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COINCIDENCE INDEX FOR FIBER-PRESERVING MAPS
AN APPROACH TO STABLE COHOMOTOPY

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A Coincidence index in any generalized (multiplicative) cohomology theory is defined for certain pairs of maps between euclidean neighborhood retracts over a metric space B.
By taking an adequate geometric equivalence relation between two such coincidence situations, groups $FIX^k(B)$ and $FIX^k(B,A)$, for A closed in B, k an integer, can be defined. The purpose of this paper is to show that these groups constitute a generalized multiplicative cohomology theory. Moreover, we show that the index determines an isomorphism between this theory and stable cohomotopy.

0. INTRODUCTION

Dold, 1974 [Dol], defined a fixed point index for fiber-preserving (self-)maps of ENR_B's, which is an endomorphism of h(B), the cohomology of the base-space B, or equivalently, an element in $h^o(B)$ if h is a multiplicative cohomology theory. If the multiplicative theory is stable cohomotopy he showed that the indexes of two maps coincide if and only if a certain geometric equivalence between the two maps exists; moreover he showed that every element of the 0th stable cohomotopy of B occurs as the index of a fiber-preserving map, thus showing that the 0th stable chomotopy group of the base space has a geometric description as a set (ring) of equivalence classes of fiber-preserving maps.

In this paper, a coincidence index for (certain) pairs of fiber preserving maps is defined. This is an endomorphism of h(B), now of degree k (depending on the pair) or an element in $h^k(B)$ if h is a multiplicative theory. A geometric equivalence between these pairs is introduced and it is shown that this gives a geometric description for the k-th stable cohomotopy group of the base space, thus extending the results of Dold. Furthermore, given this geometric description of the stable cohomotopy groups, suspension isomorphisms, as well as other structure-homomorphisms of a cohomology theory have a geometric description.

In his thesis [Sch], B. Schäfer gave a description of the stable homotopy category where the morphisms were described by fixed point situations, (Kap. 4 and 5). By specializing these morphism groups (varying the domain and letting the codomain to be a point) one gets the groups that I study here.

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1. COINCIDENCE INDEX HOMOMORPHISM

The topological background for this section can be found in [Do1] and [Do2] where the definitions not given here, appear.

1.1 Let B be a metric space and $p: E \to B$ be an ENR_B. We shall study the following situations: Let $V \subset E$ be open; a map over B

$$\begin{array}{ccc} E \supset V & \xrightarrow{f} & \mathbb{R}^k \times E \\ & {}_{p}\searrow \quad \swarrow_{p \circ proj_E} & \\ & B & \end{array}$$

is said to be *compactly* k-*fixed* if its k-*fixed point set*, $Fix^k(f) = \{v \in V : f(v) = (0,v)\}$, lies *properly* over B, i.e. if $p|Fix^k(f) : Fix^k(f) \to B$ is a proper map. Observe that $Fix^k(f)$ is the coincidence set of f with the 0-section $E \to \mathbb{R}^k \times E$, whence the terminology "coincidence index". Anyway, I will prefer to use the terminology "k-fixed point index", since first, the case $k = 0$ will be the fixed point index studied by Dold and second, we are really studying fixed points of maps defined on open sets of the 0-section of $\mathbb{R}^k \times E$, i.e. of certain partial self maps of $\mathbb{R}^k \times E$.

1.2 We begin by studying compactly k-fixed maps in the case $E = \mathbb{R}^n \times B$ with p the projection onto B. Since $p|Fix^k(f)$ is proper, we know [Do2,(1.3)], that there exists a continuous function $\rho : B \to (0,\infty) \subset \mathbb{R}$ such that

$$Fix^k(f) \subset \{(y,b) \in \mathbb{R}^n \times B : \|y\| \leqslant \rho(b)\} = E_\rho$$

Define the k-*index homomorphism* of f by

$$I_f^k : h^j(B) \overset{\sigma^{k+n}}{\simeq} h^{j+k+n}((\mathbb{R}^{k+n},\mathbb{R}^{k+n}-0)xB) \xrightarrow{(\iota-f)^*} h^{j+k+n}(V,V\text{-}Fix^k(f))$$

$$\simeq h^{j+k+n}(E,E\text{-}Fix^k(f)) \to h^{j+k+n}(E,E\text{-}E_\rho)$$

$$\simeq h^{j+k+n}((\mathbb{R}^n,\mathbb{R}^n-0)xB) \overset{\sigma^n}{\simeq} h^{j+k}(B)$$

where σ^{k+n}, resp. σ^n, is the (k+n)-, resp. n-fold suspension isomorphism and $(\iota-f) : V \to \mathbb{R}^{k+n}xB = \mathbb{R}^k xE$ is such that $(\iota-f)(y,b) = ((0,y)-\varphi(y,b),b)$ with φ the first component of f, $f(y,b) = (\varphi(y,b),b) \in (\mathbb{R}^k x\mathbb{R}^n)xB$.

This k-index has properties analogous to those of Dold's index. We list them without proofs since they are essentially the same as Dold's.

1.4. <u>Localization in E</u>. *If* W *is open in* V *and* $Fix^k(f) \subset W$, *then* $f|W$ *is compactly* k-*fixed and* $I_f^k = I_{f|W}^k$. □

1.5. <u>Localization in B</u>. *The* k-*index factors as follows:*

$$h^j(B) \to h^{j+k}(B,A) \to h^{j+k}(B)$$

for any A *such that* $p(Fix^k(f)) \subset B\text{-}A$. □

1.6. <u>Additivity</u>. *If* $V = V_1 \cup V_2$ *is a union of open sets such that* $f_{1,2} = f|V_1 \cap V_2$ *(and* f*) is compactly* k-*fixed, then* $f_1 = f|V_1$ *and* $f_2 = f|V_2$ *are compactly* k-*fixed and* $I_f^k + I_{f_{1,2}}^k = I_{f_1}^k + I_{f_2}^k$. □

1.7. <u>Units</u>. *If* $s : B \to E$ *is a section, then the* k-*index of* $(0,sp) : E \to \mathbb{R}^k xE$ *is the identity of* h(B) *if* k = 0 *and* 0 *if* k > 0. □

1.8. <u>Naturality in B</u>. *Let* B' *be metric and* $\beta : B' \to B$ *be continuous. If*

$$\begin{array}{ccc} E' \supset V' & \xrightarrow{f'} & \mathbb{R}^k xE' \\ & \searrow \quad \swarrow & \\ & B' & \end{array}$$

is the induced map after taking pullbacks over β *, then* f' *is compactly* k-*fixed and the diagram*

$$\begin{array}{ccc} h^j(B) & \xrightarrow{I_f^k} & h^{j+k}(B) \\ \beta^* \downarrow & & \downarrow \beta^* \\ h^j(B') & \xrightarrow{I_{f'}^k} & h^{j+k}(B') \end{array}$$

commutes. □

1.9. Homotopy Invariance. *If* $g : W \to \mathbf{R}^k xF$, $W \subset F$ *open, is compactly* k-*fixed over* BxI, *with projection* $F \to BxI$ $(I = [0,1])$ *and if* $g_t : W_t \to \mathbf{R}^k xF_t$ *is the part of* g *over* $Bx\{t\} \approx B$, *then* $I_{g_t}^k = I_{g_0}^k$. □

1.10. Multiplicativity. *If* $f : V \to \mathbf{R}^k xE$ *and* $f' : V' \to \mathbf{R}^{k'} xE'$ *are compactly* k- *and* k'-*fixed over* B, *then*

$$fx_B f' : Vx_B V' \to (\mathbf{R}^k xE)x_B(\mathbf{R}^{k'}xE') \approx \mathbf{R}^{k+k'}x(Ex_B E')$$

is compactly (k+k')-*fixed and*

$$I_{fx_B f'}^{k+k'} = I_f^k \circ I_{f'}^{k'} : h^j(B) \to h^{j+k+k'}(B). \quad \square$$

As in [Do1], we have the special case:

1.11. Stability. *If* $e : \mathbf{R} \to \mathbf{R}$ *is constant, then* $fxe : Vx\mathbf{R} \to \mathbf{R}^k xEx\mathbf{R}$ *is compactly* k-*fixed as a map over* B *with respect to* $Ex\mathbf{R} \to E \to B$ *and* $I_{fxe}^k = I_f^k$. □

1.12. Commutativity. *Assume* $p : E \to B$ *and* $p' : E' \to B$ *are* ENR_B*'s and* $U \subset E$ *and* $U' \subset E'$ *are open. Let* $f : U \to \mathbf{R}^k xE'$ *and* $g : U' \to \mathbf{R}^{k'} xE$ *be continuous over* B. *If*

$$(1xg)f : f^{-1}(\mathbf{R}^k xU') \to \mathbf{R}^k x\mathbf{R}^{k'} xE \approx \mathbf{R}^{k+k'} xE$$

is compactly (k+k')-*fixed, then so is*

$$(1xf)g : g^{-1}(\mathbf{R}^{k'} xU) \to \mathbf{R}^{k'} x\mathbf{R}^k xE' \approx R^{k+k'} xE'$$

and $I_{(1xg)f}^{k+k'} = (-1)^{kk'} I_{(1xf)g}^{k+k'}$.

The proof is essentially the same as for [DoO,VII,5.9]. □

1.13. Naturality in h. *If* $T : h_1 \to h_2$ *is a natural transformation of cohomology theories, then*

$$\begin{array}{ccc} h_1^j(B) & \xrightarrow{I_f^k} & h_1^{j+k}(B) \\ T\downarrow & & \downarrow T \\ h_2^j(B) & \xrightarrow{I_f^k} & h_2^{j+k}(B) \end{array}$$

is commutative.

1.14. The k-Index Element. (PROPOSITION and DEFINITION) *If* h *is a multiplicative cohomology theory, then* $I_f^k : h(B) \to h(B)$ *is a homomorphism of degree* k *of* h(B)-*modules, i.e.*

$$I_f^k(x) = x \cup I_f^k(1)$$

for all $x \in h(B)$, *where* $1 \in h^0(B)$ *is the unit element.*
The element $I_f^k(1) \in h^k(B)$ completely describes I_f^k and is called the k-*index of* f; it is denoted by $I^k(f)$. □

1.15. PROPOSITION and DEFINITION. *If* $p : E \to B$ *is any* ENR_B *and* $V \subset E$ *is open, then every continuous map* $f : V \to \mathbb{R}^k xE$ *admits a decomposition*

$$f : V \xrightarrow{\alpha} U \xrightarrow{\beta} \mathbb{R}^k xE$$

over B, *where* U *is open in some* $\mathbb{R}^n xB$. *If* f *is compactly* k-*fixed, so is* $g = (1x\alpha)\beta : \beta^{-1}(\mathbb{R}^k xV) \to \mathbb{R}^k x(\mathbb{R}^n xB)$, *hence* $I_g^k : h^j(B) \to h^{j+k}(B)$ *is defined. Moreover,* I_g^k *depends only on* f *and not on the decomposition* $f = \beta\alpha$.
By definition $I_f^k = I_{(1x\alpha)\beta}^k$ is the k-*index homomorphism of* f, (resp. $I^k(f) = I^k((1x\alpha)\beta)$ the k-*index of* f if h is multiplicative). *All the properties* 1.4. - 1.14. *still hold for this general case.*

The proof is an application of commutativity 1.12. □

2. THE IMAGE OF I^k IN $h^k(B)$

2.1. As in [Do1], let $s \in h^1(\mathbb{R},\mathbb{R}-0) \cong h^1(\mathbf{S}^1)$ denote the image of $1 \in h^0(*)$ under the isomorphisms $h^0(*) \cong h^0(\mathbb{R}-0) \overset{\delta}{\cong} h^1(\mathbb{R},\mathbb{R}-0)$, i.e. under the *suspension isomorphism*. Recall that the general suspension isomorphism in h is given by multiplication with s, and its n-th iterate σ^n by multiplication with s^n, where $s^n = \sigma^n(1)$ is the image of $1 \in h^0(*)$ under $h^0(*) \cong h^n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong \tilde{h}^n(\mathbb{S}^n)$.

2.2. There is a k-analogue of [Do1, (3.2)] through which k-*stably spherical elements* may be defined. Consider the

2.3. DEFINITION. Define the k-*stable cohomotopy group* of the space X by

$$\pi_s^k(X) = \lim_j [\Sigma^j(X^+), \mathbb{S}^{k+j}]$$

Using the k-stable cohomotopy group one can give an alternative definition of k-stably spherical elements. We have as in [Do1]

2.4 PROPOSITION. *If* h *is a multiplicative cohomology theory, then* $y \in h^k(B) = \tilde{h}^k(B^+)$ *is* k-*stably spherical if and only if* y *is in the image of the Hurewicz-homomorphism* $\varepsilon\colon \pi_s^k(B) \to h^k(B)$ (which sends $f\colon \Sigma^j(B^+) \to \mathbb{S}^{k+j}$ to $\sigma^{-j} f^*(s^{k+j})$). □

Following the same techniques as Dold we can prove the k-analogue of [(Do1, (3.7)], namely

2.5. THEOREM. *Let* h *be a multiplicative cohomology theory and* B *be metric. The elements of* $h^k(B)$ *which occur as* k-*indices of compactly* k-*fixed maps over* B *are precisely the* k-*stably spherical ones; i.e. the image of* I^k *coincides with the image of* $\varepsilon\colon \pi_s^k(B) \to h^k(B)$ □

3. HOMOTOPY INVARIANCE AS THE FUNDAMENTAL PROPERTY OF I^k; THE MONOID $FIX^k(B)$

If the cohomology theory h is stable cohomotopy π_s then every element $y \in h^k(B) = \pi_s^k(B)$ is stably spherical. This shows that $f \mapsto I^k(f)$ is surjective by 2.5. We now answer the question about injectivity; we show that $I^k(f_0) = I^k(f_1)$ if and only if f_0 and f_1 are homotopic in the sense of 1.9.

3.1. DEFINITION. If B is a metric space, let $F^k(B)$ be the set of (fiberwise homeomorphism classes of) compactly k-fixed maps over B. We call two elements $f_0: V_0 \to \mathbb{R}^k \times E_0$ and $f_1: V_1 \to \mathbb{R}^k \times E_1$ in $F^k(B)$ *equivalent*, in symbols $f_0 \sim f_1$, if there exists a compactly k-fixed map $g: W \to \mathbb{R}^k \times F$ over $B \times I$ such that $g_0 = f_0$ and $g_1 = f_1$. We denote the class of f by $[f]$ and the set of equivalence classes by $FIX^k(B) = F^k(B)/\sim = \{[f]\}$.

The main result of the first part of this paper is the following

3.2. THEOREM. *If* h *is stable cohomotopy, then* I^k *is isomorphic, i.e.*

$$I^k: FIX^k(B) \cong \pi_s^k(B)$$

for all $k \geqslant 0$.

In the last section we saw that I^k is surjective. The injectivity in the general case $(k>0)$ follows essentially in the same way as for Dold's case $(k = 0)$, therefore we omit the proof. However, we give the formulation of the series of lemmas that lead to this result since some of them will be of interest in what follows. □

3.3. LEMMA. *If* $f_0: V_0 \to \mathbb{R}^k \times E_0$ *and* $f_1: V_1 \to \mathbb{R}^k \times E_1$ *are compactly k-fixed maps over* B, *such that* E_0 *is an open subset of* E_1, $Fix^k(f_1) \subset V_0 \subset V_1$ *and* $f_0(v) = f_1(v)$ *for all* $v \in V_0$, *then* $f_0 \sim f_1$. □

3.4. LEMMA. *Let* $i: E \to D$ *be an inclusion over* B *of* ENR_B*'s and let* $r: D \to E$ *be a retraction over* B. *If* $f: V \to \mathbb{R}^k \times E$ *is a compactly k-fixed map over* B *then so is* $f' = (1 \times i) f r : r^{-1}(V) \to \mathbb{R}^k \times D$ *and* $f' \sim f$. □

3.5. PROPOSITION. *Every element* $\xi \in FIX^k(B)$ *has a representative of the form* $f: \mathbb{R}^n \times B \to \mathbb{R}^k \times \mathbb{R}^n \times B$ *such that*

(i) $\|(0,y) - \varphi(y,b)\| = \|y\|$

(ii) $\varphi_2(\lambda y,b) = \lambda \varphi_2(y,b)$

for all $b \in B$, $y \in \mathbb{R}^n$, $\lambda \geqslant 0$, *where* $f(y,b) = (\varphi(y,b),b) =$
$= (\varphi_1(y,b), \varphi_2(y,b),b)$. *In particular* $(i) \Rightarrow Fix^k(f) = \{0\}xB$. □

4. THE RELATIVE CASE. FIX* AS A COHOMOLOGY THEORY

In this paragraph we extend the definition of the functor FIX^k to pairs of metric spaces

4.1 DEFINITION. Let $A \subset B$. A k-*fixed point situation over* (B,A) is a compactly k-fixed map $f: V \to \mathbb{R}^k x E$ over B such that

$$Fix^k(f) \cap p^{-1}(A) = \phi$$

where $p: E \to B$ is the projection. Two such k-fixed point situations f_0 and f_1 over (B,A) are equivalent if there exists a k-fixed point situation g over $(B,A)xI = (BxI,AxI)$ that restricts to f_0 and f_1 at the levels 0 and 1 respectively. Let $FIX^k(B,A)$ denote the set of equivalence classes under this relation. It is an abelian group under disjoint union, the negative given by taking the exterior product by the element $2\cdot: \mathbb{R} \to \mathbb{R}$ (multiplication by 2) in $FIX^o(*)$. (*Exterior product* of an element $[f] \in FIX^k(X,A)$ with the element $[g] \in FIX^{k'}(Y,B)$ is the element $[fxg] \in FIX^{k+k'}(XxY, YxB \cup AxY)$ represented by the cartesian product fxg of f and g).

In what follows we shall show that these groups constitute a multiplicative cohomology theory and furthermore we shall see that the index I is a natural transformation of cohomology theories, thus proving that FIX* is isomorphic to stable cohomotopy.

4.2. Given an element in $FIX^k(B,A)$ we can apply to it the "forgetful" homomorphism to obtain an element in $FIX^k(B)$ which obviously restricts to the trivial element in $FIX^k(A)$. This shows that the composite

4.3 $$FIX^k(B,A) \to FIX^k(B) \to FIX^k(A)$$

is zero. We have

4.4. THEOREM. *If* $A \subset B$ *is closed, then the sequence* 4.3 *is exact.*

Proof. Let $E \supset V \xrightarrow{f} \mathbb{R}^k x E$ represent an element in $FIX^k(B)$ that goes to zero in $FIX^k(A)$. This means that the restriction $E_A \supset V_A \xrightarrow{f_A} \mathbb{R}^k x E_A$

is equivalent to a k-fixed point free situation. Let

$$\begin{array}{ccc} F \supset W & \xrightarrow{\ g\ } & \mathbb{R}^k x F \\ & \searrow \quad \swarrow & \\ & AxI & \end{array}$$

be such an equivalence, i.e. $g_{Ax\{0\}} = f_A$ and $Fix^k(g_{Ax\{1\}}) = \phi$. We can consider

4.5.
$$\begin{array}{ccc} Ex\{0\} \cup F \supset Vx\{0\} \cup W & \xrightarrow{\ fx\{0\} \cup g\ } & \mathbb{R}^k x(Ex\{0\} \cup F) \\ & \searrow \quad \swarrow & \\ & Bx\{0\} \cup AxI & \end{array}$$

by taking the obvious identifications. By 3.4. we know that $fx\{0\} \cup g$ in 4.5. is equivalent to a compactly k-fixed map

$$\begin{array}{ccc} \mathbb{R}^n x(Bx\{0\} \cup AxI) \supset U & \xrightarrow{\ h\ } & \mathbb{R}^k x \mathbb{R}^n x(Bx\{0\} \cup AxI) \\ & \searrow \quad \swarrow & \\ & Bx\{0\} \cup AxI & \end{array}$$

such that $Fix^k(h) \approx Fix^k(fx\{0\} \cup g)$ over $Bx\{0\} \cup AxI$. Moreover, we may assume that $\bar{U} \to Bx\{0\} \cup AxI$ is proper, (since, if not, there is a tubular neighborhood of $Fix^k(h)$ with proper closure that we may intersect with U, and of course, apply 3.3.).

Since $Bx\{0\} \cup AxI$ is closed in BxI we may assume that there is an open neighborhood U" of $Fix^k(h)$ in $\mathbb{R}^n xBxI$ such that U is closed in U" and, moreover, such that $\bar{U}" \to BxI$ is proper (by Tietze's lemma we can extend the function that defines the tubular neighborhood to the whole BxI).

We can now apply Tietze's lemma to extend (the euclidean part of) h to U". Let $h" : U" \to \mathbb{R}^k x \mathbb{R}^n xBxI$ be such an extension. We may assume that h" is defined also on the boundary $\partial U"$ (since if not, we can change U" by another neighborhood $U_1"$ of $Fix^k(h)$ such that $\bar{U}_1" \subset U"$). Let $C = Fix^k(h") \cap \partial U"$, since $p|C$ is proper, we have that p(C) is closed and contained in $BxI - (Bx\{0\} \cup AxI)$, because h" coincides with h over $Bx\{0\} \cup AxI$ and thus has no k-fixed points on $\partial U"$ over it.

There exists a map $\tau : B \to [0,1]$, such that $\tau|A = 1$ and $\{(b,t) \in BxI : t \leq \tilde{\tau}(b)\} \cap p(C) = \phi$. Define $\gamma : BxI \to BxI$ by $\gamma(b,t) = (b, \min\{t,\tau(b)\})$. Let $h' : U' \to \mathbb{R}^k x \mathbb{R}^n xBxI$ be the pullback of h" over γ; clearly $Fix^k(h')$ has no common points with $\partial U'$, hence $Fix^k(h') \to BxI$ is a proper map. h' is a compactly k-fixed deformation of f to a map with no k-fixed points over A, thus showing that [f] comes from $FIX^k(B,A)$. □

Before going on proving the existence of the long exact sequence we prove the Homotopy Axiom.

4.6. THEOREM. *If* $\alpha_0 \simeq \alpha_1 : (B', A') \to (B,A)$ *are homotopic maps of pairs, then* $\alpha_0^* = \alpha_1^* : FIX^k(B,A) \to FIX^k(B',A')$.

Proof. Let $\alpha : (B',A')xI \to (B,A)$ be a homotopy and let $f : V \to \mathbb{R}^k xE$ represent an arbitrary element in $FIX^k(B,A)$. The pullback of f over α is an equivalence between $\alpha_0^*(f)$ and $\alpha_1^*(f)$, thus $\alpha_0^*[f] = \alpha_1^*[f]$. □

4.7. COROLLARY. *If the pair* (B,A) *is homotopy equivalent to a closed pair (i.e.* $(B,A) \simeq (B',A')$, $A' \subset B'$ *closed), then*

$$FIX^k(B,A) \to FIX^k(B) \to FIX^k(A)$$

is exact.

Proof. This sequence is isomorphic to the corresponding sequence of (B',A'). Apply 4.4. □

Given the nature of the functors FIX^k the most natural suspension isomorphism can be given in terms of the suspension given by multiplying by $(\mathbb{R}, \mathbb{R}-0)$. We define

4.8. $$\sigma : FIX^k(B) \to FIX^{k+1}((\mathbb{R}, \mathbb{R}-0)xB)$$

as exterior product by the element $s \in FIX^1(\mathbb{R}, \mathbb{R}-0)$ given by $\Delta: \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $\Delta(t) = (-t,t)$, i.e. $\sigma(f)(t,v) = (-t,f'(v), t, f_2(v))$ if $f(v) = (f'(v), f_2(v))$.

4.9. THEOREM. σ *is an isomorphism.*

One can construct an explicit inverse to σ, but since the theorem follows automaticly from 4.23, we omit this construction. □

4.10. REMARK. It is straightforward to show the compatibility of σ with suspensions in general cohomology, namely to show that $I^{k+1}(\sigma[f]) = \sigma(I^k[f])$ holds.

We now have all the ingredients to show the existence of a long exact sequence for a pair (B,A), $A \subset B$ closed.

Observe that combining 4.6. and 4.9. we have an isomorphism

4.11. $$\sigma : FIX^{k-1}(A) \to FIX^k((I,\partial I)xA)$$

given, say, by sending $[f]$ to $[\sigma(f): IxE \to \mathbb{R} x \mathbb{R}^{k-1} x IxE]$ so that $\sigma(f)(t,v) = (1-2t, f'(v), t, \bar{f}(v))$ if $f = (f',\bar{f})$. Whence, in order to define $\delta: FIX^{k-1}(A) \to FIX^k(B,A)$ it is enough to define

4.12. $$\delta': FIX^k((I,\partial I)xA) \to FIX^k(B,A),$$

which can be done as follows.

Suppose first that one takes an element in $FIX^k((I,\partial I)xA)$ represented by

$$\mathbb{R}^n x IxA \supset V \xrightarrow{g} \mathbb{R}^k x \mathbb{R}^n x IxA$$

over IxA such that $Fix^k(g) \subset \mathbb{R}^n x (0,1)xA$. Without loss of generality we may assume that the closure of V, $\bar{V}$, is contained in $\mathbb{R}^n x (0,1)xA$, lies properly over IxA and that g can be extended to the boundary, ∂V, of V. In particular g will not have any k-fixed points on the boundary.

Let $V"$ be open in $\mathbb{R}^n x IxB$, so that $\bar{V}" \cap (\mathbb{R}^n x IxB) = \bar{V}$, $\bar{V}" \to IxB$ is proper and $\bar{V}" \subset \mathbb{R}^n x (0,1)xB$. By Tietze's lemma we may extend g to $g": V" \to \mathbb{R}^k x \mathbb{R}^n x IxB$ (compare with the proof of 4.4).

We can assume that g'' can also be extended to the boundary of V''. g'' might fail to be compactly k-fixed, i.e. it may have k-fixed points on $\partial V''$. These k-fixed points on the boundary (after shrinking V'' if necessary) are contained in $\mathbb{R}^n x I x (B-A)$ and lie properly over IxB (since $\partial V''$ does). Whence, their image on IxB is a closed set contained in $(0,1] x (B-A)$. Again, as in the proof of 4.4., there exists a continuous map $\tau : B \to I$ such that $\tau|A = 1$ and $\{(t,b) \in$ $\in IxB : t \leq \tau(b)\}$ does not intersect this image. Pulling g'' back over $\gamma : IxB \to IxB$, $\gamma(t,b) = (\min\{t,\tau(b)\},b)$, we get a compactly k-fixed map $g' : V' \to \mathbb{R}^k x \mathbb{R}^n x I x B$ extending g and having no k-fixed points over $\{0\}xB \cup \{1\}xA$. Let $\delta'(g) = [g'_{\{1\}xB}] \in FIX^k(B,A)$. Clearly

4.13. $$[g'_{\{1\}xB}] = 0 \in FIX^k(B)$$

4.14. PROPOSITION. δ' *determines a well defined homomomorphism* $FIX^k((I,\partial I)xA) \to FIX^k(B,A)$.

Proof. We prove first that two compactly k-fixed extensions g'_0 and g'_1 of g without k-fixed points over $Bx\{0\}$ are equivalent in such a way that their restrictions to $\{1\}xB$ become equivalent over (B,A). Take $IxBxI$ and define G over $IxBx\partial I \cup IxAxI$ by g'_0 over $IxBx\{0\}$, g'_1 over $IxBx\{1\}$ and by g over $IxAx\{s\}$ for all $s \in I$. As before, extend G to G'' on an open set over $(0,1] xBxI$ with proper closure, take a map $\tau : BxI \to I$ such that $\tau|\partial IxB \cup IxA = 1$ and $\{(t,b,s) \in IxBxI : t \leq \tau(b,s)\}$ contains no image of a k-fixed point of G'' that lies on the boundary. Hence, the pullback, G', of G'' over $\gamma : IxBxI \to IxBxI$, $\gamma(t,b,s) = (\min\{t,\tau(b,s)\},b,s)$ is the compactly k-fixed point deformation we were looking for.

In order to extend the definition of δ' to a general

$$\begin{array}{ccc} E \supset V & \xrightarrow{g} & \mathbb{R}^k x E \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & IxA & \end{array}$$

such that $pFix^k(g) \subset (0,1)xA$, we need the following lemma as in the case of the suspension (cf. 4.12.)

4.15. LEMMA. *Let*

$$\mathbb{R}^{n_1} x I x A \supset V_1 \overset{\alpha}{\to} \mathbb{R}^k x \mathbb{R}^{n_2} x I x A$$

and $$\mathbb{R}^{n_2}\times I\times A \supset V_2 \xrightarrow{\beta} \mathbb{R}^{n_1}\times I\times A$$

be such that

$$\mathbb{R}^{n_2}\times I\times A \supset \beta^{-1}(V_1) \xrightarrow{\alpha\beta} \mathbb{R}^k\times\mathbb{R}^{n_2}\times I\times A$$

is compactly k-*fixed over* $(I,\partial I)\times A$, *then also*

$$\mathbb{R}^{n_1}\times I\times A \supset \alpha^{-1}(\mathbb{R}^k\times V_2) \xrightarrow{(1\times\beta)\alpha} \mathbb{R}^k\times\mathbb{R}^{n_1}\times I\times A$$

is compactly k-*fixed over* $(I,\partial I)\times A$ *and*

4.16. $$\delta'(\alpha\beta) \sim \delta'((1\times\beta)\alpha)$$

holds.

Proof. We only have to prove 4.16. We shall sketch it: Extend α to α'' and β to β'' over $I\times B$ (with similar conditions on the open sets as before). If $\alpha''\beta''$ is not already compactly k-fixed, produce τ and γ (as before) such that the pullback of $\alpha''\beta''$ over it, is. Let α' and β' be the pullbacks of α'' and β'' over γ, then $\alpha'\beta'$ is precisely $\delta'(\alpha\beta)$ and is equivalent by commutativity to $(1\times\beta')\alpha'$ which is precisely $\delta'((1\times\beta)\alpha)$. This proves the lemma. □

4.17. THEOREM. *The sequence*

4.18. $$\cdots \to FIX^{k-1}(B) \xrightarrow{i^*} FIX^{k-1}(A) \xrightarrow{\delta} FIX^k(B,A) \xrightarrow{j^*} FIX^k(B) \to \cdots$$

is exact, where $\delta = \delta'$, *and* i *and* j *are inclusions*.

Proof. By the naturality of σ it is enough to prove the exactness of

4.19. $$FIX^k((I,\partial I)\times B) \xrightarrow{i^*} FIX^k((I,\partial I)\times A) \xrightarrow{\delta'} FIX^k(B,A) \xrightarrow{j^*} FIX^k(B)$$

At $FIX^k((I,\partial I)\times A)$:

$\delta' i^* = 0$ clearly,

$\delta'(g) = 0 \Rightarrow g'_{\{1\}\times B}$ can be deformed to a k-fixed point free map, whence g has an extension to $I\times B$ without any k-fixed points on $\partial I\times B$.

At $FIX^k(B,A)$:

$j^*\delta' = 0$ as we already saw,

$j^*(f) = 0 \Rightarrow \exists\, g'$ over $I\times B$ such that $g'_{\{1\}\times B} = f$ (hence has no k-fixed points on $\{1\}\times A$) and $g'_{\{0\}\times B}$ is k-fixed point free. Its restriction, g, to $I\times A$ has no k-fixed points on $\partial I\times A$, whence, $[g] \in FIX^k((I,\partial I)\times A)$ is such that $\delta'[g] = [f]$.

The exactness at $FIX^k(B)$ is the contents of 4.4. □

4.20. Take the following sequence of maps of pairs:

$$(I,\partial I)\times A \twoheadrightarrow (\Sigma A^+,*) \xleftarrow{\kappa} (CA^+ \cup B^+,*) \twoheadleftarrow (I\times A \cup \{0\}\times B,\{1\}\times A)$$
$$\to (I\times B,\{1\}\times A) \leftarrow (B,A)$$

where κ is the map of the Puppe sequence that maps canonically the mapping cone of the inclusion $A^+ \subset B^+$ ($X^+ = X \sqcup *$) onto the suspension of A^+ and the rest of them are also canonical maps that are equivalences, the last of them being inclusion on the top. By pulling back or "pushing forward" a k-fixed point situation g over $(I,\partial I)\times A$ along the sequence we obtain precisely $\delta'(g)$ over (B,A). Combining this fact with the remark 4.10. we find that

4.21. *The* k-*index is compatible with the coboundary homomorphism* δ.

4.22. PROPOSITION. *Let* $(X = X_1 \cup X_2; X_1, X_2)$ *be an excisive triple, then the canonical homomorphism*

$$FIX^k(X,X_2) \to FIX^k(X_1, X_1 \cap X_2)$$

is an isomorphism.

There is a direct proof of this fact, which we omit, since this is also a consequence of 4.23. □

With this last proposition we have our main

4.23. THEOREM. *Let* $k \geqslant 0$. *(i) The functors* FIX^k *together with the natural transformations* δ *constitute a multiplicative cohomology theory* FIX* *on the category of metric spaces.*

(ii) The indices $I^k: FIX^k \to h^k$ *define a natural transformation of multiplicative cohomology theories. And*

(iii) $I^k: FIX^k \to \pi_S^k$ *is a natural equivalence between the theory* FIX* *and stable cohomotopy* $(k \geq 0)$.

Proof. *(i)* The homotopy axiom is proved in 4.6., the excision axiom is 4.22. and the exactness axiom is 4.17, The multiplicative structure is given as follows. Let

$$\cup: FIX^k(B,A) \otimes FIX^{k'}(B,A') \to FIX^{k+k'}(B,A \cup A')$$

be defined on representatives

$$\begin{array}{ccc} E \supset V & \xrightarrow{f} & \mathbb{R}^k \times E \\ & \searrow \;(B,A)\; \swarrow & \end{array} \qquad \begin{array}{ccc} E' \supset V' & \xrightarrow{f'} & \mathbb{R}^{k'} \times E' \\ & \searrow \;(B,A')\; \swarrow & \end{array}$$

by letting $[f] \cup [f']$ be represented by

$$\begin{array}{ccc} E \times_B E' \supset V \times_B V' & \xrightarrow{f \times_B f'} & \mathbb{R}^k \times \mathbb{R}^{k'} \times E \times_B E' \\ & \searrow \;(B,A \cup A')\; \swarrow & \end{array}$$

(ii) Naturality in B, 1.8, multiplicativity, 1.10. and 4.25 show that I^k defines such a transformation.

(iii) Using *(ii)* together with theorem 3.2. and the five-lemma we get the proof that I^k is an isomorphism also for pairs. □

4.24. REMARKS.

(a) One may define FIX^k also for negative k as follows. The elements of $FIX^k(B,A)$ are represented by k-*fixed point situations*

4.25.
$$\begin{array}{ccc} \mathbb{R}^{-k} \times E \supset V & \xrightarrow{f} & E \\ & \searrow \;B\; \swarrow & \end{array}$$

such that $Fix^k(f) = \{(y,e) \in V: f(y,e) = e\}$ lies properly over B and has no points over A. The equivalence between two of these is a k-fixed point situation over $(B,A) \times I$.

We have analogously to 4.9. that there is an isomorphism

$$\sigma\colon \mathrm{FIX}^k(B,A) \to \mathrm{FIX}^{k+1}((\mathbb{R},\mathbb{R}-0)\times(B,A))$$

given again by taking the exterior product with the element $\Delta\colon \mathbb{R}\to\mathbb{R}\times\mathbb{R}$, $\Delta(t) = (-t,t)$, in $\mathrm{FIX}^1(\mathbb{R},\mathbb{R}-0)$.

(b) If k is negative the k-index can still be defined, formally in the same way as in 1.3., namely

$$\begin{aligned} I_f^k\colon h^j(B) &\overset{\sigma^n}{\simeq} h^{j+n}((\mathbb{R}^n,\mathbb{R}^n-0)\times B) \xrightarrow{(\iota-f)^*} h^{j+n}(V,V-\mathrm{Fix}^k(f)) \\ &\simeq h^{j+n}(\mathbb{R}^{-k}\times E,\mathbb{R}^{-k}\times E-\mathrm{Fix}^k(f)) \to h^{j+n}(\mathbb{R}^{-k}\times E,\ \mathbb{R}^{-k}\times E-(\mathbb{R}^{-k}\times E)_\rho) \\ &\simeq h^{j+n}((\mathbb{R}^{-k+n},\mathbb{R}^{-k+n}-0)\times B) \overset{\sigma^{-k+n}}{\simeq} h^{j+k}(B) \end{aligned}$$

for $E = \mathbb{R}^n\times B$ in 4.25., where $\iota\colon V \to \mathbb{R}^n\times B = E$ is such that $\iota(y,e) = e$ and $(\mathbb{R}^{-k}\times E)_\rho = (\mathbb{R}^{-k+n}\times B)_\rho$ is a tubus of radius $\rho\colon B\to(0,\infty)$ containing $\mathrm{Fix}^k(f)$.

4.26. I_f^k *will have similar properties to* 1.4.- 1.15.

(c) If $p\colon E\to B$ is a proper ENR_B and $f\colon E\to\mathbb{R}^k\times E$ is a continuous map, $k > 0$, then one easily proves that f is equivalent to a k-fixed point free map, hence the index $I^k(f)$ is zero for every cohomology theory; in particular there is no interesting Lefschetz-Hopf formula for the k-index if $k > 0$.

(d) More generally, if $\xi,\eta\to B$ are vector bundles one can consider (ξ,η)-*fixed point situations*

$$\begin{array}{ccccc} \eta\times_B E & \supset & V & \xrightarrow{f} & \xi\times_B E \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

such that $\mathrm{Fix}^{\xi,\eta}(f) = \{(y,e)\in V\colon f(y,e) = (0,e)\}$ lies properly over B and has no points over A. These situations define a group $\mathrm{FIX}^{\xi,\eta}(B,A)$ which can be shown to depend only on the difference $[\xi]-[\eta]\in K(B)$. If ξ,η are pullbacks of corresponding bundles $\bar\xi,\bar\eta\to X$ by some map $B\to X$

then, by varying B, these groups are seen to constitute a cohomology theory in the category of spaces over X which is graded by K(X). Details will appear elsewhere.

4.27 EXAMPLE: Let $S^2 = \mathbb{C} \cup \{\infty\}$ be the Riemann-sphere. $FIX^2(S^2) = \mathbb{Z}$ If $n \in \mathbb{Z}$ then the compactly 2-fixed map over S^2

$$S^2 \supset \mathbb{C} \xrightarrow{f} \mathbb{C} \times S^2$$
$$z \longrightarrow (z^n, z)$$

represents n. (Note that $Fix^2(f) = \{0\}$ for all n).

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