigma\bar\nu\Lambda(b,b) \overset{(*)}{\subseteq} \bar\nu\bar\rho^2\Lambda(b,b) \subseteq S(4,5)$ Here inclusion $(*)$ holds since $\Lambda(a,b)$ is cyclic over $\Lambda(b,b)$ but *not* over $\Lambda(a,a)$.

Since $\Gamma$ is wild hereditary we conclude by the foregoing remark that any form of P is wild, a contradiction. $\square$

**Lemma 7.1.2.** *Let* P *be as in lemma 7.1.1, then we have the following implications:*

$$
\vec\nu\vec\rho \neq 0 \Longrightarrow
\begin{cases}
\vec\sigma\vec\nu = 0 \quad \text{and} \begin{cases} \vec\rho^2 = 0 = \vec\gamma\vec\nu & \text{or} \\ \vec\rho^2 = \vec\gamma\vec\nu \end{cases} \text{or} \\
\vec\sigma\vec\nu = \vec\nu\vec\rho \quad \text{and} \begin{cases} \vec\sigma^2 = 0 = \vec\nu\vec\gamma & \text{or} \\ \vec\sigma^2 = \vec\nu\vec\gamma & \text{or} \\ \vec\rho^2 = 0 = \vec\gamma\vec\nu & \text{or} \\ \vec\rho^2 = \vec\gamma\vec\nu \end{cases}
\end{cases}
$$

*Proof.* The alternative $\vec\sigma\vec\nu = 0$ or $\vec\sigma\vec\nu = \vec\nu\vec\rho \neq 0$ is a consequence of lemma 7.1.1. Suppose in the first case $0 \neq \vec\rho^2 \neq \vec\gamma\vec\nu$, then we obtain the following cleaving diagram:

$$
\Gamma\colon\quad
\begin{tikzcd}
2 & 1 \arrow[l, "\bar\gamma"'] \arrow[d, "\bar\rho"] \\
4 \arrow[u, "\bar\sigma"] \arrow[r, "\bar\nu"] & 3 \arrow[d, "\bar\rho"] \\
 & 5
\end{tikzcd}
$$

The proof is the same as in lemma 7.1.1.

Now suppose $\vec{\sigma}\vec{\nu} = \vec{\nu}\vec{\rho}$ and $0 \neq \vec{\sigma}^2 \neq \vec{\nu}\vec{\gamma}$ and $0 \neq \vec{\rho}^2 \neq \vec{\gamma}\vec{\nu}$. We can choose a presentation of $\Lambda$ such that $\bar{\sigma}\bar{\nu} = \bar{\nu}\bar{\rho}$, then it is easy to verify that the following diagram is cleaving:

$$\begin{tikzcd}
 & 5 \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{\rho}"] & 4 & 3 \arrow[l, "\bar{\sigma}"'] \arrow[r, "\bar{\nu}"] & 2 & 1 \arrow[l, "\bar{\rho}"'] \\
8 \arrow[r, "\bar{\nu}"] \arrow[d, "\bar{\sigma}"'] & 6 \arrow[d, "\bar{\rho}"] & & & & \\
9 \arrow[r, "\bar{\nu}"] \arrow[d, "\bar{\sigma}"] & 7 & & & & \\
10 & & & & &
\end{tikzcd}$$

Our hypothesis on P yields directly the critical inclusions $\bar{\nu}S(6,10) \subset S(8,10)$ and $S(5,9)\bar{\nu} \subset S(5,7) \supset \bar{\gamma}S(4,7)$ for the natural choice of the cleavage $S$. $\square$

**Lemma 7.1.3.** *Let again* P *be as in lemma 7.1.1, then* $\vec{\sigma}\vec{\nu} = 0 \neq \vec{\nu}\vec{\rho}$ *and* $\vec{\rho}\vec{\gamma} = 0 \neq \vec{\gamma}\vec{\sigma}$ *implies* $\vec{\sigma}^2 = 0, \vec{\rho}^2 = 0$.

*Proof.* Suppose for example $\vec{\rho}^2 \neq 0$ then lemma 7.1.2 implies $\vec{\rho}^2 = \vec{\gamma}\vec{\nu}$. In this case we can find a presentation of $\Lambda$ with $\bar{\rho}^2 = \bar{\gamma}\bar{\nu}$ and obtain the following cleaving diagram:

$$\begin{tikzcd}
 & & & \cdot \arrow[r, "\bar{\rho}"] \arrow[d, "\bar{\gamma}"'] & \cdot \arrow[d, "\bar{\rho}"] & \cdot \arrow[l, "\bar{\nu}"'] \arrow[r, "\bar{\sigma}"] & \cdot \\
\cdot & \cdot \arrow[l, "\bar{\rho}"] \arrow[r, "\bar{\gamma}"'] & \cdot & \cdot \arrow[l, "\bar{\sigma}"] \arrow[r, "\bar{\nu}"'] & \cdot & &
\end{tikzcd}$$

This is very easy to check since taking into account lemma 7.1.2 we have for P only the alternative $\vec{\sigma}^2 = \vec{\nu}\vec{\gamma}$ or $\vec{\sigma}^2 = 0 = \vec{\nu}\vec{\gamma}$; and in both cases obviously the standard form is the only possible form for P. $\square$

7.2. **Corollary.** *Let* $\mathrm{P} = \overrightarrow{\Lambda}$ *be the ray category of a tame distributive category* $\Lambda$ *with quiver* $\sigma \circlearrowright a \underset{\gamma}{\overset{\nu}{\rightleftarrows}} b \circlearrowleft \rho$ *then it is (up to duality) isomorphic to a quotient of one of the following ray categories (which we describe by the numerical data as in 5.3 and the dimensions* $s := \#\mathrm{P}(\mathrm{a},\mathrm{a}) - 1$, $p := \#\mathrm{P}(\mathrm{a},\mathrm{b}) - 1$, $q := \#\mathrm{P}(\mathrm{b},\mathrm{a}) - 1$, $r := \#\mathrm{P}(\mathrm{b},\mathrm{b}) - 1$, *in other words the entries of the Cartan matrix* $C_{\mathrm{P}}$ *of* $\mathrm{k}(\mathrm{P})$.

| | $i$ | $j$ | $k$ | $l$ | $m$ | $n$ | $s$ | $p$ | $q$ | $r$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{P9_{p,q,r,s}}$) | 1 | 1 | 2 | 0 | 2 | 0 | $s$ | $p$ | $q$ | $r$ | |
| $\mathrm{P9'_m}$) | 1 | 1 | 2 | 0 | $m$ | 0 | 4 | 2 | 2 | $m+2$ | $m\geq 3$ |
| $\mathrm{P10_2}$) | 1 | $\infty$ | 2 | 0 | 2 | 0 | 3 | 3 | 2 | 3 | |
| $\mathrm{P10_m}$) | 1 | $\infty$ | 2 | 0 | $m$ | 0 | 3 | 2 | 2 | $m+1$ | $m\geq 3$ |
| P11) | $\infty$ | $\infty$ | $\infty$ | $l$ | $\infty$ | $n$ | 2 | 2 | 2 | 2 | |
| $\mathrm{P12_m}$) | $1/\infty$ | $\infty$ | 2 | 0 | $m$ | 0 | 4 | 2 | 2 | $m+1$ | $\vec{\gamma}\vec{\sigma}\vec{\nu}=0$ |
| $\mathrm{P13_m}$) | $1/\infty$ | $\infty$ | 2 | 0 | $m$ | 1 | 4 | 2 | 2 | $m+1$ | $\vec{\gamma}\vec{\nu}=0$ |
| $\mathrm{P14_{k,m}}$) | $\infty$ | $\infty$ | $k$ | 0 | $m$ | 0 | $k+1$ | 1 | 1 | $m+1$ | |

*In case* $\mathrm{P9_{p,q,r,s}}$) *we may suppose* $2\leq r\leq s$ *and thus* $s-1\leq \min\{p,q\}\leq \max\{p,q\}\leq r+1$.

*Proof.* Because of lemma 7.1.1 we have to study (up to duality) only quotients of ray categories with $(i,j)\in\{(1,1),(1,\infty),(\infty,\infty),(1/\infty,\infty)\}$. Using now the two other lemmas and the observations of 5.3 produces the given list. □

7.3. Now we determine the possible forms (5.6) for the ray categories described in 7.2. These will already form the elements of the table T (corresponding to the quiver $\sigma \circlearrowright \cdot \underset{\gamma}{\overset{\nu}{\rightleftarrows}} \cdot \circlearrowleft \rho$). We explain this for some of the more interesting cases.

$\mathrm{P9_{p,q,r,s}}$) Let $\Lambda$ be a fixed form of this ray category with presentation $\pi$. By 5.6 we can suppose that the following elements generate $I^\pi$: $\sigma\nu-\nu\rho P_1(\rho)$, $\rho\gamma-\gamma\sigma P_2(\sigma)$, $\nu\gamma-\sigma^2Q_1(\sigma)$, $\gamma\nu-\rho^2Q_2(\rho)$, $\sigma^p\nu,\rho^q\gamma,\sigma^s,\rho^r$ with $P_i,Q_i\in \mathrm{k[T]},\mathrm{P_i(0)}\neq 0\neq \mathrm{Q_i(0)}$. Note that $\mathrm{P9_{p,q,r,s}}$ has no singular path. Now choose a new presentation $\pi'$ with $\pi'(\sigma)=\pi(\sigma)P_2(\pi(\sigma))$, $\pi'(\nu)=\pi(\nu)Q_2^{-1}(\pi(\rho))$, $\pi'(\rho)=\pi(\rho)$, $\pi'(\gamma)=\pi(\gamma)$ where $Q_2^{-1}Q_2=1$ modulo $(T^r)$. Thus we find the following generators of $I^{\pi'}$: $\sigma\nu-\nu\rho P(\rho)$, $\rho\gamma-\gamma\sigma$ $\nu\gamma-\sigma^2Q(\sigma)$, $\gamma\nu-\rho^2$, $\sigma^p\nu,\rho^q\gamma,\sigma^s,\rho^r$ for some $P,Q\in \mathrm{k[T]}$ with $P(0)\neq 0\neq Q(0)$.

*Remark* . For $p,q,r,s$ big we find that $\mathrm{H^2(P9_{p,q,r,s},k^*)}$ vanishes, thus we can suppose in these cases $P(0)=1=Q(0)$, compare example 5.7 and 8.1.1. In general, however it seems to be cumbersome to determine the isoclasses of the $\Lambda$ with $\vec{\Lambda}=\mathrm{P9_{p,q,r,s}}$.

$\mathrm{P9_m}),\mathrm{m}\geq 3$. We find (using contour functions): $\mathrm{H^2(P9_m,k^*)}\cong \mathrm{H^2(P9_m/\langle\vec{\sigma}^3\rangle,k^*)}\cong \mathrm{H^2(P9_m/\langle\vec{\rho}^{m+1}\rangle,k^*)}\cong\{1_\mathrm{k}\}$ and also $\mathrm{H^2(P9_m/\langle\vec{\sigma}^3,\vec{\rho}^{m+1}\rangle,k^*)}\cong\mathrm{k}^*$. (Suppose for example that $\Lambda$ is a form of $\mathrm{P9_m}$ with presentation $\pi$ such that $0=\bar{\sigma}\bar{\nu}-A\bar{\nu}\bar{\rho}=\bar{\rho}\bar{\gamma}-\bar{\gamma}\bar{\sigma}=\bar{\nu}\bar{\gamma}-\bar{\sigma}^2=\bar{\gamma}\bar{\nu}-\bar{\rho}^m$, then we obtain $A=1$ since $0\neq\bar{\sigma}^3=\bar{\sigma}\bar{\nu}\bar{\gamma}=A\bar{\nu}\bar{\rho}\bar{\gamma}=A\bar{\nu}\bar{\gamma}\bar{\sigma}=A\bar{\sigma}^3$.) Thus let $\Lambda$ be a fixed form of $\mathrm{P9_m}$ with presentation $\pi$. Then we can suppose by 5.6, that $I^\pi$ is generated by $\sigma\nu-\nu\rho$, $\rho\gamma-\gamma\sigma$, $\nu\gamma-\sigma^2(1_a+A\sigma)$, $\gamma\nu-\rho^m(1_b+B\rho)$, $\sigma^2\nu,\rho^2\gamma,\sigma^4,\rho^{m+2}$. Now set $\pi'(\sigma)=\pi(\sigma)$, $\pi'(\nu)=\pi(\nu)$, $\pi'(\gamma)=\pi(\gamma)(1_a-A\pi(\sigma))$, $\pi'(\rho)=\pi(\rho)$ and we find that $I^{\pi'}$ is generated by $\sigma\nu-\nu\rho$, $\rho\gamma-\gamma\sigma$, $\nu\gamma-\sigma^2$, $\gamma\nu-\rho^m(1_b+(B-A)\rho)$, $\sigma^2\nu,\rho^2\gamma,\sigma^4,\rho^{m+2}$. Note that there are no singular paths, since all morphisms permit bitransit.

Similarly one finds that for $\mathrm{P9_m}/\langle\vec{\sigma}^2\rangle$ and $\mathrm{P9_m}/\langle\vec{\rho}^{\mathrm{m}+1}\rangle$ the only forms are the standard forms. Finally we find for a form $\Lambda$ of $\mathrm{P9_m}/\langle\vec{\sigma}^3,\vec{\rho}^{\mathrm{m}+1}\rangle$ with presentation $\pi$ the following generators of $I^\pi$: $\sigma\nu - A\nu\rho$, $\rho\gamma-\gamma\sigma$, $\nu\gamma-\sigma^2$, $\gamma\nu-\rho^m$, $\sigma^2\nu, \rho^2\gamma, \sigma^3, \rho^{m+1}$ for some $A \in \mathrm{k}^*$. Note that different $A$ correspond to nonisomorphic forms. The study of the forms for the categories $\mathrm{P10_m}$ (and their quotients) is similar but easier.

$\mathrm{P13_m}$) This is the only case with a singular path. $\mathrm{H}^2(\mathrm{P13_m},\mathrm{k}^*)$ trivially vanishes. Let now $\Lambda$ be a form of $\mathrm{P13_m}$ with presentation $\pi$. Again by 5.6 we can suppose that $I^\pi$ is generated by the following elements: $\nu\gamma-\sigma^2(1_a+A\sigma)$, $\gamma\sigma\nu - \rho^m$, $\gamma\nu - B\gamma\sigma\nu$, $\nu\rho, \sigma^2\nu, \rho\gamma, \gamma\sigma^2, \sigma^4, \rho^{m+1}$. Note, that because of $\phi^\pi(\nu\rho) = 0 = \phi^\pi(\rho\gamma)$ we must have $\phi^\pi(\nu\gamma) \in \operatorname{soc}\Lambda(b,b)$. If $B \neq A$ we set $C := B - A$, choose a new presentation with $\pi'(\sigma) = C\pi(\sigma)$, $\pi'(\nu) = C\pi(\nu)$, $\pi'(\gamma) = C\pi(\gamma)(1_a - A\pi(\sigma))$, $\pi'(\rho) = C^{3/m}\pi(\rho)$ and obtain as generators of $I^{\pi'}$: $\nu\gamma-\sigma^2$, $\gamma\sigma\nu-\rho^m)$, $\gamma\nu-\gamma\sigma\nu$ $\nu\rho, \sigma^2\nu, \rho\gamma, \gamma\sigma^2, \sigma^4, \rho^{m+1}$. If in turn $B = A$ we see $\Lambda \cong \mathrm{k}(\mathrm{P13_m})$. A further examination shows that if $\operatorname{char}\mathrm{k} \neq 2$ even the first form is isomorphic to the standard form (compare the situation of the Riedtmann contours in [BGRS]).

## 8. The tameness proofs for Table T

Section 8.1.1 and 8.1.2 are essentially taken from [GeP], the remaining parts are from [Ge3].

8.1. We start with the examination of $\Lambda 9_{p,q,r,s,P,Q}$ where $P, Q \in \mathrm{k}^*$. Recall, that we have already seen the tameness of $\Lambda 9_{p,q,r,s,1,1}$ in Example 1.6 using clannish categories, and that we obtained partial results for the case $P = Q^{-1}$ in Example 4.4.

8.1.1. *Claim.* Assume that $(P,Q) \neq (1,1) \in \mathrm{k}^* \times \mathrm{k}^*$, then $\mathcal{R}^4\Lambda 9_{p,q,r,s,P,Q} = 0$, in other words $(p,q,r,s) \leq (3,3,4,4)$. If $X$ is an indecomposable $\Lambda 9_{p,q,r,s,P,Q}$-module with $X(\mathcal{R}^3\Lambda 9_{p,q,r,s,P,Q}) \neq 0$, then $X$ is projective-injective.

*Proof.* First we observe that the following elements lie in $\mathcal{R}^3\Lambda 9_{p,q,r,s,P,Q}$:

$$\begin{array}{ccccccccccl}
\bar\gamma\bar\sigma\bar\nu &=& \bar\rho\bar\gamma\bar\nu &=& \bar\rho^3 &=& \bar\gamma\bar\nu\bar\rho &=& P^{-1}\bar\gamma\bar\sigma\bar\nu & \in \mathcal{R}^3\Lambda 9_{p,q,r,s,P,Q}(b,b),\\
\bar\nu\bar\rho\bar\gamma &=& \bar\nu\bar\gamma\bar\sigma &=& Q\bar\sigma^3 &=& \bar\sigma\bar\nu\bar\gamma &=& P\bar\nu\bar\rho\bar\gamma & \in \mathcal{R}^3\Lambda 9_{p,q,r,s,P,Q}(a,a),\\
\bar\gamma\bar\sigma^2 &=& \bar\rho\bar\gamma\bar\sigma &=& \bar\rho^2\bar\gamma &=& \bar\gamma\bar\nu\bar\gamma &=& Q\bar\gamma\bar\sigma^2 & \in \mathcal{R}^3\Lambda 9_{p,q,r,s,P,Q}(b,a),\\
\bar\nu\bar\rho^2 &=& \bar\nu\bar\gamma\bar\nu &=& Q\bar\sigma^2\bar\nu &=& QP\bar\sigma\bar\nu\bar\rho &=& QP^2\bar\nu\bar\rho^2 & \in \mathcal{R}^3\Lambda 9_{p,q,r,s,P,Q}(a,b).
\end{array}$$

We consider several cases:

i) $Q = 1 = QP^2$. Hence $P = -1$ (in particular $\operatorname{char}\mathrm{k} \neq 2$). Therefore

$$\begin{aligned}
\mathcal{R}^3\Lambda_{p,q,r,s,-1,1}(a,a) &= 0 = \mathcal{R}^3\Lambda_{p,q,r,s,-1,1}(b,b) \quad \text{and}\\
\mathcal{R}^4\Lambda_{p,q,r,s,-1,1}(a,b) &= \bar\nu\,\mathcal{R}^3\Lambda_{p,q,r,s,-1,1}(b,b) = 0,\\
\mathcal{R}^4\Lambda_{p,q,r,s,-1,1}(b,a) &= \bar\gamma\,\mathcal{R}^3\Lambda_{p,q,r,s,-1,1}(a,a) = 0
\end{aligned}$$

Thus $(p,q,r,s) \leq (3,3,3,3)$ It is easy to see that $P_a := \Lambda_{3,3,3,3,-1,1}(-,a) \cong D\Lambda_{3,3,3,3,-1,1}(b,-) =: I_b$ and $P_b \cong I_a$. Observe that $\mathcal{R}^3\Lambda_{3,3,3,3,-1,1}$ is generated by $\bar{\gamma}\bar{\sigma}^2$ and $\bar{\nu}\bar{\rho}^2$. Now, if $X$ is an indecomposable module with $X(\bar{\gamma}\bar{\sigma}^2) \neq 0$, then $X \cong P_b$. Indeed, let $\varphi : P_b \longrightarrow X$ be a morphism with $\varphi(\bar{\gamma}\bar{\sigma}^2) \neq 0$, thus $\varphi$ is injective, since $\operatorname{soc} P_b = \mathrm{k}\,\bar{\gamma}\bar{\sigma}^2$. Hence $\varphi$ is a section, because $P_b$ is injective, and finally an isomorphism since $X$ is indecomposable; similarly, if $\bar{\nu}\bar{\rho}^2 X \neq 0$ then $X \cong P_a$ (see [CR, Lemma 59.1, p.409]).

ii) $P = 1 = P^{-1}$. Hence $1 \neq Q = QP^2$. Therefore $\mathcal{R}^3\Lambda_{p,q,r,s,1,Q}(a,b) = 0 = \mathcal{R}^3\Lambda_{p,q,r,s,1,A}(b,a)$ and $\mathcal{R}^3\Lambda_{p,q,r,s,1,A}$ is generated by $\bar{\rho}^3$ and $\bar{\sigma}^3$ which lie in the socle; in other words $(p,q,r,s) \leq (2,2,4,4)$. In the case of equality $P_a \cong I_a$ and $P_b \cong I_b$.

iii) $Q = 1, P \in \mathrm{k}^* \setminus \{1,-1\}$, hence $QP^2 \neq 1$. Therefore

$$\mathcal{R}^3\Lambda_{p,q,r,s,P,1}(a,a) = 0 = \mathcal{R}^3\Lambda_{p,q,r,s,P,1}(b,b) = 0 \quad \text{and} \quad \mathcal{R}^3\Lambda_{p,q,r,s,P,1}(a,b) = 0.$$

In this case, $\mathcal{R}^3\Lambda_{p,q,r,s,P,1}$ is generated by $\bar{\rho}^2\bar{\gamma}$; in other words $(p,q,r,s) \leq (2,3,3,3)$ (moreover in the case of equality $P_a \cong I_b$).

iv) $QP^2 = 1$, $Q \neq 1 \neq P$. Thus

$$\mathcal{R}^3\Lambda_{p,q,r,s,P,P^{-2}}(a,a) = 0 = \mathcal{R}^3\Lambda_{p,q,r,s,P,P^{-2}}(b,b) \quad \text{and} \quad \mathcal{R}^3\Lambda_{p,q,r,s,P,P^{-2}}(b,a) = 0.$$

In this case, $\mathcal{R}^3\Lambda_{p,q,r,s,P,P^{-2}}$ is generated by $\bar{\sigma}^2\bar{\nu}$ and $P_b \cong I_a$.

In the remaining cases it is clear that $\mathcal{R}^3\Lambda_{p,q,r,s,P,Q} = 0$. $\square$

This result will us allow to assume $(p,q,r,s) = (2,2,3,3)$ if we are interested in the representations of $\Lambda 9_{p,q,r,s,P,Q}$ with $(P,Q) \neq (1,1) \in \mathrm{k}^* \times \mathrm{k}^*$.

8.1.2. For $\Lambda_{2,2,3,3,P,Q}$ we have a Galois-covering $F : \widetilde{\Lambda 9}_{P,Q} \longrightarrow \Lambda_{2,2,3,3,P,Q}$ defined by the action of $\mathbb{Z}$ and we describe $\widetilde{\Lambda 9}_{P,Q}$ by its quiver and generators of $I^\pi$ (we omit the obvious zero-paths):

$$\widetilde{\Lambda 9}_{P,Q} : \cdots \quad
\begin{tikzcd}
a_{-1} \arrow[r, "\sigma_{-1}"] \arrow[dr, "\nu_{-1}"'] & a_0 \arrow[r, "\sigma_0"] \arrow[dr, "\nu_0"'] & a_1 \arrow[r, "\sigma_1"] \arrow[dr, "\nu_1"'] & a_2 \\
b_{-1} \arrow[r, "\rho_{-1}"] \arrow[ur, "\gamma_{-1}"] & b_0 \arrow[r, "\rho_0"] \arrow[ur, "\gamma_0"] & a_1 \arrow[r, "\rho_1"] \arrow[ur, "\gamma_1"] & b_2
\end{tikzcd}
\quad \cdots \qquad
\left.\begin{array}{r}
Q\sigma_i\sigma_{i+1} - \nu_i\gamma_{i+1} \\
\sigma_i\nu_{i+1} - P\nu_i\rho_{i+1} \\
\gamma_i\nu_{i+1} - \rho_i\rho_{i+1} \\
\gamma_i\rho_{i+1} - \sigma_i\gamma_{i+1}
\end{array}\right\}$$

We introduce some notation. For numbers $i,j \in \mathbb{Z}$ with $i \leq j$ we define $\widetilde{\Lambda 9}^{(i,j)}_{P,Q}$ as the full subcategory of $\widetilde{\Lambda 9}_{P,Q}$ with the vertices $a_i, b_i, a_{i+1}, b_{i+1}, \ldots, a_{j+1}, b_{j+1}$. We write $\widetilde{\Lambda 9}^{(i)}_{P,Q} := \widetilde{\Lambda 9}^{(i,i+1)}_{P,Q}$.

For a scalar $C \in \mathrm{k}^*$ we define the $\widetilde{\Lambda 9}_{P,Q}$-module $M_C^{(i)}$ as follows:

$$\begin{aligned} M_C^{(i)}(x) &= \begin{cases} \mathrm{k}, & \text{if } x \in \{a_i, b_i, a_{i+1}, b_{i+1}\}; \\ 0, & \text{else.} \end{cases} \\ M_C^{(i)}(\sigma_i) &= C\,\mathrm{id}, \quad M_C^{(i)}(\nu_i) = M_C^{(i)}(\rho_i) = M_C^{(i)}(\gamma_i) = \mathrm{id}. \end{aligned}$$

Observe, that $M_C^{(i)}$ is also a $\widetilde{\Lambda 9}_{P,Q}/\mathcal{R}^2\widetilde{\Lambda 9}_{P,Q}$-module. We have:

$$\widetilde{\Lambda 9}_{P,Q}^{(i,j+1)} \cong \left(\widetilde{\Lambda 9}_{P,Q}^{(i,j)}[M_{Q^{-1}}^{(j)}]\right)[M_P^{(j)}] \cong [M_1^{(i+1)}]\left([M_{P^{-1}Q^{-1}}^{(i+1)}]\widetilde{\Lambda 9}_{P,Q}^{(i+1,j+1)}\right). \qquad (*)$$

First we consider the case $AB \neq 1$:
In this situation $M_1^{(i)}$ and $M_{AB}^{(i)}$ belong to different tubes of the tame hereditary algebra $\widetilde{\Lambda 9}_{P,Q}^{(i)}$. Hence $\widetilde{\Lambda 9}_{P,Q}^{(i,i+1)} = (\widetilde{\Lambda 9}_{P,Q}^{(i)}[M_1^{(i)}])[M_{AB}^{(i)}]$ is a tubular algebra and the modules $M_1^{(i+1)}$, $M_{AB}^{(i+1)}$ belong to different tubes of the last separating tubular family of $\widetilde{\Lambda 9}_{P,Q}^{(i,i+1)}$ ($\mathcal{T}_\infty$ in Ringel's notation [Ri2]). This process may be iterated to show that $\widetilde{\Lambda 9}_{P,Q}^{(i,j)}$ is an *iterated tubular algebra* [PT]. In particular, *indecomposable $\widetilde{\Lambda 9}_{P,Q}$-modules have a support contained in $\widetilde{\Lambda 9}_{P,Q}^{(i,i+1)}$ for some $i \in \mathbb{Z}$. That is $\widetilde{\Lambda 9}_{P,Q}$ is locally support finite.* By 4.3, the indecomposable $\Lambda 9_{p,q,r,s,P,Q}$-modules are of the form $F_\lambda(X)$ where $X$ runs over the indecomposable $\widetilde{\Lambda 9}_{P,Q}^{(0,1)}$-modules. *Since $\widetilde{\Lambda 9}_{P,Q}^{(0,1)}$ is tame (of polynomial growth), then $\Lambda 9_{p,q,r,s,P,Q}$ is so.*

Now consider the case $PQ = 1$, but $(P,Q) \neq (1,1)$, thus $Q^{-1} = P \neq 1 = P^{-1}Q^{-1}$. By the same type of arguments as in example 1.7.4 we conclude from $(*)$, that $\widetilde{\Lambda 9}_{P,Q}$ *is locally support-finite, moreover if $X$ is a indecomposable $\widetilde{\Lambda 9}_{P,Q}$-module, then* $\operatorname{supp} X$ *is contained in $\widetilde{\Lambda 9}_{P,Q}^{(i,i+1)}$ for some $i \in \mathbb{Z}$,* (see also [GeP, 1.4]).

To show that $\Lambda 9_{p,q,r,s,P,Q}$ is tame it is now enough to prove that $\widetilde{\Lambda 9}_{P,Q}^{(0,1)} \cong \widetilde{\Lambda 9}_{1,1}^{(0,1)}$ is tame. For this purpose we establish an equivalence

$$\varphi\colon \widetilde{\Lambda 9}_{1,1}^{(0,1)}\text{-mod} \longrightarrow \Gamma^{(0,1)}\text{-mod},$$

where $\Gamma^{(0,1)}$ is the clannish category given by the following quiver with relations:

$$\begin{tikzcd} p_0 \arrow[loop above, "\epsilon_0"] \arrow[r, "\delta_0"] & p_1 \arrow[loop above, "\epsilon_1"] \arrow[r, "\delta_1"] & p_2 \arrow[loop above, "\epsilon_2"] \end{tikzcd} \qquad \left.\begin{aligned} \delta_0\delta_1 &= 0 \\ q(\epsilon_i) &= (\epsilon_i - K_1)(\epsilon_i - K_2) = 0 \end{aligned}\right\}$$

where $K_1, K_2$ are different elements of $k^*$. For $X \in \widetilde{\Lambda 9}_{1,1}^{(0,1)}$-mod, we set

$$\begin{aligned}\varphi(X)(p_i) &:= X(a_i) \oplus X(b_i)\\ \varphi(X)(\epsilon_i) &:= \begin{pmatrix} K_1 \operatorname{id}_{X(a_i)} & 0 \\ 0 & K_2 \operatorname{id}_{X(b_i)} \end{pmatrix}\\ \varphi(X)(\delta_i) &:= \begin{pmatrix} X(\rho_i) & X(\nu_i) \\ -X(\gamma_i) & -X(\sigma_i) \end{pmatrix}.\end{aligned}$$

The inverse functor is defined as in (1.6). As in the other case we conclude that $\Lambda 9_{p,q,r,s,P,Q}$ *is tame.*

*Remark .* (1) It is not hard to see, that in this last case $\Lambda 9_{p,q,r,s,P,Q}$ is of exponential growth, see [GeP, 1.4,ii)].

(2) Note, that $\widetilde{\Lambda 9}_{1,1}$ is not locally support finite: Consider $\psi\colon \Gamma\text{-mod} \longrightarrow \Lambda_{1,1}\text{-mod}$ the functor defined in (1.6) and let $M_{w,K}$ be the $\Gamma$-module corresponding to the asymmetretic band-word $w = \delta\epsilon^*\delta\epsilon^*\delta^{-1}\epsilon^*$ and a simple $k[T,T^{-1}]$-module, see [CB2]. The pull-up functor $F_. : \Lambda_{1,1}\text{-mod} \longrightarrow \tilde{\Lambda}_{1,1}\text{-Mod}$ yields an indecomposable $\tilde{\Lambda}_{1,1}$-module $F_.\psi(M_{w,K})$ with infinite support).

(3) By (1.6), $\Lambda_{1,1}$ is tame. Moreover, since $\widetilde{\Lambda 9}_{1,1}^{(0,1)}$ is not of polynomial growth, $\Lambda_{1,1}$ is neither so.

(4) Given two pairs $(P,Q),(C,D) \in (k^*)^2$, the algebras $\widetilde{\Lambda 9}_{P,Q}^{(0,1)}$ and $\tilde{\Lambda}_{C,D}^{(0,1)}$ are isomorphic iff $PQ = CD$; the algebras $\widetilde{\Lambda 9}_{P,Q}^{(0,2)}$ and $\tilde{\Lambda}_{C,D}^{0,2}$ are isomorphic iff $(A,B) = (C,D)$. In particular, this shows that the algebras in the family $(\Lambda 9_{p,q,r,s,P,Q})_{A,B\in k^*}$ are pairwise not isomorphic.

8.1.3. Now let $P(T) = P_0 + P_1T + \cdots + P_nT^n \in k[T]$ then define for $A \in k$ the polynomial $P_A(T) = P_0 + (P_1A)T + \cdots + (P_nA^n)T^n \in k[T]$. With this notation we observe that the family $(\Lambda 9_{p,q,r,s,P_A,Q_A})_{A\in k}$ describes a degeneration of $\Lambda 9_{p,q,r,s,P,Q}$ into $\Lambda 9_{p,q,r,s,P_0,Q_0}$, where the tameness of the last one was shown above in 8.1.2.

Next we note that $\Lambda 9'_{m,A}$ is self-injective; modulo its socle it is isomorphic to $\Lambda 9''_{m,1}$ (thus $\Lambda 9'_{m,A}$ tame iff $\Lambda 9''_{m,1}$ tame); furthermore $\Lambda 9''_{m,A}$ degenerates into $\Lambda_A$ which is given by the following relations: $\bar{\sigma}\bar{\nu} = A\bar{\nu}\bar{\rho}$, $\bar{\rho}\bar{\gamma} = \bar{\gamma}\bar{\sigma}$, $\bar{\nu}\bar{\gamma} = \bar{\sigma}^2$ $\bar{\gamma}\bar{\nu} = 0$, $\bar{\rho}^2\bar{\gamma} = 0$, $\bar{\sigma}^2\bar{\gamma} = 0$,$\bar{\sigma}^3 = 0$, $\bar{\rho}^{m+1} = 0$. This is No.20 in Bekkert's list [Be].

8.2. Now study (still with the same quiver) the family $\Gamma_T$ of categories with presentation $\pi_T$ where we obtain the relations: $\bar{\nu}\bar{\rho} = T\bar{\sigma}\bar{\nu}$, $\bar{\sigma}^2 = T\bar{\nu}\bar{\gamma}$, $\bar{\gamma}\bar{\nu} = T^{4m-11}\bar{\rho}^m$, $\bar{\rho}\bar{\gamma} = 0$, $\bar{\gamma}\bar{\sigma}^2 = 0$, $\bar{\sigma}^3 = 0$, $\bar{\rho}^{m+1} = 0$, $\bar{\sigma}^2\bar{\nu} = 0$. We see $\Gamma_T \cong \Lambda 10''_m$ for $T \in k^*$, while $\Gamma_0$ is special biserial (for $m \geq 3$). This proves that $\Lambda 10''_m$ is tame.

$\Lambda 10$ has a projective injective module, factoring out its socle we obtain $\Lambda 10'_1$. Note that $\Lambda 10'_A$ corresponds to No.19 in Bekkert's list [Be] of tame algebras. We give however an independent tameness proof for the case $A \in \mathrm{k}^* \setminus \{1\}$:

Study the following Galois covering of $\Lambda 10_A$:

$$\widetilde{\Lambda 10}_A\colon \qquad \begin{tikzcd}
\cdots & a_{-1} \arrow[r, "\sigma_{-1}"] \arrow[dr, "\nu_{-1}"] & a_0 \arrow[r, "\sigma_0"] \arrow[dr, "\nu_0"] & a_1 \arrow[r, "\sigma_1"] \arrow[dr, "\nu_1"] & a_2 & \cdots \\
 & b_{-1} \arrow[r, "\rho_{-1}"'] \arrow[ur, "\gamma_{-1}"'] & b_0 \arrow[r, "\rho_0"'] \arrow[ur, "\gamma_0"'] & a_1 \arrow[r, "\rho_1"'] \arrow[ur, "\gamma_1"'] & b_2 &
\end{tikzcd}$$

with the obvious relations. We write $\widetilde{\Lambda 10}_A^{(i,j)}$ for the full subcategory of $\widetilde{\Lambda 10}_A$ with objects $a_i, b_i, a_{i+1}, b_{i+1}, \dots, a_{j+1}, b_{j+1}$. For a scalar $C \in \mathrm{k}^*$ we define the $\widetilde{\Lambda 10}_A$-module $M_C^{(i)}$ as follows:

$$\begin{aligned}
M_C^{(i)}(x) &= \begin{cases} \mathrm{k}, & \text{if } x \in \{a_i, b_i, a_{i+1}, b_{i+1}\}; \\ 0, & \text{else.} \end{cases} \\
M_C^{(i)}(\bar\nu_i) &= C\,\mathrm{id}, \quad M_C^{(i)}(\bar\sigma_i) = M_C^{(i)}(\bar\rho_i) = M_C^{(i)}(\bar\gamma_i) = \mathrm{id}.
\end{aligned}$$

In addition we define the $\widetilde{\Lambda 10}_A$-modules $M^{(i,+)}$ and $M^{(i,-)}$ by $M^{(i,+)}(x) = M_C^{(i)}(x) = M^{(i,-)}(x)$ for all $x \in \mathrm{Obj}\,\widetilde{\Lambda 10}_A$ and $M^{(i,+)}(\bar\rho) = 0 = M^{(i,-)}(\bar\gamma)$, while the remaining non-trivial multiplications are the identity. Note that $M^{(i,+)}$ and $M^{(i,-)}$ lie in different inhomogenous tubes of $\widetilde{\Lambda 10}_A^{(i,i)}$-mod. We find

$$\widetilde{\Lambda 10}_A^{(i,j+1)} = \left(\widetilde{\Lambda 10}_A^{(i,j)}[M_A^{(j)}]\right)[M^{(j,+)}] = [M_{1/A}^{(i+1)}]\left([M^{(i+1,-)}]\widetilde{\Lambda 10}_A^{(i+1,j+1)}\right).$$

By a similar argument as in example1.7.4 (see also [GeP, 1.4,ii)]) we find that for $A \in \mathrm{k}^* \setminus \{1\}$ the support of any indecomposable $\widetilde{\Lambda 10}_A$-module lies in some $\widetilde{\Lambda 10}_A^{(i,i+1)}$. Thus, to show that $\Lambda 10_A$ is tame ($A \neq 1$) it is sufficient to show that $\widetilde{\Lambda 10}_A^{(i,i+1)}$ is tame. This can be seen by the pattern [Ri] which appears for the one point extension of $\widetilde{\Lambda 10}_A^{(i,i)}[M_A^{(i)}]$ by $M^{(i,+)}$ or alternatively observe that $\widetilde{\Lambda 10}_A^{(i,i+1)}$ degenerates into a special biserial algebra (and thus must be tame by our result on degenerations).

8.3. The categories $\Lambda 11$, $\Lambda 13_m$ and $\Lambda 14_{m,n}$ are special biserial, thus tame.

To prove that $\Lambda 12_m$ is tame, study the following (algebraic) family of categories $\Lambda_T$ with quiver $\sigma \circlearrowright a \underset{\gamma}{\overset{\nu}{\rightleftarrows}} b \circlearrowleft \rho$ and presentation $\pi(T)$ which we determine by the following generators of $I^{\pi(T)}$: $\gamma\nu - T\rho^m, \gamma\sigma\nu, \nu\rho, \rho\gamma, \sigma^2 - T\nu\gamma, \rho^{m+1}, \nu\gamma\nu, \gamma\nu\gamma$. We find the following base $\mathcal{B}(\rangle, |)$ for $\mathcal{R}\Lambda_T(i,j)$ (independent of $T$): $\mathcal{B}(\dashv, \dashv) = \{\bar\sigma, \bar\nu\bar\gamma\}$,$\mathcal{B}(a,b) = \{\bar\nu, \bar\sigma\bar\nu\}$, $\mathcal{B}(\lfloor, \dashv) = \{\bar\gamma, \bar\gamma\bar\sigma\}$, $\mathcal{B}(\lfloor, \lfloor) = \{\bar\rho, \bar\rho^{\in}, \dots, \bar\rho^{\Updownarrow}\}$. Furthermore $\Lambda_T \cong \Lambda 12_m$ for all $T \in \mathrm{k}^*$ and $\Lambda_0$ is special biserial, thus $\Lambda 12_m$ degenerates into the tame category $\Lambda_0$. By our result on degenerations we conclude that $\Lambda 12_m$ is tame.

It is easy to see the degeneration of $\Lambda 13_m^{\mathrm{bis}}$ into $\Lambda 13_m$ (if $\operatorname{char} \mathrm{k} = 2$) which proves by the same argument the tameness.

## References

[A] M. Auslander, *Representation theory of artin algebras II,* Comm. Algebra 1 (1974), 269–310.

[B] R. Bautista, *On algebras of strongly unbounded representation type,* Comm. Math. Helvetici 60 (1985), 392–399.

[BGRS] R. Bautista, P. Gabriel, A. Roiter, and L. Salmeron, *Representation-finite algebras and multiplicative bases,* Invent. Math. 81 (1984), 217–285.

[Bo1] K. Bongartz, *A criterion for finite representation type.* Math. Ann. 269 (1984), 1–12

[Bo2] K. Bongartz, *Indecomposables are standard,* Commnet. Math. Helvtici 60 (1985), 400–410.

[Bo3] K. Bongartz, *On degenerations and extensions of finitedimensional modules.* Preprint Univ. Wuppertal.

[BoGa] K. Bongartz and P. Gabriel, *Covering spaces in representation theory,* Invent. Math. 65 (1982), 331–378.

[Be] V.I. Bekkert, *Tame two-point quivers with relations,* Izvestiya Vuz. Math. Vol.30 (1986), 62–64.

[CB1] W.W. Crawley-Boevey, *On tame algebras and Bocses,* Proc. London Math. Soc. 56 (1988), 451–483.

[CB2] W.W. Crawley-Boevey, *Functorial filtrations II: Clans and the Gelfand problem,* J. London Math. Soc. 40 (1989), 9–30.

[CR] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras,* New York, Interscience (1962).

[DlRi] V. Dlab and C.M. Ringel, *Eigenvalues of Coxeter transformations and the Gelfand-Kirillov dimension of preprojective algebras,* Proc. Amer. Math. Soc. 83 (1981), 228–232.

[DoSk1] P. Dowbor and A. Skowroński, *On the representation type of locally bounded categories.* Tsukuba J. Math. Vol.10 No.1 (1986), 63–72

[DoSk2] P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras* Arch. Math. (Basel) 44 (1985), 522–529

[DoSk3] P. Dowbor and A. Skowroński, *Galois coverings of representation infinite algebras,* Comment. Math. Helvetici 62 (1987), 311–337.

[Dr] Y.A. Drozd, *Tame and wild matrix problems.* In Representation Theory II, Springer LNM 832 (1980)

[DrGr] Y.A. Drozd and G.M. Greuel, *Tame wild dichotonomy for Cohen Macaulay Modules.* Preprint Univ. Kaiserslautern.

[E] K. Erdmann, *Blocks of tame representation type and related algebras.* Springer LNM 1428 (1990)

[F1] U. Fischbacher, *Une novelle preuve d' une thèoréme de Nazarova et Roiter,* C. R. Acad. Sci. Paris, t. 300, Série I, n$^o$9 (1985) 259–262.

[F2] U. Fischbacher, *Zur Kombinatorik der Algebren mit endlich vielen Idealen,* J. Reine und Ang. Math 370 (1986), 192–213.

[Ga1] P. Gabriel, *Finite representation type is open.* In: Representations of Algebras, Springer LNM 488 (1975), 132–155

[Ga2] P. Gabriel, *The universal cover of a representation-finite algebra,* Representation theory I, Springer Lect. Notes Math. 903 (1981), 68–105.

[Ga3] P. Gabriel, *Darstellungen endlichdimensionaler Algebren*, 17p. Report ICM 1986 (Berkley).

[Ge1] Ch. Geiß, *Darstellungsendliche Algebren und multiplikative Basen,* Diplomarbeit.
[Ge2] Ch. Geiß, *A decomposition theorem for tensor products with bimodules.* Comm. Alg. 20(10) (1992), 2991–2998
[Ge3] Ch. Geiß, *Tame distributive 2-point algebras,* To appear in the Proceedings of the 6th international conference on representations of algebras (Canadian mathematical society proceedings series).
[GeP] Ch. Geiß and J.A. de la Peña, *An interesting family of algebras.* Arch. Math. 60 (1993), 25–35.
[Ha] R. Hartshorne, *Algebraic Geometry.* Springer GTM 52 (1977)
[HoM] M. Hoshino and J. Miyachi, *Tame two point algebras.* Tsukuba J. Math. Vol.12 No.1 (1988), 65–96
[K0] H. Kraft, *Geometric methods in representation theory.* In: Representations of Algebras, Springer LNM 944 (1980)
[K] H. Kraft, *Geometrische Methoden in der Invariantentheorie.* Vieweg (1985)
[MP1] R. Martinez-Villa and J.A. de la Peña, *Automorphisms of representation-finite algebras,* Invent. Math. 72 (1983), 359–362.
[MP2] R. Martinez-Villa and J.A. de la Peña, *The universal cover of a quiver with relations,* J. Pure Appl. Alg. 30 (1983), 277–292.
[M] W. Müller, *Artinsche Ringe vom endlichen Typ* Verlag Uni-Druck München, Algebra-Berichte Nr. 24 (1974).
[NRo] L.A. Nazarova and A.V. Roiter, *Categorical matrix problems and the Brauer-Thrall conjecture,* preprint Kiev (1973); German translation: Mitt. Math. Inst. Gießen 115 (1975), 1–153.
[P1] J.A. de la Peña, *On the dimension of module varieties of tame and wild algebras.* Comm. Alg. 19(6) (1991), 1795–1805
[P2] J.A. de la Peña, *Functors preserving tameness,* Fundamenta Math. 137 (1991), 177–185.
[P3] J.A. de la Peña, *Quadratic forms and the representation type of an algebra,* Sonderforschungsbereich Diskrete Strukturen in der Mathematik. Ergänzungsreihe 90.003 (1990).
[PT] J.A. de la Peña and B. Tomé, *Iterated tubular algebras,* J. Pure Appl. Alg. 64 (1990), 303–314.
[Ri1] C.M. Ringel, *The representation type of local algebras,* Representations of Algebras. Springer Lect. Notes Math. 488 (1975), 283–305.
[Ri] C.M. Ringel, *Tame algebras,* Representation Theory I. Springer LNM 831 (1980), 134–287
[Ri2] C.M. Ringel, *Tame algebras and integral quadratic forms,* Springer Lect. Notes Math. 1099 (1984).
[Ri3] C.M. Ringel, *Recent advances in the representation theory of finite dimensional algebras,* Representation theory of finite groups and finite dimensional algebras, Birkhäuser (1991), 141–192.
[Sk1] A. Skowroński, *Group algebras of polynomial growth,* Manuscripta Math. 59 (1987), 499–516.
[WW] B. Wald and J. Waschbsch, *Tame biserial algebras,* J. Algebra 95 (1985), 480–500.
[ZS] O. Zariski and P. Samuel, *Commutative Algebra, Volume I.* Springer GTM 28 (1979)

Universität Bayreuth, Fakultät für Mathematik und Physik, 95440 Bayreuth
*E-mail address*: `Christof.Geiss@uni-bayreuth.d400.de`