A SUM FORMULA FOR STABLE EQUIVARIANT MAPS

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Key words and phrases. Stable equivariant homotopy, generalized fixed point situations, fixed point transfer, Lie-group actions, metric spaces, categories.

1. INTRODUCTION

The purpose of this report is to show a formula to decompose equivariant stable maps as a sum of its restrictions to certain fixed point sets under the action.

More precisely, H. Ulrich [3] showed a decomposition formula for the equivariant fixed point index which expresses it as a sum of indices of restrictions to certain fixed point classes under subgroups of the compact Lie group acting. In this paper, we show a similar formula, but instead of for the equivariant index, it is proved for any equivariant stable map $\alpha : X \to Y$.

2. PRESENTATION OF THE MAIN RESULT

Throughout the paper, $G$ will represent a compact Lie group and all spaces will be $G$-metric spaces.

Definition 2.1. Let $X$ and $Y$ be metric $G$-spaces and let $M, N$ and $K$ be $G$-modules, that is real, finite dimensional vector spaces provided with a linear $G$-action. Consider equivariant maps of pairs

$$f : (N \oplus K, N \oplus K - 0) \times X \to (M \oplus K, M \oplus K - 0) \times Y$$

and let $K$ vary in a cofinal set of $G$-modules closed under direct sum, leaving $M$ and $N$ fixed.

If

$$f' : (N \oplus K', N \oplus K' - 0) \times X \to (M \oplus K', M \oplus K' - 0) \times Y$$

is another such map, we declare them as stably equivalent if there exist $G$-modules $L$ and $L'$ in the cofinal set such that

$$K \oplus L \cong K' \oplus L'$$

and the suspensions

$$f \circ 1_L : (N \oplus K \oplus L, N \oplus K \oplus L - 0) \times X \to (M \oplus K \oplus L, M \oplus K \oplus L - 0) \times Y$$

and

$$f' \circ 1_{L'} : (N \oplus K' \oplus L', N \oplus K' \oplus L' - 0) \times X \to (M \oplus K' \oplus L', M \oplus K' \oplus L' - 0) \times Y$$

are $G$-homotopic, up to the canonical homeomorphism. We denote the class of $f$ by $\{f\}$. 

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The main theorem of this report now reads as follows.

**Theorem 2.2.** Let $X$ be a $G$-trivial space and let $\alpha : X \to Y$ be $k$-stable $G$-map. Then

$$\{\alpha\} = \sum \{\alpha^{(n)}\} - \{\alpha^{(m)}\},$$

where $\{\alpha^{(n)}\} : X \to Y^{(n)} \subset Y$, $\{\alpha^{(m)}\} : X \to Y^{(m)} \subset Y$, both stably.

In the next section, we shall settle the elements to understand (see 4.1) and prove the main theorem, and in section 4 we sketch the proof of the result.

### 3. The RO(G)-Graded Categories

In this section we shall sketch the definition of the RO(G)-graded equivariant stable homotopy category and the RO(G)-graded equivariant fixed point category, as defined in [2]. This category is equivalent to the more usual equivariant stable homotopy category defined in terms of regular suspensions defined by smashing pointed $G$-spaces with $G$-spheres, that is, with one-point compactifications of $G$-modules.

In the previous section we already defined the equivariant stable equivalence class of a map $\{\alpha\}$ 2.1. We show here how it fits into a category.

First of all, we shall understand under an RO(G)-graded category, a category whose morphism sets are graded by the elements of the real representation ring RO(G) of the compact Lie group $G$ and the composite of a morphism of degree $\rho \in RO(G)$ and a morphism of degree $\sigma \in RO(G)$, if defined, has degree $\rho + \sigma \in RO(G)$.

**Definition 3.1.** Two classes $\{f\} : X \to Y$ of degree $[N] - [M]$ and $\{g\} : Y \to Z$ of degree $[L] - [K]$ are composed as follows.

Let $\{f\}$ be represented by

$$f : (M, M - 0) \times X \to (N, N - 0) \times Y$$

and $\{g\}$ be represented by

$$g : (K, K - 0) \times Y \to (L, L - 0) \times Z.$$

Then the composite $\{g\} \circ \{f\} : X \to Z$ is constructed as follows. Let $P$ and $Q$ be $G$-modules such that $P \oplus N$ and $Q \oplus K$ are isomorphic. Then $\{g\} \circ \{f\}$ is represented by

$$(1_Q \oplus g) \circ (1_P \oplus f) : (P \oplus M, P \oplus M - 0) \times X \to (Q \oplus L, Q \oplus L - 0) \times Z,$$

the composite taken up to the induced homeomorphism between $(P \oplus N, P \oplus N - 0) \times Y$ and $(Q \oplus K, Q \oplus K - 0) \times Y$. This morphism has degree $[Q \oplus L] - [P \oplus M] = ([N] - [M]) + ([L] - [K])$.

Then we have a category $G\text{-}\text{Stab}^*$, whose morphisms are stable $G$-homotopy classes of $G$-maps.

The other category relevant for this report is the RO(G)-graded equivariant fixed point category $G\text{-}\text{Fix}^*$ built up with fixed point situations.
First recall the concept of a $G$-Euclidean neighborhood retract over $X$, a $G$-ENRX, for $X$ a metric $G$-space. It is, namely, a continuous $G$-map $p: E \to X$ such that there exists a (real) $G$-module $L$, a $G$-invariant open set $U \subseteq L \times X$ and $G$-maps $i: E \to U$ and $r: U \to E$ commuting with the projections into $X$, that is, $\text{proj}_X \circ i = p$ and $p \circ r = \text{proj}_X$, and are such that $r \circ i = \text{id}_E$.

We shall be dealing with fixed point situations according to the following definition.

**Definition 3.2.** A fixed point situation over $X$ is a commutative diagram

$$
\begin{array}{ccc}
N \times E & \supset V & M \times E \\
\downarrow f & & \downarrow \text{proj}_E \\
X & & \text{proj}_E \\
\end{array}
$$

(3.3)

where $p: E \to X$ is a $G$-ENRX, $M$ and $N$ are $G$-modules, $V$ is an invariant open subset of $N \times E$ and $f$ is compactly fixed, that is, the coincidence set $\text{Fix}(f) = \{(y, e) \in V \mid f(y, e) = (0, e)\}$ lies properly over $X$, that is, the preimage of every compact set in $X$ is compact in $\text{Fix}(f)$.

In order to define the category $G\text{-Fix}^*$ we need a little more. Let $\rho = [M] - [N] \in RO(G)$ be given and let

$$
\begin{array}{ccc}
N \times E & \supset V & M \times E \\
\downarrow f & & \downarrow \text{proj}_E \\
X & & \text{proj}_E \\
\end{array}
$$

be a fixed point situation. On the other hand, take a (nonstable) map $\varphi: \text{Fix}(f) \to Y$ and consider the pair $(f, \varphi)$. Two such given pairs $(f_0, \varphi_0)$ and $(f_1, \varphi_1)$ are said to be homotopic if there exists a fixed point situation over $X \times I$.

$$
\begin{array}{ccc}
N \times E & \supset V & M \times E \\
\downarrow f & & \downarrow \text{proj}_E \\
X \times I & & \text{proj}_E \\
\end{array}
$$

and a map $\Phi: \text{Fix}(F) \to Y$, such that the pair $(F, \Phi)$, when restricted to each bottom and top of the cylinder $X \times I$, yields the two given pairs. Denote the homotopy class of $(f, \varphi)$ by $\{f, \varphi\}$ and call it class of degree $\rho$.

**Definition 3.4.** The category $G\text{-Fix}^*$ is defined as follows:

Its objects are $G$-ENRs. If $X$ and $Y$ are two objects, then a morphism of degree $\rho$ from $X$ to $Y$ is a class $\{f, \varphi\}: X \to Y$ of degree $\rho$, as defined above. The identity morphism is simply the class of degree 0 $\{\text{id}_X, \text{id}_Y\}$, but the composition operator is delicate to define and we refer the reader to [2].
The important fact is that both categories $G$-$\text{Stab}^*$ and $G$-$\text{Fir}^*$ are isomorphic (see [2] Theorem 3.2). In fact, we can give a functor

$$u : G$-$\text{Fir}^* \rightarrow G$-$\text{Stab}^*$$

which is the identity on objects and on morphisms it sends a class $\{f, \varphi\}$ to the composite

$$X \xrightarrow{\tau(f)} \text{Fix}(f) \xrightarrow{\varphi} Y,$$

where $\tau(f)$ denotes the equivariant fixed point transfer of $f$, which is an equivariant stable map of degree $\rho$, as defined in [1], and $\varphi$ can be considered as an equivariant stable map of degree 0. (To be more precise, since, in fact, $\tau(f) : X \rightarrow W$, for any neighborhood $W$ of $\text{Fix}(f)$, one has to extend, by the Tietze-Gleason Lemma, $\varphi$ to an equivariant map defined over $W$, and then compose.)

In particular, this proves the following result.

**Proposition 3.5.** For any stable map $\{\alpha\} \in G$-$\text{Stab}^{(M)} (X, Y)$ there exists a unique class $\{f, \varphi\}$ of degree $[M] - [N]$, where $f : V \rightarrow M \times E$ is a fixed point situation over $X$ and $\varphi : \text{Fix}(f) \rightarrow Y$ is an equivariant nonstable map, such that $\{\alpha\}$ factors as

$$\{\alpha\} = \varphi \circ \tau(f).$$

This result has as a consequence that several properties of the transfer can be shown for more general stable maps. This is what we shall apply in the next section.

**4. Proof of the main theorem**

We shall prove the main theorem 2.2 in what follows. First of all, let us consider some definitions.

**Definition 4.1.** Let $Y$ be any $G$-space, $H \subset G$ a closed subgroup and $(H)$ its conjugation class. We use the following notations.

$$Y^H = \{y \in Y \mid G_y \supset H\}, \quad Y^{(H)} = \{y \in Y \mid (G_y) \supset (H)\},$$

$$Y^H = \{y \in Y \mid G_y \supseteq H\}, \quad Y^\mathcal{L} = \{y \in Y \mid (G_y) \supseteq (H)\},$$

$$Y^H = \{y \in Y \mid G_y = H\}, \quad Y^\mathcal{L} = \{y \in Y \mid (G_y) = (H)\},$$

where $G_y$ is the isotropy group of the point $y$ and $(H) \subset (K)$ means $H$ is contained in some conjugate of $K$. Therefore $Y^H = Y^H - Y^\mathcal{L}$ and $Y^{(H)} = Y^{(H)} - Y^{(\mathcal{L})}$. For equivariant maps $f : X \rightarrow Y$, the maps $f^H, f^\mathcal{L}, f^{(H)}$ and $f^{(\mathcal{L})}$ are the corresponding restrictions.

Theorem 2.13 in [1] is the natural model for the important lemma to our proof. It reads as follows.

**Theorem 4.2.** Let $E \rightarrow X$ be a $G$-ENR with $X$ a $G$-space with trivial action and let $V \subset \mathbb{R}^n \times E$ be open and invariant. Let, moreover, $f : V \rightarrow \mathbb{R}^m \times E$ be compactly fixed. Then the equivariant fixed point index of $f$ decomposes as follows.

$$I(f) = \sum I(f^{(H)}) - I(f^{(\mathcal{L})}) \in h^m_\alpha(X),$$

the sum taken over the finitely many orbit types around $\text{Fix}(f)$, where $h^*_G$ is an $RO(G)$-graded $G$-cohomology theory.
Our lemma, which extends 4.2, is the following (cf. 4.2 in [2]).

**Lemma 4.4.** Let $E \rightarrow X$ be a $G$-ENRX with $X$ a $G$-space with trivial action and let $V \subset \mathbb{R}^n \times E$ be open and invariant. Let, moreover, $f : V \rightarrow \mathbb{R}^n \times E$ be compactly fixed. Then the equivariant transfer of $f$ decomposes as follows.

$$
\tau(f) = \sum (\tau(f^{(H)}) - \tau(f^{(\mu)})) : h^*(W) \rightarrow h^*_G(X),
$$

for $W$ an invariant neighborhood of $\text{Fix}(f)$ in $\mathbb{R}^n \times E$, the sum taken over the finitely many orbit types around $\text{Fix}(f)$, where $h^*_G$ is any $RO(G)$-graded $G$-cohomology theory. In particular, the decomposition formula holds for the transfer $\tau(f)$ seen as a stable map $X \rightarrow \text{Fix}(f)$, and the sum is taken in $G$-$\text{Stab}^{m-n}(X, \text{Fix}(f))$.

Now our proposition 3.5 can be applied to prove the desired result 2.2.

**Proof:** Decompose $\{\alpha\}$ as $\varphi \circ \tau(f)$. By the last lemma,

$$
\tau(f) = \sum (\tau(f^{(H)}) - \tau(f^{(\mu)})).
$$

Therefore,

$$
\varphi \circ \tau(f) = \sum (\varphi \circ \tau(f^{(H)}) - \varphi \circ \tau(f^{(\mu)}))
$$

hence

$$
\{\alpha\} = \sum \{\alpha^{(H)}\} - \{\alpha^{(\mu)}\},
$$

where $\{\alpha^{(H)}\} : X \rightarrow Y^{(H)} \subset Y$, $\{\alpha^{(\mu)}\} : X \rightarrow Y^{(\mu)} \subset Y$, since $\tau(f^{(H)}) : X \rightarrow \text{Fix}(f)$ factors through $\text{Fix}(f^{(H)}) \subset \text{Fix}(f)$ and $\tau(f^{(\mu)}) : X \rightarrow \text{Fix}(f)$ factors through $\text{Fix}(f^{(\mu)}) \subset \text{Fix}(f)$;

$$
\varphi \circ \tau(f^{(H)}) = \varphi^{(H)} \circ \tau(f^{(H)}),
$$

$$
\varphi \circ \tau(f^{(\mu)}) = \varphi^{(\mu)} \circ \tau(f^{(\mu)});
$$

$$
\{\alpha^{(H)}\} = \varphi^{(H)} \circ (f^{(H)} : X \rightarrow Y^{(H)} \subset Y),
$$

$$
\{\alpha^{(\mu)}\} = \varphi^{(\mu)} \circ (f^{(\mu)} : X \rightarrow Y^{(\mu)} \subset Y).
$$

Let $X = \ast$, $Y = S^L$, where $L$ is a $G$-module and let $\{\alpha\}$ be as before. An interesting corollary of the main theorem is the next.

**Corollary 4.6.** $\{\alpha\} = \sum (\{\alpha^{(H)}\} - \{\alpha^{(\mu)}\}) \in G$-$\text{Stab}^k(\ast, S^L) = \pi^G_k(S^L)$.

This shows a decomposition formula for elements in the $G$-equivariant stable homotopy groups of $G$-spheres.
REFERENCES


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