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GENERALIZED LEFSCHETZ NUMBERS FOR EQUIVARIANT MAPS

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1. Introduction

In this paper we discuss a kind of Lefschetz number defined for a generalized multiplicative G-equivariant cohomology theory $h_G^* = \{h_G^i\}$, where G is a compact Lie group, and $i \in \mathcal{I}$ runs along some at most countable set of indices $\mathcal{I}$ (cf. [22, 3]).

For every pair $Y \subset X$ of finite G-CW-complexes the cohomology algebra $h_G^*(X, Y)$ has the structure of an $h_G^*(\mathrm{pt})$-module and, consequently, also of an $h_G^0(\mathrm{pt})$-module, given by the multiplicative structure. Assume that $h_G^*(X, Y)$ is either
(a) a finitely generated projective $h_G^*(\mathrm{pt})$-module, or
(b) a finitely generated projective $h_G^0(\mathrm{pt})$-module respectively.

Let $f\colon (X, Y) \longrightarrow (X, Y)$ be a G-equivariant map. Under assumption (a) a generalized trace, $\operatorname{tr} f^*$, of the induced map $f^*\colon h_G^*(X, Y) \longrightarrow h_G^*(X, Y)$ is well defined (cf. [23, 4]) and will be called the *full generalized Lefschetz number* and denoted by

$$L_{h_G^*}(f) \in h_G^*(\mathrm{pt}),$$

(in fact, one may prove that $L_{h_G^*}(f) \in h_G^0(\mathrm{pt})$). Taking $\mathcal{I} = \mathbb{N} \cup \{0\}$ and using the same argument ([23, 4]), assumption (b) guarantees the existence of each trace $\operatorname{tr}_{h_G^0}(f^i)$ of the induced homomorphisms $f^i\colon h_G^i(X, Y) \longrightarrow h_G^i(X, Y)$ and consequently allows us to take their alternate sum to define another generalized Lefschetz number

$$L_{h_G^0}(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}_{h_G^0}(f^i) \in h_G^0(\mathrm{pt}),$$

called the *cut generalized Lefschetz number*. Note that, since $h_G^0(\mathrm{pt}) \subset h_G^*(\mathrm{pt})$ is a subring, assumption (b) is much more restrictive. On the other hand, since the ring $h_G^0(\mathrm{pt})$ is smaller and simpler than $h_G^*(\mathrm{pt})$, the Lefschetz number is easier to handle with in it.

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An illustrative example of this is the stable (equivariant) cohomotopy theory, which we shall discuss later in more detail.

In this paper we study general properties of the two notions mentioned above, pointing out their differences and giving applications to the problem of the existence of fixed orbits of a G-equivariant map, in the case that G is finite. To that end we make use of the equivariant stable cohomotopy theory tensored with the rational numbers. More precisely, in Section 3 we discuss the nonequivariant case ($G = \mathbf{e}$). A direct use of the Lefschetz-Hopf-Dold formula shows in this case that the full generalized Lefschetz number is given by

$$L_{h^*}(f) = \varepsilon(L(f)),$$

where $L(f)$ is the *classical Lefschetz number*, derived in the singular cohomology with rational coefficients, and $\varepsilon\colon \mathbb{Z} \longrightarrow h^0(\mathrm{pt})$ is the homomorphism given by the natural transformation from stable cohomotopy to the given generalized cohomology theory h^* (Theorem 3.1), or, equivalently, given by just mapping $1 \in \mathbb{Z}$ to $1 \in h^0(\mathrm{pt})$. Next we prove Theorem 3.3 which asserts that in the nonequivariant case the cut generalized Lefschetz number is given by

$$L_{h^0}(f) = \varepsilon(L(f))\, L_{h^0}(\mathrm{id}_{\mathrm{pt}}),$$

where $L(f)$ is the classical Lefschetz number, derived in singular cohomology with rational coefficients, and $\varepsilon\colon \mathbb{Z} \longrightarrow h^0(\mathrm{pt})$ is as above. Our proof of this fact is based on the Atiyah-Hirzebruch-Whitehead spectral sequence converging to $h^*(X, Y)$. This shows that in the nonequivariant case the full and the cut generalized Lefschetz numbers are nonzero only if the classical Lefschetz number is different from zero. This is not the situation in the equivariant case

The generalized Lefschetz numbers are interesting in the equivariant case. First the equivariant version of the Lefschetz-Hopf-Dold formula proved by the second author ([15, 16, 19]) allows us to show that for every equivariant cohomology theory, which is stable with respect to suspensions given by representations, and for any equivariant map $f\colon (X, Y) \longrightarrow (X, Y)$, the full generalized Lefschetz number is equal to

$$L_{h^*_G}(f) = \varepsilon(L_{\omega^*_G}(f)),$$

where $\varepsilon\colon \omega^*_G(\mathrm{pt}) \longrightarrow h^*_G(\mathrm{pt})$ is the natural homomorphism, now from the equivariant stable cohomotopy theory to the given cohomology theory (Theorem 4.1). Using the mentioned Lefschetz-Hopf-Dold theorem for the equivariant stable cohomotopy theory and the Ulrich formula ([24]), the right hand side can be expressed by the Lefschetz numbers of the restrictions of f to the fixed point sets of subgroups of G, as was already studied by tom Dieck, Marzantowicz and Ulrich ([3, 11, 24]). In particular it vanishes if f has no fixed point.

In an earlier paper by the first author ([12]) the generalized Lefschetz number in the equivariant K-theory tensored with the complex numbers has been studied. The trace, and consequently the Lefschetz number, is a complex-valued class function $L_{K_G^*}(f)(g)$ on G. The main result of [12] states that if f is an equivariant selfmap of a finite G-CW-complex X, satisfying $L_{K_G^*}(f)(g) \neq 0$, then there exists a point $x \in X^g$ such that

$$f(x) = hx\,,$$

for some $h \in G$; that is, f maps the orbit of x into itself.

In what follows, we use the equivariant stable cohomotopy theory ω_G^*, for a finite group G, graded by the elements of the real representation ring RO(G), or by the nonnegative integers, respectively.

The geometric meaning of the cut generalized Lefschetz number can be drastically different. To show this we study the equivariant stable cohomotopy theory ω_G^*, graded by the nonnegative integers. Tensoring ω_G^* with $\mathbb{Q}$ we get an equivariant cohomology theory, also graded by $\mathbb{N} \cup \{0\}$, for which the cut generalized Lefschetz number is well-defined. This is due to the following two facts.

(i) For G finite, there is the Segal theorem which asserts that $\omega_G^0(\mathrm{pt}) = A(G)$ is the Burnside ring and $\omega_G^i(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ if $i \neq 0$ ([21]). More generally, a theorem of Kosniowski ([9]) states that for every G-space X we have an isomorphism

$$\omega_G^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus \omega^i\left(X^H/W(H)\right) \otimes_{\mathbb{Z}} \mathbb{Q}\,,$$

where the sum is taken over the conjugacy classes of subgroups H of G and $W(H)$ denotes the Weyl group $N(H)/H$ of H.

(ii) The Dress description of the ring $A(G)\otimes\mathbb{Q}$ as the ring of rational-valued functions on the set $\mathcal{S}_G$ of one representative in each conjugacy class of subgroups of G.

From (i) it follows that $\omega_G^*(\mathrm{pt})$ is a finitely generated $A(G) \otimes \mathbb{Q}$-module for every finite CW-complex. (ii) implies that every ideal in $A(G) \otimes \mathbb{Q}$ is projective and consequently every $A(G) \otimes \mathbb{Q}$-module has a projective resolution of length less than or equal to 2. Theorem 4.5 states that the cut generalized Lefschetz number $L_{\omega_G^0 \otimes \mathbb{Q}}(f) \in A(G) \otimes \mathbb{Q}$ is a function on $\mathcal{S}_G$ such that for every subgroup $H \subset G$ we have

$$L_{\omega_G^0 \otimes \mathbb{Q}}(f)(H) = L(f^{(H)}/G)\,,$$

where $f^{(H)}$ denotes the restriction of f to $X^{(H)} = GX^H$ and $f^{(H)}/G$ is the map induced by $f^{(H)}$ on the orbit space. This shows that here the cut generalized Lefschetz number is different from the full generalized Lefschetz number. Moreover, the cut generalized Lefschetz number is an invariant measuring the existence, and structure, of orbits mapped by f into itself. In particular it can be nonzero for a fixed point free equivariant map. In the proof of this theorem we use the above mentioned Segal-Kosniowski theorem (cf. (i)).

We would like to emphasize the following.

(1) The equality $\omega_G^i(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $i > 0$ is not true in general for a compact Lie group G (see [7] for a description of $\omega_G^1(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q}$ when G is an infinite abelian compact Lie group).

(2) The equivariant cohomotopy theory $\bigoplus_{i=0}^{\infty} \omega_G^i$ is only a portion of the full equivariant cohomotopy theory $\bigoplus \omega_G^{\alpha}$, graded by the elements α of the real representation ring $\mathrm{RO}(G)$ (cf. [9, 15]). In particular, the former admits suspension isomorphisms only with respect to trivial representations $V = \mathbb{R}^n$.

2. Generalized Lefschetz numbers for equivariant maps

Let G be a compact Lie group and $\{h_G^i\}$ be a generalized equivariant multiplicative cohomology theory, e.g. stable equivariant cohomotopy theory, equivariant K-theory, equivariant cohomology theory in the sense of Segal, Illman and Matumoto, tom Dieck and others (see [3, 2, 6, 9, 13, 15, 21] for further discussion and examples). We say that $h_G^* = \bigoplus_{\alpha \in \mathcal{I}} h_G^{\alpha}$ is $\mathrm{RO}(G)$-*graded* if $\mathcal{I} = \mathrm{RO}(G)$. If, otherwise, $\mathcal{I} = \mathbb{N}^+ \cup \{0\}$ then we say that h_G^* is $\mathbb{N}$-*graded*; in case that $\mathcal{I} = \mathbb{Z}_2$ then h_G^* is $\mathbb{Z}_2$-*graded*.

Given a pair of finite G-CW-complexes (X, Y), $Y \subset X$, since h_G^* is multiplicative, one has that $h_G^*(X, Y)$ is an $h_G^*(\mathrm{pt})$-module, and thus, for every α, $h_G^{\alpha}(X, Y)$ is an $h_G^0(\mathrm{pt})$-module. Let now $f\colon (X, Y) \longrightarrow (X, Y)$ be an equivariant selfmap of the pair (X, Y). Our aim is to define a trace of the induced homomorphism $f^*\colon h_G^*(X, Y) \longrightarrow h_G^*(X, Y)$ as endomorphism of an $h_G^*(\mathrm{pt})$-module and of the homomorphism $f^i\colon h_G^i(X, Y) \longrightarrow h_G^i(X, Y)$ as endomorphism of an $h_G^0(\mathrm{pt})$-module, thus defining the generalized full and cut Lefschetz numbers of f. In general, the ring $h_G^0(\mathrm{pt})$, and hence also the superring $h_G^*(\mathrm{pt})$, is not a field, but one can use a general definition of trace introduced by Thomas ([23]) or, independently, in a different but more general context, by Dold and Puppe ([4]). To do it we need the following alternative finiteness assumptions on the theory h_G^*.

1_F For every pair (X, Y) of finite G-CW-complexes, $h_G^*(X, Y)$ is a finitely generated $h_G^*(\mathrm{pt})$-module.

1_C For every pair (X, Y) of finite G-CW-complexes, $h_G^i(X, Y)$ is a finitely generated $h_G^0(\mathrm{pt})$-module for every $i \in \mathbb{N}$.

2_F Every finitely generated $h_G^*(\mathrm{pt})$-module M has a finitely generated projective resolution of finite length, which exists if in particular M is a finitely generated projective module over $h_G^*(\mathrm{pt})$.

2_C Every finitely generated $h_G^0(\mathrm{pt})$-module M has a finitely generated projective resolution of finite length, which exists if in particular M is a finitely generated projective module over $h_G^0(\mathrm{pt})$.

Under the assumptions 1_F or 2_F, for every $h_G^*(\mathrm{pt})$-endomorphism φ of an $h_G^*(\mathrm{pt})$-module M, there exists a well-defined element of $h_G^*(\mathrm{pt})$, denoted by $\mathrm{tr}_{h_G^*(\mathrm{pt})}(\varphi)$, or shortly $\mathrm{tr}^{\mathrm{f}}(\varphi)$ if there is no danger of confusion, which we call the *full trace* of φ ([4, 23]).

The assignment $\varphi \mapsto \mathrm{tr}_{h_G^*(\mathrm{pt})}(\varphi)$ has the following properties.

(**P1**) **Exactness**. For every short exact sequence of $h_G^*(\mathrm{pt})$-modules and endomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_2} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0
\end{array}$$

we have $\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi_1) + \mathrm{tr}(\varphi_2)$.

(**P2**) **Commutativity**. For every two $h_G^*(\mathrm{pt})$-endomorphisms φ, ψ of an $h_G^*(\mathrm{pt})$-module M, we have $\mathrm{tr}(\varphi \circ \psi) = \mathrm{tr}(\psi \circ \varphi)$.

DEFINITION 2.1. Let (X, Y) be a pair of finite G-CW-complexes and $f : (X, Y) \longrightarrow (X, Y)$ be an equivariant selfmap of this pair. Let also h_G^* be an equivariant multiplicative cohomology theory satisfying 1_F or 2_F, and $f^* : h_G^*(X, Y) \longrightarrow h_G^*(X, Y)$ be the induced homomorphism. Under this assumption the element

$$L_{h_G^*}(f) = \mathrm{tr}_{h_G^*(\mathrm{pt})}(f^*) \in h_G^*(\mathrm{pt})$$

is well-defined and is called the *generalized full Lefschetz number* of f in h_G^*.

Since $h_G^0(\mathrm{pt})$ is a ring with 1, there exists a natural homomorphism $\varepsilon : \mathbb{Z} \longrightarrow h_G^0(\mathrm{pt})$, defined by

$$\varepsilon(1) = 1 .$$

Note that ε is the restriction of the natural transformation from the stable cohomotopy theory ω_G^* to the theory h_G^* evaluated at a point, since $\mathbb{Z} \subset \omega_G^0(\mathrm{pt})$.

Analogously, under assumptions 1_C or 2_C, for every $h_G^0(\mathrm{pt})$-endomorphism φ of an $h_G^0(\mathrm{pt})$-module M, there exists a well-defined element of $h_G^0(\mathrm{pt})$, denoted by $\mathrm{tr}_{h_G^0(\mathrm{pt})}(\varphi)$, or shortly $\mathrm{tr}^{\mathrm{c}}(\varphi)$ if there is no danger of confusion, which we call the *cut trace* of φ ([4, 23]).

The assignment $\varphi \mapsto \mathrm{tr}_{h_G^0(\mathrm{pt})}(\varphi)$ also has the Properties **P1** and **P2** given above.

DEFINITION 2.2. Let (X, Y) be a pair of finite G-CW-complexes and $f : (X, Y) \longrightarrow (X, Y)$ be an equivariant selfmap of this pair. Let also h_G^* be an $\mathbb{N}\cup\{0\}$- or $\mathbb{Z}_2$-graded equivariant multiplicative cohomology theory, satisfying 1_C or 2_C, and for each degree i, let $f^i : h_G^i(X, Y) \longrightarrow h_G^i(X, Y)$ be the induced homomorphism. Suppose also that

3_C $h_G^*(X, Y)$ is a finitely generated $h_G^0(\mathrm{pt})$-module.

Under these assumptions, the element

$$L_{h^0_G}(f) = \sum(-1)^i \operatorname{tr}_{h^0_G(\mathrm{pt})}(f^i) \in h^0_G(\mathrm{pt})$$

is well-defined and will be called the *cut generalized Lefschetz number* of f.

3. Universal property of the classical Lefschetz number in the nonequivariant case

We show now that in the nonequivariant case the given variations of the Lefschetz number are not essential from the point of view of the fixed point theory. We assume, therefore, that $G = \mathbf{e}$ is the trivial group. We have the following.

Theorem 3.1. *Let* (X, Y) *be a pair of finite CW-complexes and* $h^* = \{h^i\}_{i\in\mathcal{I}}$ *be a generalized multiplicative cohomology theory satisfying assumptions* 1_F *or* 2_F. *Then the full generalized Lefschetz number exists and satisfies*

$$L_{h^*}(f) = \varepsilon^*(L(f)),$$

where $L(f)$ *is the classical Lefschetz number of* f *obtained by using singular* (*ordinary*) *cohomology with rational coefficients, and* $\varepsilon^* \colon \mathbb{Z} \longrightarrow h^0(\mathrm{pt})$ *is the natural ring homomorphism.*

Proof. The statement of this theorem is part of Corollary 4.5 in [4] which is obtained by purely algebraic means. For convenience to the reader we include a proof outline based on the Lefschetz-Hopf-Dold theorem (cf. [15]). Assume first that $Y = \emptyset$. Indeed, from this formula we get $L_{h^*}(f) = I(f, h^*)$, where the $I(f, h^*) \in h^0(\mathrm{pt})$ is the fixed point index of f on X. From the functoriality of the index we have $I(f, h^*) = \varepsilon^*(I(f, \omega^*))$, where ε^* is the natural homomorphism from the stable cohomotopy theory ω^* to any multiplicative cohomology theory. On the other hand, the same argument applied to the singular cohomology theory $H^*(\ \ ;\mathbb{Q})$ with rational coefficients shows that $I(f, \omega^*) = L(f)$, since $\varepsilon^0 \colon \omega^0(\mathrm{pt}) \longrightarrow H^0(\mathrm{pt};\mathbb{Q})$ is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. The relative case follows from property **P1**. □

Corollary 3.2. *Let* (X, Y), f *and* h^* *be as before. Then there exists* $m \in \mathbb{N} \cup \{0\}$ *such that* $L_{h^*}(f)$ *is equal to the remainder of* $L(f)$ *modulo* m. (*If* $m = 0$, *then the remainder is equal to the given integer, if* $m = 1$, *then it is equal to* 0 *for every number*).

Proof. The image $\operatorname{im}(\varepsilon^*(\mathbb{Z})) \subset h^0(\mathrm{pt})$ is equal to $\mathbb{Z}$, $\mathbb{Z}_m$, or zero respectively. □

Now we give a formula expressing the cut generalized Lefschetz number $L_{h^0}(f)$

in terms of the classical Lefschetz number $L(f)$.

Let h^* be any multiplicative cohomology theory. Since $h^0(\mathrm{pt})$ is a ring with 1, there exists a natural homomorphism $\varepsilon^0 \colon \mathbb{Z} \longrightarrow h^0(\mathrm{pt})$, defined by

$$\varepsilon^0(1) = 1\,.$$

Note that ε^0 corresponds to the zero-th level of the natural homomorphism ε from the stable cohomotopy theory ω^* to h^* introduced above.

We have the following theorem.

Theorem 3.3. *Let h^* be a generalized cohomology theory satisfying assumptions 1_C, 2_C and 3_C, and $\varepsilon^0 \colon \mathbb{Z} \longrightarrow h^0(\mathrm{pt})$ be as above; let $f \colon (X, Y) \longrightarrow (X, Y)$ be a self-map of a pair of finite CW-complexes. Then the cut generalized Lefschetz number exists and we have the equality*

$$L_{h^0}(f) = \varepsilon(L(f))L_{h^0}(\mathrm{id}_{\mathrm{pt}}).$$

For a *proof* we use the Atiyah-Hirzebruch-Whitehead spectral sequence converging to $h^*(X, Y)$ (see [14]). This and the Hopf lemma stated below (3.4) will reduce the computation of $L_{h^0}(f)$ to deriving the Lefschetz number of the homomorphism induced by f on the E_2-terms of the spectral sequence. Consequently, replacing f by a homotopic cellular map, we can use the E_1-terms of the spectral sequence to deduce $L_{h^0}(f)$, and the statement will follow by an algebraic argument.

As it is for the classical trace, we have the following fact, called the Hopf lemma (cf. [4, 23]).

Lemma 3.4. *Let (C^i, d_i) be a chain complex of finitely generated $h^0_G(\mathrm{pt})$-modules, $f = \{f_i\}$, $f_i \colon C^i \longrightarrow C^i$, be an endomorphism of this complex and*

$$H^i = \ker d_i / \operatorname{im} d_{i+1}$$

be the homology of the complex. If assumption 2_C is satisfied and $C^i = 0$ for almost every i, then

$$\sum (-1)^i \operatorname{tr}_{h^0(\mathrm{pt})}(f_i) = \sum (-1)^i \operatorname{tr}_{h^0(\mathrm{pt})} H^i(f)\,.$$

Let h^* be a multiplicative cohomology theory and (X, Y) be a pair of finite CW-complexes. Then the Atiyah-Hirzebruch-Whitehead spectral sequence $\{E_r^{p,q},\, d_r\}$ converges to $h^*(X, Y)$, i.e.

$$E_\infty^{p,q} = \frac{\ker(h^{p+q}(X, Y) \longrightarrow h^{p+q}(X^{(p-1)}, Y))}{\ker(h^{p+q}(X, Y) \longrightarrow h^{p+q}(X^{(p)}, Y))}\,.$$

Moreover $E_1^{p,q} = h^{p+q}(X^{(p)}, X^{(p-1)} \cup Y)$ and the first differential d_1 is equal to the connecting homomorphism

$$d_1 = \delta\colon h^{p+q}(X^{(p)}, X^{(p-1)} \cup Y) \longrightarrow h^{p+q+1}(X^{(p+1)}, X^{(p)} \cup Y)$$

of the exact sequence of the triple. The Atiyah-Hirzebruch-Whitehead theorem states that every E_2-term $E_2^{p,q}$ is equal to the p-th singular cohomology group $H^p(X, Y; h^q(\mathrm{pt}))$, of (X, Y) with coefficients in $h^q(\mathrm{pt})$.

Furthermore, any continuous selfmap $f\colon (X, Y) \longrightarrow (X, Y)$ induces an endomorphism $f_2^{p,q}\colon H^p(X, Y; h^q(\mathrm{pt})) \longrightarrow H^p(X, Y; h^q(\mathrm{pt}))$ commuting with differentials d_2, thus providing a homomorphism of spectral sequences. If, moreover, f is a cellular map, then f induces also endomorphisms of the E_1-terms $E_1^{p,q}$.

In case that the theory h^* is $\mathbb{Z}_2$-graded, the second variable q in the bigrading of the above spectral sequence runs also over the elements of $\mathbb{Z}_2$. Then, the symbol $p+q$, if $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_2$, means the sum of the remainder of p modulo 2 and q in $\mathbb{Z}_2$.

Since h^* is a multiplicative theory, $\{E_r^{p,q}, d_r\}$ is a multiplicative spectral sequence. Consequently each group $E_r^{p,q}$ is a module over $E_r^{0,0} \cong E_1^{0,0}$. But $E_1^{0,0} = h^0(X^{(0)}, Y)$ is itself a module over $h^0(X^{(0)})$. Furthermore, the homomorphism $h^0(\mathrm{pt}) \longrightarrow h^0(X^{(0)})$, induced by the map $X^{(0)} \longrightarrow \mathrm{pt}$, endows $h^0(X^{(0)})$ with the structure of an $h^0(\mathrm{pt})$-module. Therefore, finally, each $E_r^{p,q}$ has the structure of an $h^0(\mathrm{pt})$-module.

The structure described above induces an $h^0(\mathrm{pt})$-module structure on $E_\infty^{p,q}$ coinciding with the structure of $h^0(\mathrm{pt})$-module on $h^*(X, Y)$. The existence and convergence of the Atiyah-Hirzebruch-Whitehead spectral sequence allow us to verify assumption 3_C using the following proposition.

Proposition 3.5. *Let h^* be a generalized cohomology theory and $Y \subset X$ be a pair of finite CW-complexes. Then $h^*(X, Y)$ is a finitely generated $h^0(\mathrm{pt})$-module if and only if $h^*(\mathrm{pt})$ is a finitely generated $h^0(\mathrm{pt})$-module.*

Consequently, instead of assumption 3_C, we can put the following equivalent assumption.

$3'_C$ $h^*(\mathrm{pt})$ is a finitely generated $h^0(\mathrm{pt})$ module.

Suppose that for a given cohomology theory h^*, assumption $3'_C$ is satisfied. For a selfmap $f\colon (X, Y) \longrightarrow (X, Y)$ of a pair of finite CW-complexes we define, for each r,

$$\Lambda^r_{h^*}(f) = \sum_{p,q} (-1)^{p+q} \operatorname{tr}_{h^0(\mathrm{pt})}(f_r^{p,q}),$$

where $f_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p,q}$ is the endomorphism induced by f in the E_r-terms of the spectral sequence. For convenience, we might denote $\Lambda^r_{h^*}(f)$ also by $\Lambda(\{f_r^{p,q}\}, \{E_r^{p,q}\})$.

$$\begin{array}{cccc}
F^{-1\,q}/F^{0\,q-1}\,, & F^{0\,q}/F^{1\,q-1}\,, & F^{1\,q}/F^{2\,q-1}\,, & \cdots \\
F^{-1\,q+1}/F^{0\,q}\,, & F^{0\,q+1}/F^{1\,q}\,, & F^{1\,q+1}/F^{2\,q}\,, & \cdots \\
F^{-1\,q+2}/F^{0\,q+1}\,, & F^{0\,q+2}/F^{1\,q+1}\,, & F^{1\,q+2}/F^{2\,q+1}\,, & \cdots \\
\vdots\,, & \vdots\,, & \vdots\,, & \ddots\,.
\end{array}$$

Table 1.

Proposition 3.6. *Let h^* be a generalized cohomology theory such that $h^*(\mathrm{pt})$ is a finitely generated $h^0(\mathrm{pt})$-module and let $f\colon (X,Y) \longrightarrow (X,Y)$ be a map of a pair of finite CW-complexes. Then for every $r \geq 2$ (or $r \geq 1$ if f is a cellular map) we have*

$$\Lambda^r_{h^*}(f) = L_{h^0}(f)\,.$$

Proof. From the Hopf lemma, it follows that

$$\Lambda^r_{h^*}(f) = \Lambda(\{f^{p,q}_r\},\{E^{p,q}_r\}) = \Lambda(\{f^{p,q}_{r+1}\},\{E^{p,q}_{r+1}\}) = \Lambda^{r+1}_{h^*}(f)$$

for $r \geq 1$, because $E^{p,q}_{r+1}$ is the homology at $E^{p,q}_r$ of the corresponding complex of E_r-terms of the spectral sequence. Since $\{E^{p,q}_r\}$ is strongly convergent to $h^*(X,Y)$, there exists r_0 such that for every $r \geq r_0$

$$\{E^{p,q}_{r_0}\} = \{E^{p,q}_r\} = \{E^{p,q}_\infty\}\,,$$

and consequently

$$\Lambda^r_{h^*}(f) = \Lambda^\infty_{h^*}(f)\,.$$

It is sufficient to show that

$$L_{h^0}(f) = \Lambda^\infty_{h^*}(f)\,.$$

But $E^{p,q}_\infty$ is the associated module and $f^{p,q}_\infty$ is equal to the homomorphism $\tilde{f}^{p,q}_{(p-1)}$ induced by $f^{p+q}_{(p-1)}$ on

$$E^{p,q}_\infty = \frac{F^{p,q}}{F^{p+1,q-1}}\,,$$

where $F^{p,q} = \ker(h^{p+q}(X,Y) \longrightarrow h^{p+q}(X^{(p-1)},Y))$. Consequently, it is enough to show that

$$\sum (-1)^{p+q}\,\mathrm{tr}_{h^0(\mathrm{pt})}(\tilde{f}^{p+q}_{(p)}) = L_{h^0}(f)\,.$$

Indeed, writing out the E_∞-terms $\{E^{p,q}_\infty\}$ in tabular form, we obtain the Table 1.

By the above, taking an alternate sum with appropriate signs and using the additivity of the trace we get

$$\begin{aligned}\sum(-1)^{p+q}\,\mathrm{tr}_{h^0(\mathrm{pt})}(f_\infty^{p,q}) &= \sum(-1)^{p+q}\,\mathrm{tr}_{h^0(\mathrm{pt})}(f_{(p)}^{p+q})\\ &= \sum_q(-1)^q\,\mathrm{tr}_{h^0(\mathrm{pt})}(f_{(-1)}^q)\\ &= L_{h^0}(f)\,,\end{aligned}$$

since $F^{0\,n} = h^n(X,Y)$. This proves the desired result. $\square$

We now pass to the proof of Theorem 3.3.

Proof. We may assume that our map f is cellular. Otherwise, we may replace it up to homotopy by one if necessary. The map $f\colon (X,Y)\longrightarrow (X,Y)$ induces a endomorphism

$$\{f_1^{p,q}\}\colon \{E_1^{p,q}\}\longrightarrow\{E_1^{p,q}\}$$

of the E_1-terms of the Atiyah-Hirzebruch-Whitehead spectral sequence. By Proposition 3.6, it is sufficient to derive $\Lambda^1_{h^0(\mathrm{pt})}(f)$. We have

$$\begin{aligned}E_1^{p,q} &= h^{p+q}(X^{(p)}, X^{(p-1)}\cup Y)\\ &= C^p(X,Y;h^q(\mathrm{pt}))\\ &= C^p(X,Y;\mathbb{Z})\otimes_{\mathbb{Z}} h^q(\mathrm{pt})\,.\end{aligned}$$

Since $C^p(X,Y;\mathbb{Z}) = \bigoplus_{\sigma\in X^{(p)}\setminus Y}\mathbb{Z}_\sigma$, $\mathbb{Z}_\sigma=\mathbb{Z}$, where the sum is taken over all p-cells which are not in Y, we have

$$\mathrm{tr}_{h^0(\mathrm{pt})}(f_1^{p,q}) = \sum_{\sigma\in X^{(p)}\setminus Y}\mathrm{tr}(\overline{f}|_{\mathrm{Hom}(\mathbb{Z}_\sigma,\,\mathbb{Z})\otimes h^q(\mathrm{pt})})\,.$$

But the map $\overline{f}$ induced by f on $\mathrm{Hom}(\mathbb{Z}_\sigma,\mathbb{Z})\cong\mathbb{Z}$ is multiplication by the incidence number $\mathrm{inc}(\sigma,\sigma) = m_\sigma$. Therefrom it follows that it is enough to compute the trace over $h^0(\mathrm{pt})$ of the endomorphism of $h^q(\mathrm{pt})$ given by multiplication with $m_\sigma\in\mathbb{Z}$.

(i) If $h^q(\mathrm{pt})$ is a free $h^0(\mathrm{pt})$-module, then $\mathrm{tr}_{h^0(\mathrm{pt})}(m_\sigma) = \varepsilon(m_\sigma)\,\mathrm{tr}(\mathrm{id}_{h^q(\mathrm{pt})})$.

(ii) Suppose that $h^q(\mathrm{pt})$ is a projective $h^0(\mathrm{pt})$-module. Then there exist an $h^0(\mathrm{pt})$-module I and an integer n such that $I\oplus h^q(\mathrm{pt})\cong(h^0(\mathrm{pt}))^n$. Then the following diagram is commutative

$$\begin{tikzcd} I\oplus h^q(\mathrm{pt}) \arrow[r, "\cong"] \arrow[d, "0\oplus m_\sigma"'] & (h^0(\mathrm{pt}))^n \arrow[d, "m_\sigma\alpha"] \\ I\oplus h^q(\mathrm{pt}) \arrow[r, "\cong"] & (h^0(\mathrm{pt}))^n \end{tikzcd},$$

where $\alpha\colon (h^0(\mathrm{pt}))^n \longrightarrow (h^0(\mathrm{pt}))^n$ is the endomorphism induced by $0 \oplus \mathrm{id}_{h^q(\mathrm{pt})}$ on $(h^0(\mathrm{pt}))^n$.

(iii) The general case, when $h^q(\mathrm{pt})$ has a finite projective resolution, follows from (ii). We obtain $\mathrm{tr}_{h^0(\mathrm{pt})}(m_\sigma) = \varepsilon(m_\sigma)\,\mathrm{tr}(\mathrm{id}_{h^q(\mathrm{pt})})$. Adding up over the cells, we have

$$\mathrm{tr}(f_1^{p,q}) = \sum_{\sigma\in X^{(p)}\setminus Y} \varepsilon(m_\sigma)\,\mathrm{tr}(\mathrm{id}_{h^q(\mathrm{pt})})\,.$$

Taking the alternate sum over the index p we have

$$\begin{aligned}\sum(-1)^p\,\mathrm{tr}(f_1^{p,q}) &= \sum(-1)^p\varepsilon\left(\sum_{\sigma\in X^{(p)}\setminus Y} m_\sigma\right)\mathrm{tr}(\mathrm{id}_{h^0(\mathrm{pt})})\\ &= \varepsilon\left(\sum(-1)^p\sum_{\sigma\in X^{(p)}\setminus Y} m_\sigma\right)\mathrm{tr}(\mathrm{id}_{h^q(\mathrm{pt})})\\ &= \varepsilon(L(f))\,\mathrm{tr}(\mathrm{id}_{h^q(\mathrm{pt})})\,.\end{aligned}$$

Taking once more the alternate sum, now with respect to the index q, we get

$$\begin{aligned}\sum_q(-1)^q\sum_p(-1)^p\,\mathrm{tr}(f_1^{p,q}) &= \sum_q(-1)^q\varepsilon(L(f))\,\mathrm{tr}_{h^0(\mathrm{pt})}(h^q(\mathrm{pt}))\\ &= \varepsilon(L(f))L_{h^*}(\mathrm{id}_{pt})\,.\end{aligned}$$

Now the proof of Theorem 3.3 is complete. □

The following is an example of a theory for which $L_{h^*}(f) \neq L_{h^0}(f)$.

EXAMPLE 3.7. Let us fix a pair of CW-complexes (A, B), $B \subset A$. Then define a generalized cohomology theory h^* by the formula

$$h^i(X, Y) = H^i(X \times A, Y \times B; \mathbb{Q})\,,$$

where $H^*(-;\mathbb{Q})$ is the singular cohomology theory with rational coefficients. It is not difficult to check that for every fixed pair (A, B), h^* is a generalized cohomology theory. For example, taking the pair $(A, \emptyset)$ we get a generalized cohomology theory such that $h^*(X) = H^*(X;\mathbb{Q}) \otimes H^*(A;\mathbb{Q})$. In particular, $h^*(\mathrm{pt}) = H^*(A;\mathbb{Q})$, which implies that $L_{h^0}(\mathrm{id}_{pt}) = \chi(A)$ is the Euler characteristic of A, thus one can obtain any integer by an adequate choice of A.

Applying Theorems 3.1 and 3.3 we get formulas for the full and cut generalized Lefschetz numbers in this theory, namely

$$L_{h^*}(f) = L(f) \neq L_{h^0}(f) = \chi(A)L(f)\,,$$

if $\chi(A) \neq 1$.

4. Generalized Lefschetz numbers for stable cohomotopy

To start, recall that for a finite group G, the full generalized Lefschetz number of an equivariant map f in the stable equivariant cohomotopy theory graded by $i \in \mathcal{I}$, or in the same theory tensored with the rational numbers, $\omega_G^* \otimes \mathbb{Q}$, is equal to the equivariant fixed point index. This follows from the Lefschetz-Hopf-Dold theorem (cf. [16, 19]). This means that it detects equivariantly the fixed points of f. Next, we shall prove the main result of this paper which states that for a finite group G, the cut generalized Lefschetz number of an equivariant map f in the stable equivariant cohomotopy theory tensored with the rational numbers, $\omega_G^* \otimes \mathbb{Q}$, is an invariant which detects the fixed orbits of f, i.e. orbits which are mapped by f into itself.

Let G be a finite group, i be an element of

$$\mathcal{I} = \begin{cases} \mathbb{N} \cup \{0\} & \text{or} \\ \mathrm{RO}(G)\,, & \end{cases}$$

and (X, Y) be a pair of G-spaces. Note that $\mathbb{N} \cup \{0\} \subset \mathrm{RO}(G)$ corresponds to the trivial representations of arbitrary dimension.

For $\alpha = [W] - [W'] \in \mathrm{RO}(G)$, we mean by $\omega_G^\alpha(X, Y)$ the α-th equivariant cohomotopy group of (X, Y) in the sense of [21], namely,

$$\omega_G^\alpha(X, Y) = \mathrm{colim}_V \left[S^{V \oplus W} \wedge (X/Y); S^{V \oplus W'} \wedge (X/Y) \right]_G \,,$$

where S^L means the one-point compactification of the G-module L, and the colimit is taken over a cofinite set of real representations V of G ordered by inclusion. If $\mathcal{I} = \mathbb{N} \cup \{0\}$, by definition, $\{\omega_G^i\}$ constitutes an equivariant cohomology theory which is stable with respect to suspensions by trivial representations (i.e. $\omega_G^i(X, Y) \cong \widetilde{\omega}_G^{i+n}(\mathbb{S}^n \wedge (X/Y))$ for every $n \in \mathbb{N}$), or with respect to the suspension by any representation if $\mathcal{I} = \mathrm{RO}(G)$ (see [21]). Note that $\omega_G^i(X, Y)$ is a module over the ring $\omega_G^0(\mathrm{pt}) = \mathrm{colim}_V\, [S^V; S^V]$ with a module structure induced by the equivariant map $X/Y \longrightarrow \mathrm{pt}$.

The fundamental property of the equivariant cohomotopy is given by Segal's theorem ([21], see also [9, 20] for a proof), which states

$$\omega_G^0(\mathrm{pt}) \cong A(G)\,,$$

where $A(G)$ is the Burnside ring of G (cf. [3] for its definition).

Recall also that for any equivariant cohomology theory h^*, there is an equivariant fixed point index $I_G(f, h^*)$, given by the Dold diagram, provided it is stable with respect to every orthogonal representation. We have the following theorem (cf. [15, 16]).

Theorem 4.1. *Let G be a finite group and ω_G^* be the equivariant stable cohomotopy theory. Assume that X is a finite G-CW-complex, $f\colon X \longrightarrow X$ is an equivariant map, and that $\omega_G^*(X)$ is a finitely generated $\omega_G^*(\mathrm{pt})$-module. Furthermore, assume that $\omega_G^*(X)$ is a projective, or flat in the sense of* [4], *$\omega_G^*(\mathrm{pt})$-module. Then the full Lefschetz number $L_{\omega_G^*}(f)$ exists, and we have the equality*

$$L_{\omega_G^*}(f) = I(f, \omega_G^*) \in \omega_G^0(\mathrm{pt}) = A(G)\,.$$

In particular, if $\mathrm{Fix}(f) = \emptyset$, *then* $L_{\omega_G^*}(f) = 0$.

Define $\mathcal{S}_G$ as the the set consisting of one representative of each conjugacy class of subgroups of G. The set $\mathcal{S}_G$ is partially ordered with the order given by

$$K \geq H \iff \text{there exists} \quad g \in G \quad \text{such that} \quad K \subset gHg^{-1}\,,$$
$$K > H \iff K \geq H \quad \text{and} \quad K \neq H\,.$$

For every subgroup $H \subset G$ let $\chi^H\colon A(G) \longrightarrow \mathbb{Z}$ denote the homomorphism defined by

$$\chi^H(G/K) = |(G/K)^H|\,,$$

on every elementary G-set (orbit) G/K. We use the standard notation $X^{(H)} = GX^H$, and $X^{\overline{(H)}} = G\big(\bigcup_{K>H} X^{\overline{(K)}}\big) \subset X^{(H)}$ (see [2, 3]). Using this notation, given an equivariant map $f\colon X \longrightarrow X$ and a subgroup $H \subset G$, we define $f_{(H)}$ to be the selfmap of the pair $(X^{(H)}, X^{\overline{(H)}})$ induced by f. As a corollary of Theorem 4.1, we obtain the following.

Corollary 4.2. *Let X and $f\colon X \longrightarrow X$ be as in* Theorem 4.1. *Then $L_{\omega_G^*}(f)$ is determined by the classical Lefschetz numbers of the restrictions of f to all subsets $X^{(H)}$. Moreover, as an element of the Burnside ring $A(G)$, it is equal to*

$$L_{\omega_G^*}(f) = \sum_{H \in \mathcal{S}_G} \frac{L(f_{(H)})}{|G/H|}\,(G/H)\,.$$

Proof. From Theorem 4.1 and [24], it follows that

$$\chi^H(L_{\omega_G^*(f)}) = \chi^H(I(f, \omega_G^*)) = I(f^H) = L(f^H)$$

for every subgroup $H \subset G$. On the other hand, the right hand side of the formula is an element of $A(G)$ and

$$\chi^H\left(\sum_{K \in \mathcal{S}_G} \frac{L(f_{(K)})}{|G/K|}\,(G/K)\right) = L\big(f^H\big)$$

(cf. [3, 12]); the statement is a consequence of the fact that the homomorphism $(\chi^H)_{H\in\mathcal{S}_G}\colon A(G)\longrightarrow\prod_{H\in\mathcal{S}_G}\mathbb{Z}$ is a monomorphism (cf. [5, 3]). $\square$

Now we turn to study the cut generalized Lefschetz number in the rationalized equivariant stable cohomotopy theory. This will let us get rid of the algebraic assumption in the hypothesis of Theorem 4.1.

In order to reduce the algebraic considerations in our problem, we have to take the torsion-free part of ω_G^* by tensoring it with the rational numbers. Note that for every $i\in\mathcal{I}$ and any pair (X,Y) of G-spaces, the group $\omega_G^i(X,Y)$, being abelian, can be seen as a $\mathbb{Z}$-module.

DEFINITION 4.3. Let (X,Y) be a pair of G-spaces. By $\overline{\omega}_G^*(X,Y)$ we denote the equivariant cohomology theory, graded by $\mathbb{N}\cup\{0\}$, defined by

$$\overline{\omega}_G^i(X,Y)=\omega_G^i(X,Y)\otimes_{\mathbb{Z}}\mathbb{Q}\,.$$

Let us recall that in the nonequivariant case, the nontorsion part of the stable cohomotopy theory after tensoring it with $\mathbb{Q}$ is isomorphic to the singular (cellular) cohomology with rational coefficients. In the equivariant case the situation is more complicated and, as we already said, an answer is given in the Segal and Kosniowski theorems ([21, Proposition 5], [9, 4.10]).

To state it, we recall that for a given subgroup $H\subset G$ and $N(H)=\{g\in G\mid gHg^{-1}\subset H\}$ its normalizer, the group $W(H)=N(H)/H$ might be called, by extension, the *Weyl group* of H in G.

Also, as we already said, a result of Dress states that the ring $A(G)\otimes\mathbb{Q}$ is equal to the function ring $\{\varphi\colon\mathcal{S}_G\longrightarrow\mathbb{Q}\}$, and for every subgroup $H\in\mathcal{S}_G$ the homomorphism $\chi_H\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{Q}}$ has the form $\varphi\mapsto\varphi(H)$ (cf. [5]). Denote it by $\overline{A}(G)$.

Theorem 4.4 (Segal, Kosniowski). *Let G be a finite group. For every $i\in\mathrm{RO}(G)$, $\dim(i)\neq 0$, the group $\omega_G^i(\mathrm{pt})$ is finite and, if $\dim(i)<0$, it is equal to 0. In particular, if $i\in\mathbb{Z}$ then*

$$\overline{\omega}_G^i(\mathrm{pt})=\begin{cases}0 & \text{if}\quad i\neq 0\,,\\ \overline{A}(G) & \text{if}\quad i=0\,.\end{cases}$$

Moreover, for any compact G-space X and every $n\in\mathbb{Z}$, we have

$$\overline{\omega}_G^n(X)=\bigoplus_{H\in\mathcal{S}_G}\omega^n\big(X^H/W(H)\big)\otimes\mathbb{Q}\,,$$

where the term on the right hand side is the classical nonequivariant stable cohomotopy theory.

We would like to describe the module structure of every $\overline{A}(G)$-module $\overline{\omega}_G^n(X)$, $n \in \mathbb{Z}$. First we recall that for every G-space X and every subgroup $H \subset G$, the inclusion $X^H \hookrightarrow X^{(H)}$ induces a homeomorphism $(X^H/W(H)) \approx X^{(H)}/G$.

Obviously, the isomorphism of 4.4 is induced by the inclusions $\iota_H \colon X^H \longrightarrow X$; more precisely, for every $n \in \mathbb{Z}$, $H \subset G$, and for any equivariant map $f\colon S^{V+n} \wedge X \longrightarrow S^V \wedge X$ representing an element in $\omega_G^n(X)$, its restriction $f^H\colon S^{V^H+n} \wedge X^H \longrightarrow S^{V^H} \wedge X^H$ defines a $W(H)$-equivariant map. If X has one orbit type (G/H) or, equivalently, if X^H is a free $W(H)$-complex, then f^H defines a map $f^H/W(H)\colon (S^{V^H+n} \wedge X^H)/W(H) \longrightarrow (S^{V^H} \wedge X^H)/W(H)$. Anyway, it defines an element of $\omega_{W(H)}^n(X^H)$ or, in the case of a free $W(H)$-action, an element of $\omega^n(X^H/W(H))$. One can show that for a given $\alpha \in \overline{A}(G)$ and $a = \{a_H\}_{H \in \mathcal{S}_G} \in \bigoplus_{H \in \mathcal{S}_G} \omega^n(X^H/W(H)) \otimes \mathbb{Q}$, we have that

$$\alpha \cdot a = \sum_{H \in \mathcal{S}_G} \chi^H(\alpha)\, a_H\,.$$

We are now in position to state the main theorem.

Theorem 4.5. *Let G be a finite group, X a finite G-CW-complex, $f\colon X \longrightarrow X$ an equivariant selfmap, and $\overline{\omega}_G^*$ the rationalized equivariant stable cohomotopy theory graded by $\mathbb{N} \cup \{0\}$. Then the generalized cut Lefschetz number $L_{\overline{\omega}_G^0}(f) \in \overline{A}(G)$ exists and is given by the formula*

$$L_{\overline{\omega}_G^0}(f)(H) = L(f^H/W(H)) = L(f^{(H)}/G)\,,$$

where $L(f^{(H)}/G)$ is the classical Lefschetz number of the map $f^{(H)}/G$ induced by f on the orbit space $X^{(H)}/G = X^H/W(H)$. In particular, if $L_{\overline{\omega}_G^0}(f)(H) \neq 0$ then there exist $x \in X^H$ and $h \in N(G_x)$ such that $f(x) = hx$ or, respectively, such that there exists an orbit $Gx \approx G/G_x$ of the action of G with $H \leq G_x$, satisfying $f(Gx) \subset Gx$.

Before proving Theorem 4.5, we give a geometric explanation of the formula for the generalized cut Lefschetz number $L_{\overline{\omega}_G^0}(f)$.

Corollary 4.6. *Let X and $f\colon X \longrightarrow X$ be as in* Theorem 4.5. *Then $L_{\overline{\omega}_G^0}(f)$ is determined by the classical Lefschetz numbers of the restrictions of the quotient map f/G to all subsets $X^{(H)}/G$. Moreover, as an element of the rational Burnside ring $\overline{A}(G)$, it is equal to*

$$L_{\omega_G^0}(f) = \sum_{H \in \mathcal{S}_G} L(f^H/W(H))\,\chi_H = \sum_{H \in \mathcal{S}_G} L(f^{(H)}/G)\,\chi_H\,,$$

where $\chi_H\colon \mathcal{S}_G \longrightarrow \mathbb{Q}$ is a function equal to 1 at H and 0 otherwise.

We can now pass to the proof of Theorem 4.5.

Proof. First we show the existence of the generalized Lefschetz number $L_{\overline{\omega}_G^0}(f)$ in the sense of [23, 4]. To do this, let us first remark that

$$\overline{A}(G) = \bigoplus_{H \in \mathcal{S}_G} \mathbb{Q}\,, \tag{4.7}$$

where the decomposition is as a direct sum of ideals, i.e. the summand corresponding to $H \in \mathcal{S}_G$ is equal to the ideal $I_H = \{\varphi\colon \mathcal{S}_G \longrightarrow \mathbb{Q} \mid \varphi(H) = 0\}$. Moreover, every ideal $I \subset \overline{A}(G)$ is given by

$$I = I_B = \{\varphi\colon \mathcal{S}_G \longrightarrow \mathbb{Q} \mid \varphi|_B = 0\}\,,$$

for some subset $B \subset \mathcal{S}_G$. Consequently, for every ideal $I = I_B$ there exists an ideal I', namely $I_{\mathcal{S}_G \setminus B}$, such that $\overline{A}(G) = I \oplus I'$. This shows that every submodule of a free $\overline{A}(G)$ module is projective, because every ideal of $\overline{A}(G)$ is a direct summand in $\overline{A}(G)$. To see this it is enough to take the resolution $0 \longrightarrow \ker\alpha \longrightarrow F \longrightarrow M \longrightarrow 0$ of the module M. Note that if M is finitely generated then so is also F. By the above, $\ker\alpha$ is projective and finitely generated if F is finitely generated, since $\overline{A}(G)$ is a noetherian ring. It is enough to show that for a finite G-CW-complex X, $\overline{\omega}_G^*(X)$ is a finitely generated $(\overline{\omega}_G^0(\mathrm{pt}) = \overline{A}(G))$-module. The latter follows from the fact that X has a finite cover consisting of G-sets, G-homotopy equivalent to orbits, and that for every subgroup $H \subset G$, $\overline{\omega}_G^*(G/H) = \overline{\omega}_H^*(\mathrm{pt}) = \overline{\omega}_H^0(\mathrm{pt}) = \overline{A}(H)$, as it follows from the Segal theorem. Obviously, $\overline{A}(H)$, with the $\overline{A}(G)$-module structure given by the homomorphism $\overline{\mathrm{Res}}_G^H = \mathrm{Res}_G^H \otimes \mathbb{Q}\colon \overline{A}(G) \longrightarrow \overline{A}(H)$, is a finitely generated $\overline{A}(G)$-module.

We are still left with the task of deriving $\mathrm{tr}_{\overline{A}(G)}(\overline{f}^n)$ for the endomorphism $\overline{f}^n\colon \overline{\omega}_G^n(X) \longrightarrow \overline{\omega}_G^n(X)$ induced by f. This is essentially a consequence of Theorem 4.4, but in its statement there is no information about the induced map. To compute this trace we have to calculate $\mathrm{tr}_{\overline{A}(G)}(\overline{f}^n)(H)$, for every $H \in \mathcal{S}_G$ and $n \in \{0\} \cup \mathbb{N}$. We shall do it by carefully following the argument of the proof of Theorem 2.4 of [9], adapting and restricting it to our case. Kosniowski used the localization technique and got a more general statement on the localization of equivariant stable cohomotopy theory at any prime ideal of $A(G)$; we only need the special case when the ideal is maximal, i.e. when its characteristic is equal to 0 (cf. [9]).

First note that the localization $A(G)_{(I_H)}$ of $A(G)$ at a maximal ideal I_H is equal to $\mathbb{Q}$ and we have the isomorphism

$$A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{H \in \mathcal{S}_G} \mathbb{Q} \cong \bigoplus_{H \in \mathcal{S}_G} I'_H\, A(G) \otimes \mathbb{Q} \cong \bigoplus_{H \in \mathcal{S}_G} A(G)_{(I_H)}\,. \tag{4.8}$$

This ring decomposition leads to the decomposition

$$\omega_G^n(X) \otimes \mathbb{Q} \cong \bigoplus_{H \in \mathcal{S}_G} I'_H \omega_G^n(X) \otimes \mathbb{Q} \cong \bigoplus_{H \in \mathcal{S}_G} \omega_G^n(X)_{(I_H)}\,. \tag{4.9}$$

The above isomorphisms are functorial, thus they can be used to derive the trace $\mathrm{tr}_{\overline{A}(G)}(\overline{f}^n)$. Since the isomorphism in (4.8), and consequently (4.9), comes from the embedding $A(G) \longrightarrow \bigoplus_{H\in\mathcal{S}_G}\mathbb{Z}$, we have

$$\begin{aligned}\mathrm{tr}_{\overline{A}(G)}(\overline{f}^n) &= \sum_{H\in\mathcal{S}_G} \mathrm{tr}_{\overline{A}(G)/I'_H\overline{A}(G)}\left(\overline{f}^n \otimes \mathrm{id}_{\mathbb{Q}\,|\overline{\omega}^n_G(X)/I'_H\overline{\omega}^n_G(X)}\right)\chi_H \\ &= \sum_{H\in\mathcal{S}_G} \mathrm{tr}_{\overline{A}(G)_{(I_H)}}(f^n_{(I_H)})\,\chi_H\,, \end{aligned} \tag{4.10}$$

where the summands in the first sum are the traces of linear endomorphisms of the vector spaces $\overline{\omega}^n_G(X)/I'_H\overline{\omega}^n_G(X)$ over $\mathbb{Q} = \overline{A}(G)/I'_H\overline{A}(G)$ and, in the second, they are the traces of linear endomorphisms of the vector spaces $\omega^n_G(X)_{(I_H)}$ over $\mathbb{Q} = A(G)_{(I_H)}$.

As a consequence of (4.9) and (4.10), for a G-map f we get

$$\mathrm{tr}_{\overline{\omega}^0_G}(\overline{f}^n)(H) = \mathrm{tr}_{A(G)_{(I_H)}}(f^n_{(I_H)})$$

for every $H \in \mathcal{S}_G$. Kosniowski ([9, Section 4]) shows that for every subgroup $H \subset G$, the natural homomorphism $\omega^n_G(X) \longrightarrow \omega^n_{N(H)}(X^H)$ given by restriction becomes an isomorphism

$$\omega^n_G(X)_{(I_H)} \cong \omega^n_G(X^H)_{(I_{\mathbf{e}})}\,,$$

where $\mathbf{e}$ is the trivial subgroup. This yields $\mathrm{tr}_{\overline{\omega}^0_G}(\overline{f}^n) = \mathrm{tr}_{\mathbb{Q}}\,(f^H)^n_{(I_{\mathbf{e}})}$.

In the next step in Kosniowski's paper it is shown that the natural projection $p\colon X^H \longrightarrow X^H/W(H)$ becomes an isomorphism after localization

$$\omega^n_{N(H)}(X)_{(I_e)} \cong \omega^n_{N(H)}\big(X^H/W(H)\big)_{(I_e)}\,.$$

Consequently we get $\mathrm{tr}_{\overline{\omega}^0_G}(\overline{f}^n)(H) = \mathrm{tr}_{\omega_{N(H)_{(I_{\mathbf{e}})}}}(f^H/W(H))^n_{(I_{\mathbf{e}})}$.

Finally, in the last step of Kosniowski's considerations, it is shown that for a trivial G-space X the map which forgets the G-structure becomes an isomorphism after localization at the trivial subgroup. In particular we have

$$\omega^n_{N(H)}\big(X^H/W(H)\big)_{(I_H)} \cong \omega^n_{N(H)}\big(X^H/W(H)\big)_{(I_e)} \cong \omega^n\big(X^H/W(H)\big)\otimes\mathbb{Q}\,.$$

Comparing this with the previous formula, we get

$$\mathrm{tr}_{\overline{\omega}^0_G}(\overline{f}^n)(H) = \mathrm{tr}_{\mathbb{Q}}\big(f^H/W(H)\big)^n\,,$$

which proves the statement. $\square$

Now we give a formula for the cut generalized Lefschetz number analogous to that of Corollary 4.2 for the full generalized Lefschetz number.

Let $f_{\overline{(H)}}$ denote the selfmap of the pair $(X^{(H)}/G,\, X^{\overline{(H)}}/G)$ induced by f.

Let next $\mu\colon \mathcal{S}_G \times \mathcal{S}_G \longrightarrow \mathbb{Z}$ be the generalized Möbius function of the partially ordered set $\mathcal{S}_G$ (see [8] for references).

Corollary 4.11. *Let $f\colon X \longrightarrow X$ be a G-equivariant selfmap of a finite G-CW-complex X. Then*

$$L(f_{\overline{(H)}}) = \sum_{K \le H \in \mathcal{S}_G} \mu(H,K)\, L(f^H/W(H)) = \sum_{K \le H \in \mathcal{S}_G} \mu(H,K)\, L(f^{(H)}/G)\,.$$

Proof. Using the partial order of $\mathcal{S}_G$ and the induction argument we have

$$L(f^{(H)}/G) = \sum_{K \le H \in \mathcal{S}_G} L(f_{\overline{(H)}})\,,$$

by the additivity of Lefschetz number. The statement follows from the generalized Möbius formula applied to the partially ordered set $\mathcal{S}_G$. $\square$

Theorem 4.5 leads to the following simple example for which the full and cut equivariant generalized Lefschetz numbers are different.

EXAMPLE 4.12. Let $G = \mathbb{Z}_2 = \{-1, 1\}$, $X = \{-1, 1\}$ with the obvious G-structure, and let $f\colon X \longrightarrow X$ be the G-equivariant map defined by

$$f(-1) = 1\,, \qquad f(1) = -1\,.$$

From Theorems 4.1 and 4.5 it follows that

$$L_{\overline{\varpi}_G^*}(f) = I_G(f) = 0 \in \overline{A}(G)\,,$$

but, as a class function,

$$L_{\overline{\varpi}_G^0}(f)(H) = \begin{cases} 1 & \text{if} \quad H = \mathbf{e} \\ 0 & \text{if} \quad H = G \in \overline{A}(G)\,. \end{cases}$$

Moreover, the same example holds for $\mathbb{Z}_m$ instead of $\mathbb{Z}_2$. We have the following.

EXAMPLE 4.13. Let $G = \mathbb{Z}_m$ be the cyclic group of order m, $X = G$ and $g \in G$ be its generator. Then the translation $f : x \mapsto gx$ is an equivariant fixed point free selfmap of X. Therefore, we have

$$L_{\overline{\varpi}_G^*}(f) = I_G(f) = 0 \in \overline{A}(G)\,.$$

On the other hand, from Theorem 4.5 it follows that

$$L_{\overline{\omega}_G^0}(f)(H) = \begin{cases} 1 & \text{if} \quad H = \mathbf{e} \\ 0 & \text{if} \quad H \neq \mathbf{e} \,. \end{cases}$$

This is equal to the nonzero class function $|G|^{-1}(G) \in \overline{A}(G)$.

EXAMPLE 4.14. Let V be an orthogonal representation of a group G, $S(V)$ be the unit sphere therein and $f : S(V) \to S(V)$ be an equivariant map. We have

$$L_{\overline{\omega}_G^*}(f)(H) = L(f^H) = 1 - \nu(H)\deg(f^H)\,,$$

where $\nu(H) = (-1)^{\dim V^H - 1}$.

$$L_{\overline{\omega}_G^0}(f)(H) = L(f^{(H)}/G) = L\big(f^H/W(H)\big)\,. \tag{4.15}$$

To compute $\deg(f^{(H)}/G)$ we use the following result of [2]. For every finite group and a finite G-CW-complex X there is an isomorphism

$$H^*(X/G) \cong H^*(X)^G\,, \tag{4.16}$$

where H^* represents singular cohomology theory with rational coefficients, and the right hand side means the fixed point (linear) subspace of the action of G on $H^*(X)$ induced by the action of G on X.

If the action in V is given by $\rho\colon G \longrightarrow \mathrm{Iso}(V)$ and $H \subset G$ is a subgroup, there is a group homomorphism $\sigma\colon W(H) \longrightarrow \mathbb{Z}_2$ given by $\sigma(h) = \det(\rho(h))$, i.e. $\sigma(h) = 1$ if h preserves the orientation of $S(V^H)$, and $\sigma(h) = -1$ if h reverses the orientation of $S(V^H)$. Note that σ can only be nontrivial if $|W(H)|$ is even. Next define $\xi(H) = 0$ if σ is the trivial homomorphism, and $\xi(H) = -1$ if σ is nontrivial. Using this notation, (4.15) and (4.16) we have

$$L_{\overline{\omega}_G^0}(f)(H) = L\big(f^H/W(H)\big) = 1 - \xi(H)\nu(H)\deg(f^H)\,.$$

This last is different from $L_{\overline{\omega}_G^*}(f)(H) = 1 - \nu(H)\deg(f^{(H)})$ in general.

If, in particular, we take $V = \mathbb{R}^{2n+1}$ with the antipodal action of $G = \mathbb{Z}_2$ and an equivariant map $f : S(V) \to S(V)$ of degree -1, then $L_{\overline{\omega}_G^*}(f) = 0$, but $L_{\overline{\omega}_G^0}(f)(\mathbf{e}) = 1$.

Nowadays, we know more about the $W(H)$-modules $H^*(X^H;\mathbb{Q})$ and $H^*(X^{(H)})$, than about abelian groups $\omega_G(X)$ or $K_G(X)$, thus one can hardly expect applications of Theorem 4.5, which give new information about the fixed points of an equivariant map f, or of the map f/G induced by it on the orbit space. However, in some special cases, this theorem can be useful to study the induced maps f^* in a given cohomology theory as $\omega_G(X)$, or $K_G(X)$, and consequently to study the image of $[X, X]_G$

in $[X, X]$ for simple G-spaces, such as spheres or projective spaces. An attempt in this direction was made in [11] (Corollary. 7).

REMARK 4.17. We have restricted ourselves to the case of finite G-CW-complexes, because results follow there in a very convenient form. However, one might as well reproduce all results, either for compact G-spaces having the same G-homotopy type of G-CW-complexes or, more generally, for compact G-ENRs (and, with due care, also for compact G-ANRs).

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