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Degree and fixed point index. An account *

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Abstract

In this account, a development of the concepts of Brouwer degree and Lefschetz-Hopf fixed point index is discussed in the light of work done mainly by A. Dold, H. Ulrich and the author. Generalizations to certain coincidence situations including the equivariant cases are presented, as well as how to deal with the infinite dimensional cases. In two appendices a proof of the Lefschetz-Hopf theorem for these indices is referred to, as well as a generalization of Dold's fixed point transfer is sketched.

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1 Introduction

1.0 Consider a system of equations

$$\begin{array}{rcl} g_1(x_1,\dots,x_k) & = & a_1 \\ \vdots & \vdots & \vdots \\ g_l(x_1,\dots,x_k) & = & a_l \end{array} \tag{1.1}$$

where the unknown are restricted by some conditions. These restrictions can be more precisely described by saying that the point $(x_1,\dots,x_k)$ has to belong to a certain subset V of the euclidean space $\mathbb{R}^k$. Thus, we may see the system as an equation of the form

$$g(x) = a\,, \tag{1.2}$$

*Invited article.

where $g\colon V \longrightarrow \mathbb{R}^l$, and $V \subset \mathbb{R}^k$.

In the case $k = l$, V open and bounded in $\mathbb{R}^k$ and g continuous, such that it can be extended to the boundary of V and has no solution in this boundary, Brouwer [2] defined in 1911 the concept of *degree*, which to such g assigns an integer, $\deg(g)$, such that if it is nonzero, then the equation has a solution.

The problem can be modified as follows. We shall consider two cases.

1. $k \leq l$. In this case, the system (1.1) can be rewritten as

$$\begin{aligned} f_1(x_1,\ldots,x_k) &= g_1(x_1,\ldots,x_k) - a_1 + x_1 = x_1 \\ &\vdots \\ f_k(x_1,\ldots,x_k) &= g_k(x_1,\ldots,x_k) - a_k + x_k = x_k \\ f_{k+1}(x_1,\ldots,x_k) &= g_{k+1}(x_1,\ldots,x_k) - a_{k+1} = 0 \\ &\vdots \\ f_l(x_1,\ldots,x_k) &= g_l(x_1,\ldots,x_k) - a_l = 0 \end{aligned}$$

or, written in vector form, we have a map

$$f\colon V \longrightarrow \mathbb{R}^l = \mathbb{R}^k \times \mathbb{R}^{l-k}\,, \quad V \subset \mathbb{R}^k\,,$$

such that $g(x) = a$ if and only if $f(x) = (x,0)$; hence, we look for solutions $x \in V$ for the equation

$$f(x) = (x,0) \in \mathbb{R}^k \times \mathbb{R}^{l-k}\,.$$

This is a *generalized fixed point problem.* For the classical problem, $k = l$, Lefschetz [17] defined in 1926 an invariant, $L(f)$, with integral values, called the *Lefschetz number*, defined for V a polyhedron and f such that $f(V) \subset V$. This number, which is easy to compute, has the property that $L(f) \neq 0$ implies the existence of a fixed point of f, i.e. a solution for the equation $f(x) = x$.

On the other hand, Hopf [12] and [13], a couple of years later defined another integral invariant for the case $k = l$, V open and bounded and such that f can be extended to the boundary of V without fixed points, called the *fixed point index*, $I(f)$, which fulfills the same theorem as the Lefschetz number, namely, $I(f) \neq 0$ implies that f has fixed points. This index deals with more general situations, but it is also more difficult to compute. Their relationship is given by the so-called Lefschetz-Hopf theorem which states that in the case that both $L(f)$ and $I(f)$ are defined, then $I(f) = L(f)$.

The other case of our more general set up is the following:
2. $k \geq l$. In this case, the system (1.1) can be written as

$$\begin{array}{rcccl} f_1(x_1,\dots,x_k) & = & g_1(x_1,\dots,x_k)-a_1+x_1 & = & x_1 \\ \vdots & & & & \vdots \\ f_l(x_1,\dots,x_k) & = & g_l(x_1,\dots,x_k)-a_l+x_l & = & x_l \end{array}$$

or, put in vector form, we have a map

$$f\colon V \longrightarrow \mathbb{R}^l\,, \quad V \subset \mathbb{R}^k = \mathbb{R}^l \times \mathbb{R}^{k-l}\,,$$

such that, if $x=(x',x'')\in V$, $g(x)=a$ if and only if $f(x)=x'$; hence, we look for solutions $x=(x',x'')\in V$ for the equation

$$f(x',x'') = x'\,.$$

This is another *generalized fixed point problem.*

Both cases 1. and 2. can be put together into the following problem. Take

$$f\colon V \longrightarrow \mathbb{R}^k \times \mathbb{R}^m\,, \quad V \subset \mathbb{R}^k \times \mathbb{R}^n \tag{1.3}$$

and we ask for the existence of *generalized fixed points*, namely, points $(x,x')\in V$ such that $f(x,x')=(x,0)$.

We shall describe in the next sections, for cases with increasing generality, fixed point indices which decide the existence of solutions for this problem.

The first case we shall consider is when f not only is a map as in (1.3), but a family f_b parametrized by the points b in a metric space B, in whose case we substitute the space $\mathbb{R}^k$ also by a family of more general spaces E_b, which include finite polyhedra and smooth manifolds, which in their time were considered by Lefschetz and Hopf. The problem is now the following. Let

$$f\colon V \longrightarrow E \times M\,, \quad V \subset E \times N \ \text{ open}\,, \tag{1.4}$$

where E is a *euclidean neighborhood retract over* B, an ENR_B for short, namely a continuous family given by $p\colon E \longrightarrow B$, of retracts $E_b = p^{-1}(b)$ of open sets in $\mathbb{R}^k$ (see 2.0), M and N are euclidean spaces ($M=\mathbb{R}^m, N=\mathbb{R}^n$), f preserves parameters (i.e. $f(v)\in E_b\times M$ if $v\in E_b\times N$) and is *properly fixed*, namely the solutions $\mathrm{Fix}(f) = \{(e,y)\in V \mid f(e,y)=f(e,0)\}$ lie properly over B; in particular, the fixed point set of the restriction f_b of f to each fiber over b is compact (see 2.0). In this case there is an invariant $I(f)$, which lives in the (generalized) cohomology –or homology– of B in dimension $m-n$ and has, among others, the property that $I(f) \neq 0$ implies $\mathrm{Fix}(f) \neq \emptyset$.

The case $M = N = \mathbb{R}^0 = \{0\} = 0$ was studied by Dold in [7], where he generalized the work of Lefschetz and Hopf as well as previous work of himself, [4] and [6]. In these last, he studied the case $B = \{*\}$ (see also [5]). The general case was studied by the author in [20]. This case will be discussed in section 2.

Very frequently the problem presents symmetries, that is, all the spaces E, B, M, N admit *group actions* for a group G, and p and f are compatible with those actions, i.e. they are *equivariant*. The solution of the problem in this case is sharper, and if $M = N = 0$ it has been given basically by Dold in [9] and by Ulrich in [30, 31], although tom Dieck has said something about it too [3]. Its generalization for real G-modules of finite dimension M and N was given by Ulrich and the author in [26]. This case we shall discuss in section 3.

There are generalizations of the problem in another direction, namely, in the case that E has infinite dimension, of great importance in several questions in nonlinear analysis. The development of this problem is as follows. Leray and Schauder [18] 1934 defined an index for the case $B = \{*\}$, $M = N = 0$ and E a separable Banach space, requiring f to be such that the closure of the image of V under f, $\overline{f(V)} \subset E$ is compact. Granas [11] generalized this to the case in which E is an *absolute neighborhood retract* (an ANR) and Ulrich [29] did it in the parametrized case ($B \neq \{*\}$). The general case (ANR_Bs and M and N finite dimensional G-modules) will be shortly discussed below in 3.4.

2 Fixed point index

2.0 Let B be a metric space. We shall be concerned with the following commutative diagrams, called *fixed point situations* over B

(2.1)
$$\begin{array}{ccc} E \times N \supset V & \xrightarrow{\ f\ } & E \times M \\ & {\scriptstyle p\circ\mathrm{proj}_1} \searrow \quad \swarrow {\scriptstyle p\circ\mathrm{proj}_1} & \\ & B\,, & \end{array}$$

where $p\colon E \longrightarrow B$ is an ENR_B, i.e. a *vertical* retract (meaning that the retraction commutes with the projections p and proj_B) of an open set in $B \times K$, K a euclidean space ($K = \mathbb{R}^k$), M and N are euclidean spaces ($M = \mathbb{R}^m, N = \mathbb{R}^n$) too and f is properly fixed over B, i.e. the restriction of the projection into B, $p \circ \mathrm{proj}_1$, to the fixed point set, $\mathrm{Fix}(f) = \{(e, y) \in V \mid f(e, y) = (e, 0)\}$, is proper, in other words, for each compact set $C \subset B$, the set $(p^{-1}(C) \times N) \cap \mathrm{Fix}(f)$ is compact.

We first study the case $E = B \times K$. The properness of $\mathrm{Fix}(f) \longrightarrow B$, that is, the continuous compactness of $F = \mathrm{Fix}(f)$ implies the validity of a parametrized Heine-Borel theorem; namely, there exists a function $\rho\colon B \longrightarrow \mathbb{R}^+ = (0, +\infty)$, such that $F \subset \mathbb{B}_\rho = \{(b, z, y) \in B \times K \times N \mid \|(z, y)\| \le \rho(b)\}$ (the set $\mathbb{B}_\rho$ can be described as a continuous family of balls in $K \times N = \mathbb{R}^{k+n}$ of radius varying according to ρ).

Consider the following sequence of maps of pairs

$$
\text{(2.2)}\qquad
\begin{array}{ccc}
 & (V, V - F) \xrightarrow{\;i-f\;} & B \times (K \times M, K \times M - 0) \\
 & \;\;{\scriptstyle (1)}\uparrow & \Vert \\
(E \times N, E \times N - \mathbb{B}_\rho) \hookleftarrow & (E \times N, E \times N - F) & \Vert \\
{\scriptstyle (2)}\uparrow & & \Vert \\
B \times (K \times N, K \times N - 0) & & \Vert \\
\Vert & & \Vert \\
B \times (\mathbb{R}^{k+n}, \mathbb{R}^{k+n} - 0) & \dashrightarrow & B \times (\mathbb{R}^{k+m}, \mathbb{R}^{k+m} - 0)\,,
\end{array}
$$

where $(i - f)(b, z, y) = (b, (z, 0) - f_2(b, z, y))$, if $f(b, z, y) = (b, f_2(b, z, y))$. The vertical inclusions are, respectively, (1) an excision and (2) a homotopy equivalence (of the second spaces of the pairs), and all maps are over B (i.e., they preserve the fibers). Thus they induce a *stable map*

$$
\text{(2.3)}\qquad I_f\colon B \longrightarrow B
$$

of degree $(k+m) - (k+n) = m - n$ (see [10], [24] or Appendix A (A.3)). Equivalently, (2.2) induces a homomorphism

$$
\text{(2.4)}\qquad I_f\colon h^*(B) \longrightarrow h^{*+m-n}(B)\,.
$$

for any cohomology theory h^*. More precisely, applying h^* to (2.2) we get a homomorphism

$$
h^{i+k+m}(B \times (\mathbb{R}^{k+m}, \mathbb{R}^{k+m} - 0)) \longrightarrow h^{i+k+m}(B \times (\mathbb{R}^{k+n}, \mathbb{R}^{k+n} - 0)), \quad i \in \mathbb{Z}\,,
$$

which, after desuspending $k + m$ times on the left side and $k + n$ times on the right side, gives

$$
h^i(B) \longrightarrow h^{i+m-n}(B), \quad i \in \mathbb{Z}\,,
$$

and thus (2.4). This homomorphism is called the *index homomorphism* of f. Important examples of h^* are ordinary cohomology, K-theory, or stable cohomotopy. All these examples are multiplicative theories having an element $1 \in h^0(B)$; hence, for these theories, we may define the *fixed point index* of f as

$$I(f) = I_f(1) \in h^{m-n}(B)\,. \tag{2.5}$$

Since the index map factors through the pair $(E \times N, E \times N - F)$, it vanishes when $F = \emptyset$, therefore, it has the fundamental property

$$I(f) \neq 0 \Longrightarrow \mathrm{Fix}(f) \neq \emptyset\,. \tag{2.6}$$

Before passing to other important properties of the index, let us see some special cases.

Let $B = \{*\}$ and $m = n(= 0)$; the three cohomology theories mentioned above are such that $h^0(*) = \mathbb{Z}$. In this case, the index $I(f)$ is an integer, which is the same in all cases; this is the *classical fixed point index*, or *Lefschetz-Hopf index*, [4].

If $B = \{*\}$ and $n > m = 0$, then, taking h^* as stable cohomotopy, the index $I(f)$ becomes an element of the *n-stem*, i.e. of the group Π_n^{st} of stable homotopy classes of maps $\mathbb{S}^{k+n} \longrightarrow \mathbb{S}^k$ of spheres. In fact, in [6] and [20] it is proved that every element in Π_n^{st} is the index of some f.

The fixed point index has, among others, the following properties.

Homotopy 2.7. *Let $f\colon V \longrightarrow E \times M$, $V \subset E \times N$ be properly fixed over $B \times I$ ($I = [0,1]$). Then its restrictions $f_0\colon V_0 \longrightarrow E_0 \times M$ and $f_1\colon V_1 \longrightarrow E_1 \times M$ to bottom $B \times \{0\} \approx B$ and top $B \times \{1\} \approx B$ of the cylinder $B \times I$ are properly fixed and $I(f_0) = I(f_1) \in h^{m-n}(B)$.*

Additivity 2.8. *Let $f\colon V \longrightarrow E \times M$, $V \subset E \times N$ be properly fixed over B. Let $V = V_1 \cup V_2$ with V_1 and V_2 open. If $f_1 = f|V_1$, $f_2 = f|V_2$ and $f_{12} = f|V_1 \cap V_2$ are such that two of them are properly fixed, then so is also the third and $I(f) = I(f_1) + I(f_2) - I(f_{12}) \in h^{m-n}(B)$.*

Excision 2.9. *Let $f\colon V \longrightarrow E \times M$, $V \subset E \times N$ be properly fixed over B. If $V' \subset V$ is open and such that* $\mathrm{Fix}(f) \subset V'$*, then $f' = f|V'$ is properly fixed and $I(f') = I(f) \in h^{m-n}(B)$.*

The next property allows us to define the index for general ENR_Bs. It is this property which constitutes the main difference between index and degree and shows the convenience to work with the index rather than with the degree, which, in general, can not be defined for arbitrary euclidean neighborhood retracts.

Commutativity 2.10. *Let $E = B \times L \longrightarrow B$ and $E' = B \times L' \longrightarrow B$ with L and L' euclidean spaces, and let $U \subset E$, $U' \subset E' \times N$ be open. If $\varphi\colon U' \longrightarrow E \times M$ and $\psi\colon U \longrightarrow E'$ are maps over B such that the composite*

$$(\psi \times 1_M)\varphi\colon \varphi^{-1}(U \times M) \longrightarrow E' \times M\,, \quad \varphi^{-1}(U \times M) \subset U' \subset E' \times N$$

is properly fixed, then also the composite

$$(\psi \times 1_N)^{-1}(U') \xrightarrow{\varphi(\psi\times 1_N)} E \times M\,, \quad (\psi \times 1_N)^{-1}(U') \subset U \times N \subset E \times N$$

is properly fixed and $I((\psi \times 1_M)\varphi) = I(\varphi(\psi \times 1_N)) \in h^{m-n}(B)$.

We show now how the commutativity allows us to generalize the index:

Proposition and Definition 2.11. *If $p\colon E \longrightarrow B$ is an ENR_B, M and N are euclidean spaces and $V \subset E \times N$ is open, then every map over B, $f\colon V \longrightarrow E \times M$ admits a decomposition*

$$f\colon V \xrightarrow{\alpha\times 1_N} U \xrightarrow{\beta} E \times M\,,$$

where U is open in $B \times K \times N$ for some euclidean space K, and $\alpha\colon E \longrightarrow B \times K$. If f is properly fixed, then

$$g = (\alpha \times 1_M)\beta\colon U \longrightarrow B \times K \times M$$

is also properly fixed. Hence, $I(g) \in h^{m-n}(B)$ is defined and depends only on f and not on the factorization $f = \beta(\alpha \times 1_N)$. Thus we define the *fixed point index* of f as $I(f) = I(g)$.

Proof: (sketch) Let

$$E \underset{i}{\overset{r}{\rightleftarrows}} W \subset B \times K$$

be a representation of E as an ENR_B and define $U = (r \times 1_N)^{-1}V \subset W \times N \subset B \times K \times N$. So let $\alpha = i$ (then $\alpha \times 1_N\colon V \longrightarrow U$, since

$(r\times 1_N)(i\times 1_N) = \mathrm{id}\colon V \longrightarrow V)$ and $\beta = f(r\times 1_N)\colon U \longrightarrow E\times M$. Then $\beta(\alpha \times 1_N) = f$. The rest of the proof is a straightforward application of the commutativity property 2.10.

Properties 2.7 to 2.10 remain true for the general index. □

Comment 2.12. There is a Lefschetz-Hopf formula relating the index with a trace (Lefschetz number) in the case $m = n\ (= 0)$; see [8] or [10]. For the case $m > n\ (= 0)$, the formula holds trivially; see [20]. The case $m < n$ has also a formula which follows from a more general one; see [24] and appendix A.

Examples 2.13.

(a) [7, 5.3] Let $B = \mathbb{S}^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ and consider the map $f\colon B \times \mathbb{S}^1 \longrightarrow B \times \mathbb{S}^1$, $f(b, z) = (b, b \cdot z)$. This is a properly fixed map over B (for the projection $B \times \mathbb{S}^1 \longrightarrow B$ and $M = N = 0$). If one takes stable cohomotopy as the cohomology theory, then $I(f)$ is the nontrivial element of $\pi^0_{\mathrm{st}}(B) = \Pi_1^{\mathrm{st}} = \mathbb{Z}/2$ which also is the Lefschetz trace of $f^*\colon \pi^*_{\mathrm{st}}(B \times \mathbb{S}^1) \longrightarrow \pi^*_{\mathrm{st}}(B \times \mathbb{S}^1)$, seen as a homomorphism of $\pi^*_{\mathrm{st}}(B)$-modules.

(b) [20, 4.27] Let $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. If $k \in \mathbb{Z}$, then the map

$$\mathbb{S}^2 \supset \mathbb{C} \xrightarrow{f} \mathbb{S}^2 \times \mathbb{C}\ , \quad f(z) = (z, z^k)$$

is properly fixed over $B = \mathbb{S}^2$ ($E = B$, $p = \mathrm{id}$, $M = \mathbb{C} = \mathbb{R}^2$), and $I(f) = k \in \pi^2_{\mathrm{st}}(B) = \Pi_0^{\mathrm{st}} = \mathbb{Z}$.

3 Generalizations of the index

3.0. Very often the situations one studies present some kind of symmetries; if these are given by the action of a compact Lie group G, there are cohomology theories which are fine enough to detect the symmetries. More precisely we shall be concerned in first place with the equivariant index, which will be defined for situations like (2.1), but now assuming that G acts on all spaces involved and that every map in question commutes with the group action. To be precise, $p\colon E \longrightarrow B$ will be a G-ENR$_B$, i.e., G acts on both E and B, p is G-equivariant and E is a vertical equivariant retract of an open (invariant) set $B \times K$, where now K is a *G-module*. In fact, K, M and N are now all G-modules, that is, euclidean spaces with a linear action of G. Observe that in this case the fixed point set F of f is G-invariant and hence it is possible to choose $\rho\colon B \longrightarrow \mathbb{R}^+$ also G-invariant, i.e. $\rho(\gamma b) = \rho(b)$ for all $\gamma \in G$, $b \in B$. Therefore, $\mathbb{B}_\rho$ becomes G-invariant too. One has thus that the sequence of maps (2.2) consists of equivariant maps (over B) and thus produces an *equivariant stable map*

$$I_f\colon B \longrightarrow B \tag{3.1}$$

of degree $[K \oplus M] - [K \oplus N] = [M] - [N] \in \mathrm{RO}(G)$ (A.3) where $\mathrm{RO}(G)$ denotes the *real representation ring* (or ring of G-modules) of the group G, (see e.g. Appendix A).

As before, this sequence induces, equivalently to (2.1), an *equivariant index homomorphism* of f as

$$I_f\colon h_G^*(B) \longrightarrow h_G^{*+[M]-[N]}(B) \tag{3.2}$$

for any $\mathrm{RO}(G)$-graded G-equivariant cohomology theory h_G^* (see [16] or [21]). More precisely, applying h_G^* to the, now equivariant, sequence (2.2), we get, for any $\rho \in \mathrm{RO}(G)$, a homomorphism

$$h^{\rho+[K]+[M]}(B \times (K \oplus M, K \oplus M - 0)) \longrightarrow h^{\rho+[K]+[M]}(B \times (K \oplus N, K \oplus N - 0)),$$

which, after desuspending, by $K \oplus M$ on the left and by $K \oplus N$ on the right, yields

$$h^\rho(B) \longrightarrow h^{\rho+[M]-[N]}(B), \quad \rho \in \mathrm{RO}(G),$$

and thus (3.1).

In analogy to the nonequivariant case, important examples for h_G^* are the equivariant ordinary cohomology of Lewis, May and McClure [19], equivariant K-theory ([1] or [27]), equivariant stable cohomotopy ([28], [16], [10]) or its approach via fixed point theory, FIX ([30, 31], [21]). All these theories are multiplicative and have an element $1 \in h_G^0(B)$. We define the *equivariant fixed point index* of f as the element

$$I_G(f) = I_f(1) \in h_G^{[M]-[N]}(B). \tag{3.3}$$

Once again, this index has the properties 2.5. through 2.10. and can thus be extended to general G-ENR_Bs exactly in the same way as before, (2.11).

The case $B = \{*\}$ and $M = N (= 0)$ is interesting. Equivariant stable cohomotopy π_G^* is such that $\pi_G^0(*) \cong A(G)$, the *Burnside ring* of

G (of finite G-sets, if G is finite; see [3] for a thorough study of $A(G)$ for G a compact Lie group). Thus the equivariant index in this case is an element of $A(G)$. In [8] it is proved that every element of $A(G)$ is the equivariant index of some equivariant f. In fact, more generally, in [30, 31] and [21] it is proved that every element in $\pi_G^{[M]-[N]}(B)$ is the index of some equivariant f (as described in 3.0).

In [8] it is proved that for $M = N = 0$ and $B = \{*\}$, the equivariant index is determined by the "classical" indices $I(f^H)$ of the restrictions $f^h \colon V^H \longrightarrow E^H$ of f to the spaces whose points remain fixed under the action of the elements of the closed subgroups $H \subset G$. In [30], relationships between $I_G(f)$ and $\left\{I(f^H)\right\}$ are thoroughly studied.

3.4. The adequate set up to speak about the fixed point index in infinite dimensions is that of (separable) Banach spaces or, more generally, that of the absolute neighborhood retracts over B, the ANR_Bs. We shall sketch here in a very short way a generalization of Ulrich's work [29] in this direction.

An *absolute neighborhood retract over* B, an ANR_B, $p \colon E \longrightarrow B$ is defined as a vertical retract of an open set in $B \times K$, where K now denotes a separable Banach space. We consider fixed point situations, that is, commutative diagrams

$$\begin{array}{ccc} E \times N \supset V & \xrightarrow{\quad f \quad} & E \times M \\ & \searrow \quad \swarrow & \\ & B & \end{array} \tag{3.5}$$

where $E \longrightarrow B$ is an ANR_B, M and N are euclidean spaces (possibly with a group action, in whose case f has to be equivariant) and f is *strongly fixed*, which means that besides being properly fixed, the closure, $\overline{f(V)}$, of its image (or at least to the image, $f(W)$, of some open neighborhood W of $\mathrm{Fix}(f)$) lies properly over B. In the case that $E = B \times K \longrightarrow B$, it is possible to approximate f by maps $f' \colon V' \longrightarrow B \times P \times M$, properly fixed over B, where V' is open in $B \times P \times N$ and P is a finite polyhedron contained in K. Since then P is an ENR, then $B \times P$ is an ENR_B and the index $I(f')$ is defined. If two approximations f' and f'' are close enough to f then they are homotopic (in the sense of 2.7; thus $I(f') = I(f'')$. Therefore, we may define the index of f, $I(f)$, as $I(f')$ for f' close enough to f.

Since, if $\mathrm{Fix}(f) = \emptyset$ we may assume $V = \emptyset$ and so $V' = \emptyset$, we have

$$\mathrm{Fix}(f) = \emptyset \Longrightarrow I(f) = 0\,. \tag{3.6}$$

Properties 2.7. through 2.10. remain true and so the possibility of defining the index of the situation (3.4) for a general ANR_B $p\colon E \longrightarrow B$ holds.

With due care all this can be carried out equivariantly too.

4 Equivariant degree

4.0. In this section we describe a special case of the index which refers to an important class of equations (cf. [14, 15]). We shall discuss the degree, which we shall define via the index, and using the properties of this last, we shall show its fundamental properties.

Let G be a compact Lie group and M, N (finite dimensional) real G-modules, and let K be a euclidean space. If B is a metric G-space and

$$B \times K \times N \supset V \xrightarrow{\;g\;} K \times M \tag{4.1}$$

is an equivariant map, with V open and invariant in $B \times K \times N$, and is such that the set of solutions $g^{-1}(0)$ of the equation $g(b,z,y) = 0$ lies properly over B (e.g. is compact, if $B = \{*\}$ or B itself is compact); for instance, if the closure $\overline{V}$ lies properly over B and g can be extended to $\overline{V}$ in such a way that no zeroes appear on the boundary, then we define the *degree* of g as

$$\deg(g) = I(i - g') \in \pi_G^{[M]-[N]}(B)\,, \tag{4.2}$$

where $g'\colon V \longrightarrow B \times K \times M$, $g'(b,z,y) = (b, g(b,z,y))$ and $(i-g')(b,z,y) = (b,(z,0) - g(b,z,y)) \in B \times K \times M$. This can be defined because $i - g'$ is properly fixed, since $\mathrm{Fix}(i - g') = g^{-1}(0)$.

The degree is an invariant which detects solutions of the equation

$$g(b,z,y) = 0\,, \tag{4.3}$$

which can be seen as a family of equations in the sense of [14, 15] parametrized, equivariantly, by the metric G-space B. It has the following properties.

4.4. $\deg(g) \neq 0 \Longrightarrow$ (4.3) *has a solution.*

(This follows from the equivariant version of 2.6).

Excision 4.5. $g^{-1}(0) \subset W \subset V$, *$W$ open in $B \times K \times N$* $\Longrightarrow \deg(g) = \deg(g|W)$.

(This follows from the equivariant version of 2.9).

Additivity 4.6. $V = V_1 \cup V_2$, V_1, V_2 *open in* $B \times K \times N$ *and* $g^{-1}(0) \cap V_1 \cap V_2$ *proper over* $B \Longrightarrow \deg(g) = \deg(g|V_1) + \deg(g|V_2) - \deg(g|V_1 \cap V_2)$.

(This follows from the equivariant version of 2.8).

Homotopy Invariance 4.7. *If g_t is a homotopy between g_0 and g_1 such that for every t, $g_t^{-1}(0)$ lies properly over B, then* $\deg(g_0) = \deg(g_1)$.

(It follows from the equivariant version of 2.7. In fact, the inverse is also true, if we allow the domain V_t of g_t to vary along with t).

Clearly, it is not necessary to assume in (4.1) that G acts trivially on K. On the other hand, as described in 3.4, we may more generally assume that K is a separable Banach space. The situation is as follows. Let

$$B \times K \times N \supset V \xrightarrow{g} K \times M \tag{4.8}$$

be equivariant and such that $g^{-1}(0) \subset V$, as well as the closure of

$$\{(b, z, y) \mid (z, y) = (z', 0) - g(b, z', y') \in K \times M, \ \ (b, z', y') \in V\}$$

in $B \times K \times N$ lie properly over B. Then $i - g' \colon (b, z, y) \mapsto (b, (z, 0) - (g(b, z, y))$, $(b, z, y) \in V$, is strongly fixed and thus its Leray-Schauder type index, $I(i - g')$ is defined. Hence we define the *degree* of g by

$$\deg(g) = I(i - g') \in \pi_G^{[M]-[N]}(B) \tag{4.9}$$

as before. It has the same properties as the finite dimensional index.

As in [14], K may have an action of G by isometries. There, the authors consider the case $B = \{*\}$, in which our degree lies in the *stable equivariant stem* $\Pi_G^{[M]-[N]}$ of stable homotopy classes of equivariant maps between the G-spheres $\mathbb{S}^M$ and $\mathbb{S}^N$, given by the one-point compactifications of the G-modules M and N, respectively.

As a last comment in this section it should be remarked that the degree in [14, 15] is defined in a nonstable equivariant homotopy group of G-spheres. After stabilizing, their degree becomes ours. This explains, in particular, that their additivity (property (e) of the degree in [14], p. 445) only holds up to one suspension, whereas ours is plain.

A The Lefschetz-Hopf theorem

In this appendix, a short account of the results in [24] is given. There we give a conceptual proof of a Lefschetz-Hopf trace formula for computing the index of a globally defined fixed point situation. We prove the following.

Theorem A.1. *Let* $p\colon E \longrightarrow B$ *be a proper* G-ENR_B *such that* $h_G^*(E)$ *is a projective, finitely generated* $h_G^*(B)$*-module, and let* M *and* N *be* G*-modules. Then, if* $f\colon E\times N \longrightarrow E\times M$ *is an equivariant map over* B *such that* $\mathrm{Fix}(f) \longrightarrow B$ *is proper and* $f^{-1}(E\times 0)\subset E\times\mathbb{B}$ *for some ball* $\mathbb{B}\subset N$*, then*

$$I(f) = \mathrm{trace}(f^*:\ h_G^*(E) \longrightarrow h_G^*(E)) \in h_G^{[M]-[N]}(B)\,,$$

where f^* *is, up to suspension, the endomorphism of degree* $[M]-[N]$ *induced by (the stable map)*

$$f\colon E\times(N, N-\mathbb{B}) \longrightarrow E\times(M, M-0)\,. \tag{A.2}$$

Proof: It is an application of Proposition 4.4 in [10]. We sketch it.

There is s category B-$\mathfrak{Stab}_G$, whose objects are triples $(X, X'; \rho)$, where X is a G-space over B, X' is an invariant subspace and ρ is an element of the real representation ring $\mathrm{RO}(G)$.

Its morphisms are the *stable maps* given by

$$\begin{aligned} &B\text{-}\mathfrak{Stab}_G((X,X';\rho),(Y,Y';\sigma)) = \\ &\quad = \operatorname*{colim}_K \left[(X,X')\times(K\oplus\rho, K\oplus\rho-0),(Y,Y')\times(K\oplus\sigma,K\oplus\sigma-0)\right]\,, \end{aligned} \tag{A.3}$$

also called *stable maps from* (X, X') *to* (Y, Y') *of degree* $\sigma-\rho\in\mathrm{RO}(G)$, where $[\cdot]$ denotes G-homotopy classes of G-maps over B of pairs, and K varies in the category (made small) of unitary (complex) representations of G, the direction given by $K\le L \iff \exists\, M$ such that $K\oplus M\cong L$. (By taking K large enough, $K\oplus\rho$ and $K\oplus\sigma$ become G-modules).

This category can be endowed with the structure of a monoidal category, and inside it the proper G-ENR_Bs, E, are strongly dualizable, whose dual is $(B\times L, B\times L - E)$, if E is a G-neighborhood retract in $B\times L$.

Under the assumptions of A.1, the defining sequence (2.1) of the index, defines the trace, (2.2), of the morphism (A.3) in the category G-$\mathfrak{Stab}_B$. Since the cohomology h_G^* defines a functor from this category to the category of $h_G^*(B)$-modules, which satisfies the hypothesis of [10, 4.4], then it preserves traces, thus sending (2.2) to the trace we seek. □

For all details of the proof see [24].

B The transfer

Given an equivariant fixed point situation as (2.0) there is another homomorphism related to the index homomorphism $I_f\colon h_G^*(B) \longrightarrow h_G^{*+[M]-[N]}(B)$, called the *transfer homomorphism* of f. To define it, consider, as before, first the case $E = B \times K$ and take the sequence of equivariant maps of pairs

(B.1)

$$\begin{array}{ccccc}
 & & (V, V - F) & \xrightarrow{(\mathrm{id}, i-f)} & V \times_B (E \times M, E \times M - 0) \\
 & & \downarrow\curvearrowleft & & \| \\
(E \times N, E \times N - \mathbb{B}_\rho) & \to & (E \times N, E \times N - F) & & \| \\
\downarrow\curvearrowleft & & & & \| \\
(E \times N, E \times N - 0) & & & & \| \\
\| & & & & \| \\
B \times (K \times N, K \times N - 0) & \dashrightarrow & & & V \times (K \times M, K \times M - 0)\,.
\end{array}$$

This, again, induces a stable map as (A.3)

$$\tau_f\colon B \longrightarrow V$$

of degree $[M] - [N] \in \mathrm{RO}(G)$, or equivalently, a homomorphism

$$\tau_f^V : h_G^*(V) \longrightarrow h_G^*(B)$$

for any $\mathrm{RO}(G)$-graded G-equivariant cohomology theory h_G^*, called a *transfer homomorphism* of f.

Since we may restrict f to any $W \subset V$, as to approach $\mathrm{Fix}(f)$, then all transfers τ_G^W fit together to pass to the limit and yield the *minimal transfer*

$$\check{\tau}_f\colon \check{f}_G^*(\mathrm{Fix}(f)) \longrightarrow h_G^{*+[M]-[N]}(B)\,,$$

through which all other transfers factor. This shows, in particular, that

(B.2) $$\mathrm{Fix}(f) = \emptyset \Longrightarrow \tau_f^W = 0 \quad \text{for every} \quad W\,.$$

These transfers have all properties, which generalize the ones in [8] as can be seen in [26]. Thus they can provide applications of fixed point theory to algebraic topology. As an example of an application to this last, in [25] it is proved that any equivariant stable map $\alpha : X \longrightarrow Y$ between pointed G-spaces of degree $[M] - [N] \in \mathrm{RO}(G)$, that is, a map in the category $\mathfrak{Stab}_G$, factors through a transfer of some G-fixed point situation f over X and a nonstable map, namely, one has

$$\alpha : X \xrightarrow{\tau_f} V \xrightarrow{\psi} Y\,.$$

Here V is an open invariant neighborhood of the fixed point set $\mathrm{Fix}(f)$ and $\psi : V \longrightarrow Y$ is an equivariant (nonstable) map.

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References

[1] Atiyah, M. F. and Segal, G. B., Equivariant K-theory, mimeographed notes, Warwick, 1965.

[2] Brouwer, LEJ, *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1911), 97–115.

[3] tom Dieck, T., Transformation groups and representation theory, Lecture Notes in Math. **766**, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[4] Dold, A., *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology, **4** (1985), 1–8.

[5] Dold, A., Lectures on Algebraic Topology, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 200, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[6] Dold, A., *Eine geometrische Beschreibung des Fixpunktindexes*, Arch. Math. **25** (1974), 297–302.

[7] Dold, A., *The fixed point index of fibre-preserving maps*, Invent. Math. **25** (1974), 281–297.

[8] Dold, A., *The fixed point transfer for fibre-preserving maps*, Math. Z. **148** (1976), 215–244.

[9] Dold, A., *Fixed point theory and homotopy theory*, Contemporary Math. **12** (1982), 105–115.

[10] Dold, A., Puppe, D., *Duality, trace and transfer*, Proc. of the Internat. Conference on Geometric Topology PWN Warszawa (1980), 81–102.

[11] Granas, A., *The Leray-Schauder index and the fixed point theory for arbitrary* ANR*s*, Bull. Soc. Math. France, **100** (1972), 209–228.

[12] Hopf, H., *Eine Verallgemeinerung der Euler-Poincaréschen Formel*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl II (1928), 127–136.

[13] Hopf, H., *Über die algebraische Anzahl von Fixpunkten*, Math. Z. **29** (1929), 493–524.

[14] Ize, J., Massabó, I. and Vignoli, A., *Degree theory for equivariant maps*, I, Trans. Amer. Math. Soc. **315** (1989), 433–510.

[15] Ize, J., Massabó, I., Vignoli, A., *Degree theory for equivariant maps, the general* $\mathbb{S}^1$*-action*, Memoirs Amer. Math. Soc., **100**, No. 481, (1992).

[16] Kosniowsky, C., *Equivariant cohomology and stable cohomotopy*, Math. Ann. **210** (1974), 83–104.

[17] Lefschetz, S., *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. **28** (1926), 1–49.

[18] Leray, J. and Schauder, J., *Topologie et équations fonctionelles*, Ann. Scient. Éc. Norm. Sup. **51** (1934), 45–78.

[19] Lewis, G., May, P. and McClure, J., *Ordinary* $\mathrm{RO}(G)$*-graded cohomology*, Bull. Am. Math. Soc. **4** (1981), 208–212.

[20] Prieto, C., *Coincidence index for fiber-preserving maps. An approach to stable cohomotopy*, manuscripta math. **47** (1984), 233–249.

[21] Prieto, C., $KO(B)$*-graded stable cohomotopy over* B *and* $\mathrm{RO}(G)$*-graded* G*-equivariant stable cohomotopy: A fixed point theoretical approach to the Segal Conjecture*, Contemporary Math. **58**, II (1987), 89–108.

[22] Prieto, C., *Teoría* FIX *de diagramas*, en “Conferencias del Taller de Topología Algebraica” IV Coloquio, Depto. de Matemáticas, CINVESTAV, México (1985).

[23] Prieto, C., FIX-*Theory of diagrams*, Comtemporary Math. **72** (1988), 207–224.

[24] Prieto, C., *Una fórmula de Lefschetz-Hopf para el índice de coincidencia equivariante parametrizado*, Aportaciones Mat. **5** (1988), 73–88.

[25] Prieto, C., *Transfers generate the equivariant stable homotopy category*, Topology & its Appl. **58** (1994), 181–191.

[26] Prieto, C. and Ulrich, H., *Equivariant fixed point index and fixed point transfer in nonzero dimensions*, Trans. Amer. Math. Soc. **328** (1991), 731–745.

[27] Segal, G. B., *Equivariant K-theory*, Publ. Math. I.H.E.S. **34** (1968), 128–151.

[28] Segal, G. B., *Equivariant stable homotopy theory*, Proc. Internat. Congress Math. Nice 1970, **2** París (1971), 59–63.

[29] Ulrich, H., *Der Fixpunktindex fasernweiser Abbildungen*, Diplomarbeit, Heidelberg (1976).

[30] Ulrich, H., *Der äquivariante Fixpunktindex vertikaler G-Abbildungen*, Dissertation, Heidelberg (1983).

[31] Ulrich, H., *Fixed Point Theory of Parametrized Equivariant Maps*, Lect. Notes in Math., **1343**, Springer-Verlag, Berlin-Heidelberg-New York (1988).