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A SUM FORMULA FOR STABLE EQUIVARIANT MAPS

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1. Introduction

The purpose of this report is to show a formula to decompose equivariant stable maps as a sum of its restrictions to certain fixed point sets under the action.

More precisely, H. Ulrich [3] showed a decomposition formula for the equivariant fixed point index which expresses it as a sum of indices of restrictions to certain fixed point classes under subgroups of the compact Lie group acting. In this paper, we show a similar formula, but instead of for the equivariant index, it is proved for any equivariant stable map $\alpha : X \to Y$.

2. Presentation of the main result

Throughout the paper, G will represent a compact Lie group and all spaces will be G-metric spaces.

Definition 2.1. Let X and Y be metric G-spaces and let M, N and K be G-*modules*, that is real, finite dimensional vector spaces provided with a linear G-action. Consider equivariant maps of pairs

$$f : (N \oplus K, N \oplus K - 0) \times X \to (M \oplus K, M \oplus K - 0) \times Y$$

and let K vary in a cofinal set of G-modules closed under direct sum, leaving M and N fixed. If

$$f' : (N \oplus K', N \oplus K' - 0) \times X \to (M \oplus K', M \oplus K' - 0) \times Y$$

is another such map, we declare them as *stably equivalent* if there exist G-modules L and L' in the cofinal set such that

$$K \oplus L \cong K' \oplus L'$$

and the *suspensions*

$$f \oplus 1_L : (N \oplus K \oplus L, N \oplus K \oplus L - 0) \times X \to (M \oplus K \oplus L, M \oplus K \oplus L - 0) \times Y$$

and

$$f' \oplus 1_{L'} : (N \oplus K' \oplus L', N \oplus K' \oplus L' - 0) \times X \to (M \oplus K' \oplus L', M \oplus K' \oplus L' - 0) \times Y$$

are G-homotopic, up to the canonical homeomorphism. We denote the class of f by $\{f\}$.

The main theorem of this report now reads as follows.

THEOREM 2.2. *Let X be a G-trivial space and let $\alpha : X \to Y$ be k-stable G-map. Then*

$$\{\alpha\} = \sum\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\},$$

where $\{\alpha^{(H)}\} : X \to Y^{(H)} \subset Y$, $\{\alpha^{(\underline{H})}\} : X \to Y^{(\underline{H})} \subset Y$, both stably.

In the next section, we shall settle the elements to understand (see 4.1) and prove the main theorem, and in section 4 we sketch the proof of the result.

3. THE $RO(G)$-GRADED CATEGORIES

In this section we shall sketch the definition of the $RO(G)$-graded equivariant stable homotopy category and the $RO(G)$-graded equivariant fixed point category, as defined in [2]. This category is equivalent to the more usual equivariant stable homotopy category defined in terms of regular suspensions defined by smashing pointed G-spaces with G-spheres, that is, with one-point compactifications of G-modules.

In the previous section we already defined the equivariant stable equivalence class of a map $\{\alpha\}$ 2.1. We show here how it fits into a category.

First of all, we shall understand under an *$RO(G)$-graded category*, a category whose morphism sets are graded by the elements of the real representation ring $RO(G)$ of the compact Lie group G and the composite of a morphism of degree $\rho \in RO(G)$ and a morphism of degree $\sigma \in RO(G)$, if defined, has degree $\rho + \sigma \in RO(G)$.

Definition 3.1. Two classes $\{f\} : X \to Y$ of degree $[N] - [M]$ and $\{g\} : Y \to Z$ of degree $[L] - [K]$ are composed as follows.

Let $\{f\}$ be represented by

$$f : (M, M - 0) \times X \to (N, N - 0) \times Y$$

and $\{g\}$ be represented by

$$g : (K, K - 0) \times Y \to (L, L - 0) \times Z.$$

Then the composite $\{g\} \circ \{f\} : X \to Z$ is constructed as follows. Let P and Q be G-modules such that $P \oplus N$ and $Q \oplus K$ are isomorphic. Then $\{g\} \circ \{f\}$ is represented by

$$(1_Q \oplus g) \circ (1_P \oplus f) : (P \oplus M, P \oplus M - 0) \times X \to (Q \oplus L, Q \oplus L - 0) \times Z,$$

the composite taken up to the induced homeomorphism between $(P \oplus N, P \oplus N - 0) \times Y$ and $(Q \oplus K, Q \oplus K - 0) \times Y$. This morphism has degree $[Q \oplus L] - [P \oplus M] = ([N] - [M]) + ([L] - [K])$. Then we have a category G-$\mathfrak{Stab}^*$, whose morphisms are stable G-homotopy classes of G-maps.

The other category relevant for this report is the $RO(G)$-graded equivariant fixed point category G-$\mathfrak{Fix}^*$ built up with fixed point situations.

First recall the concept of a *G-Euclidean neighborhood retract over X*, a G-ENR$_X$, for X a metric G-space. It is, namely, a continuous G-map $p : E \to X$ such that there exists a (real) G-module L, a G-invariant open set $U \subset L \times X$ and G-maps $i : E \to U$ and $r : U \to E$ commuting with the projections into X, that is, $\mathrm{proj}_X \circ i = p$ and $p \circ r = \mathrm{proj}_X$, and are such that $r \circ i = \mathrm{id}_E$.

We shall be dealing with fixed point situations according to the following definition.

Definition 3.2. A *fixed point situation over X* is a commutative diagram

$$\begin{array}{ccc} N \times E \supset V & \xrightarrow{\quad f \quad} & M \times E \\ & {\scriptstyle p\circ\mathrm{proj}_E} \searrow \quad \swarrow {\scriptstyle p\circ\mathrm{proj}_E} & \\ & X & \end{array} \tag{3.3}$$

where $p : E \to X$ is a G-ENR$_X$, M and N are G-modules, V is an invariant open subset of $N \times E$ and f is *compactly fixed*, that is, the *coincidence set* $\mathrm{Fix}(f) = \{(y, e) \in V \mid f(y, e) = (0, e)\}$ lies properly over X, that is, the preimage of every compact set in X is compact in $\mathrm{Fix}(f)$.

In order to define the category G-$\mathfrak{Fix}^*$ we need a little more. Let $\rho = [M] - [N] \in RO(G)$ be given and let

$$\begin{array}{ccc} N \times E \supset V & \xrightarrow{\quad f \quad} & M \times E \\ & {\scriptstyle p\circ\mathrm{proj}_E} \searrow \quad \swarrow {\scriptstyle p\circ\mathrm{proj}_E} & \\ & X & \end{array}$$

be a fixed point situation. On the other hand, take a (nonstable) map $\varphi : \mathrm{Fix}(f) \to Y$ and consider the pair (f, φ). Two such given pairs (f_0, φ_0) and (f_1, φ_1) are said to be *homotopic* if there exists a fixed point situation over $X \times I$.

$$\begin{array}{ccc} N \times E \supset V & \xrightarrow{\quad F \quad} & M \times E \\ & {\scriptstyle p\circ\mathrm{proj}_E} \searrow \quad \swarrow {\scriptstyle p\circ\mathrm{proj}_E} & \\ & X \times I & \end{array}$$

and a map $\Phi : \mathrm{Fix}(F) \to Y$, such that the pair (F, Φ), when restricted to each bottom and top of the cylinder $X \times I$, yields the two given pairs. Denote the homotopy class of (f, φ) by $\{f, \varphi\}$ and call it *class of degree ρ*.

Definition 3.4. The category G-$\mathfrak{Fix}^*$ is defined as follows:

Its objects are G-ENRs. If X and Y are two objects, then a morphism of degree ρ from X to Y is a class

$$\{f, \varphi\} : X \to Y$$

of degree ρ, as defined above. The identity morphism is simply the class of degree 0 $\{\mathrm{id}_X, \mathrm{id}_X\}$, but the composition operator is delicate to define and we refer the reader to [2].

The important fact is that both categories G-$\mathfrak{Stab}^*$ and G-$\mathfrak{Fix}^*$ are isomorphic (see [2] Theorem 3.2). In fact, we can give a functor

$$u : G\text{-}\mathfrak{Fix}^* \to G\text{-}\mathfrak{Stab}^*$$

which is the identity on objects and on morphisms it sends a class $\{f, \varphi\}$ to the composite

$$X \xrightarrow{\tau(f)} \mathrm{Fix}\,(\mathrm{f}) \xrightarrow{\varphi} \mathrm{Y},$$

where $\tau(f)$ denotes the *equivariant fixed point transfer* of f, which is an equivariant stable map of degree ρ, as defined in [1], and φ can be considered as an equivariant stable map of degree 0. (To be more precise, since, in fact, $\tau(f) : X \to W$, for any neighborhood W of $\mathrm{Fix}\,(\mathrm{f})$, one has to extend, by the Tietze-Gleason Lemma, φ to an equivariant map defined over W, and then compose.)

In particular, this proves the following result.

PROPOSITION 3.5. *For any stable map $\{\alpha\} \in G\text{-}\mathfrak{Stab}^{[M]-[N]}(X, Y)$ there exists a unique class $\{f, \varphi\}$ of degree $[M] - [N]$, where $f : V \to M \times E$ is a fixed point situation over X and $\varphi : \mathrm{Fix}\,(\mathrm{f}) \to \mathrm{Y}$ is an equivariant nonstable map, such that $\{\alpha\}$ factors as*

$$\{\alpha\} = \varphi \circ \tau(f)\,.$$

This result has as a consequence that several properties of the transfer can be shown for more general stable maps. This is what we shall apply in the next section.

4. Proof of the main theorem

We shall prove the main theorem 2.2 in what follows. First of all, let us consider some definitions.

Definition 4.1. Let Y be any G-space, $H \subset G$ a closed subgroup and (H) its conjugation class. We use the following notations.

$$\begin{aligned} Y^H &= \{y \in Y \mid G_y \supset H\}, & Y^{(H)} &= \{y \in Y \mid (G_y) \supset (H)\}, \\ Y^{\underline{H}} &= \{y \in Y \mid G_y \supsetneq H\}, & Y^{(\underline{H})} &= \{y \in Y \mid (G_y) \supsetneq (H)\}, \\ Y_H &= \{y \in Y \mid G_y = H\}, & Y_{(H)} &= \{y \in Y \mid (G_y) = (H)\}, \end{aligned}$$

where G_y is the isotropy group of the point y and $(H) \subset (K)$ means H is contained in some conjugate of K. Therefore $Y_H = Y^H - Y^{\underline{H}}$ and $Y_{(H)} = Y^{(H)} - Y^{(\underline{H})}$. For equivariant maps $f : X \to Y$, the maps f^H, $f^{\underline{H}}$, $f^{(H)}$ and $f^{(\underline{H})}$ are the corresponding restrictions.

Theorem 2.13 in [1] is the natural model for the important lemma to our proof. It reads as follows.

THEOREM 4.2. *Let $E \to X$ be a G-ENR_X with X a G-space with trivial action and let $V \subset \mathbb{R}^n \times E$ be open and invariant. Let, moreover, $f : V \to \mathbb{R}^m \times E$ be compactly fixed. Then the equivariant fixed point index of f decomposes as follows.*

$$I(f) = \sum (I(f^{(H)}) - I(f^{(\underline{H})})) \in h_G^{m-n}(X), \tag{4.3}$$

the sum taken over the finitely many orbit types around $\mathrm{Fix}(f)$, where h_G^* is an $RO(G)$-graded G-cohomology theory.

Our lemma, which extends 4.2, is the following (cf. 4.2 in [2]).

LEMMA 4.4. *Let* $E \to X$ *be a* G-ENR$_X$ *with* X *a* G*-space with trivial action and let* $V \subset \mathbb{R}^n \times E$ *be open and invariant. Let, moreover,* $f : V \to \mathbb{R}^m \times E$ *be compactly fixed. Then the equivariant transfer of* f *decomposes as follows.*

$$\tau(f) = \sum(\tau(f^{(H)}) - \tau(f^{(\underline{H})})) : h^*(W) \to h_G^{*+m-n}(X), \tag{4.5}$$

for W *an invariant neighborhood of* Fix (f) *in* $\mathbb{R}^n \times E$*, the sum taken over the finitely many orbit types around* Fix (f)*, where* h_G^* *is any* $RO(G)$*-graded* G*-cohomology theory. In particular, the decomposition formula holds for the transfer* $\tau(f)$ *seen as a stable map* $X \to$ Fix (f)*, and the sum is taken in* G-$\mathfrak{Stab}^{m-n}(X, \text{Fix}(f))$.

Now our proposition 3.5 can be applied to prove the desired result 2.2.

Proof: Decompose $\{\alpha\}$ as $\varphi \circ \tau(f)$. By the last lemma,

$$\tau(f) = \sum(\tau(f^{(H)}) - \tau(f^{(\underline{H})})).$$

Therefore,

$$\varphi \circ \tau(f) = \sum(\varphi \circ \tau(f^{(H)}) - \varphi \circ \tau(f^{(\underline{H})}))$$

hence

$$\{\alpha\} = \sum\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\},$$

where $\{\alpha^{(H)}\} : X \to Y^{(H)} \subset Y$, $\{\alpha^{(\underline{H})}\} : X \to Y^{(\underline{H})} \subset Y$, since $\tau(f^{(H)}) : X \to \text{Fix}(f)$ factors through $\text{Fix}(f^{(H)}) \subset \text{Fix}(f)$ and $\tau(f^{(\underline{H})}) : X \to \text{Fix}(f)$ factors through $\text{Fix}(f^{(\underline{H})}) \subset \text{Fix}(f)$;

$$\begin{aligned} \varphi \circ \tau(f^{(H)}) &= \varphi^{(H)} \circ \tau(f^{(H)}), \\ \varphi \circ \tau(f^{(\underline{H})}) &= \varphi^{(\underline{H})} \circ \tau(f^{(\underline{H})}); \end{aligned}$$

$$\begin{aligned} \{\alpha^{(H)}\} &= \varphi^{(H)} \circ \tau(f^{(H)}) : X \to Y^{(H)} \subset Y, \\ \{\alpha^{(\underline{H})}) &= \varphi^{(\underline{H})} \circ \tau(f^{(\underline{H})}) : X \to Y^{(\underline{H})} \subset Y. \end{aligned}$$

Let $X = *$, $Y = \mathbb{S}^L$, where L is a G-module and let $\{\alpha\}$ be as before. An interesting corollary of the main theorem is the next.

COROLLARY 4.6. $\{\alpha\} = \sum(\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\}) \in G\text{-}\mathfrak{Stab}^k(*, \mathbb{S}^L) = \pi_{-k}^G(\mathbb{S}^L)$.

This shows a decomposition formula for elements in the G-equivariant stable homotopy groups of G-spheres.

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