Topological abelian groups and equivariant homology

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Abstract We prove an equivariant version of the Dold-Thom theorem by giving an explicit isomorphism between Bredon-Illman homology $\widetilde{H}_*^G(X;L)$ and equivariant homotopical homology $\pi_*(F^G(X,L))$, where G is a finite group and L is a G-module. We use the homotopical definition to obtain several properties of this theory and we do some calculations.

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0 Introduction

The presentation of homology using the Dold-Thom construction has been very useful in algebraic geometry. Lawson homology (see [14, 15, 9, 10]) was defined using this approach. In this paper we study the Dold-Thom-McCord theorem (see [19]) in the equivariant case.

Let G be a finite group. If L is a G-module, then one can define a coefficient system $\overline{L}$ on the category of canonical orbits of G by $\overline{L}(G/H) = L^H$, where L^H is the subgroup of fixed points of L under $H \subset G$. One then has an ordinary equivariant homology theory $H_*^G(-;\overline{L})$, called Bredon-Illman homology, whose associated coefficient system is precisely $\overline{L}$. Let X be a (pointed) G-space and let $F(X,L)$ be the topological abelian group generated by the points of X, with coefficients in L. Consider the subgroup $F^G(X,L)$ of equivariant elements, that is, the elements $\sum l_x x$ in $F(X,L)$ such that $l_{gx} = g \cdot l_x$. Then one can associate to X the homotopy groups $\pi_q(F^G(X,L))$, and one has that, if X is a G-CW-complex, then $\widetilde{H}_q^G(X;\overline{L})$ is isomorphic to $\pi_q(F^G(X,L))$. When G is the trivial group, $H_*^G(-;\overline{L})$ is singular homology and this statement is the classical Dold-Thom theorem [7], which was extended to the equivariant case by Lima-Filho [16] (when $L = \mathbb{Z}$ with trivial G-action) and by dos Santos [21] (when

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L is any G-module). Both the original result and its equivariant generalization were proved by showing that the homotopical definition satisfies the axioms of an ordinary or an equivariant homology theory, and then using a uniqueness theorem for homology theories.

In this paper, we prove the equivariant Dold-Thom theorem by giving an explicit isomorphism $H_q^G(X;\overline{L}) \longrightarrow \pi_q(F^G(X,L))$ for all q and any X of the homotopy type of a G-CW-complex (Theorem 1.2). This isomorphism is constructed in Section 2 in two steps as follows. Let $F(\mathcal{S}_*(X),L)$ be the (reduced) singular chain complex of X with coefficients in the G-module L. Since X is a G-space, we have an action of G on the singular simplexes of X, which we denote by $\sigma \mapsto g\cdot\sigma$. Let $F^G(\mathcal{S}_*(X),L)$ be the subcomplex of equivariant chains, that is, chains $\sum l_\sigma \sigma$ such that $l_{g\cdot\sigma} = g\cdot l_\sigma$. Using the theory of simplicial sets, we give an isomorphism between the homology of this chain complex $H_*(F^G(\mathcal{S}_*(X),L))$ and $\pi_*(F^G(X,L))$. Then we show that both the chain complex $F^G(\mathcal{S}_*(X),L)$ and Illman's chain complex (see Section 2), which defines $H_*^G(X;\overline{L})$ have the same universal property (Propositions 2.10 and 2.11) so that they are canonically isomorphic. For a different approach to the nonequivariant Dold-Thom theorem due to Friedlander and Mazur see [11].

In Section 1, we state the main theorem and prove that the theory $\mathbb{H}_*^G(X;\overline{L}) = \pi_*(F^G(X,L))$ is additive. In Section 3 we study the theory $\mathbb{H}_*^G(X;\overline{L})$ in the general context of equivariant homology theories and coefficient systems. To each equivariant homology theory $h_*^G(-)$ one can associate the G-module $h_0^G(G)$. We show (Theorem 3.5) that there is an isomorphism between the group of natural transformations $\mathrm{Nat}(h_*^G, \mathbb{H}_*^G(-;\overline{L}))$, and the group of G-homomorphisms $\mathrm{Hom}_G(h_0^G(G),L)$. We also show an analogous result for the classical equivariant homology theory of Eilenberg and Steenrod.

Finally, in Section 4 we show that there is some interesting information in the groups $\mathbb{H}_0^G(X;\overline{L})$, and using the transfer for ramified covering maps, we calculate $\mathbb{H}_q^{\mathbb{Z}_2}(X;\overline{L})$ for some $\mathbb{Z}_2$-spaces X.

In this paper we shall work in the category of *compactly generated weak Hausdorff spaces* (see e.g. [18]).

1 Equivariant McCord's topological groups and equivariant homology

In [19], for a pointed topological space X and an abelian group L, McCord introduced topological groups $F(X,L)$ consisting of functions $u : X \longrightarrow L$

such that $u(*)=0$ and $u(x)=0$ for all but a finite number of elements $x \in X$ (see [1] for further details). If G is a finite group that acts continuously on X leaving the base point fixed, and additively on L (on the left), then $F(X,L)$ has a natural (left) action of G given by defining $(g \cdot u)(x) = gu(g^{-1}x)$. This turns $F(X,L)$ into a topological $\mathbb{Z}[G]$-module.

Definition 1.1 Let G be a finite group, X a pointed G-space, and L a $\mathbb{Z}[G]$-module. Define

$$F^G(X,L) = \{u \in F(X,L) \mid u(gx) = gu(x) \text{ for all } x \in X, g \in G\}\,.$$

In other words, $F^G(X,L)$ consists of the functions $u \in F(X,L)$ that are G-functions, and coincides with the subspace $F(X,L)^G$ of fixed points under the G-action on $F(X,L)$ given above.

Given a $\mathbb{Z}[G]$-module L, there is a covariant coefficient system $\overline{L}$ called the system of invariants of L, which is given by taking the fixed point subgroups L^H of L under all subgroups $H \subset G$ (see 3.2 3 below).

If X is a pointed G-space, then we denote by $\mathcal{S}_q(X)$ the set of singular q-simplexes in X, which has an obvious G-action. We may define a chain complex by taking as q-chains the elements of $F^G(\mathcal{S}_q(X), L)$ (here $\mathcal{S}_q(X)$ is taken with the discrete topology; see next section). The boundary operator is given as the restriction of the boundary operator of the singular chain complex of X. We show below (Theorem 2.9) that this chain complex is naturally isomorphic to Illman's chain complex [13], which defines the Bredon-Illman equivariant homology of X with coefficients in $\overline{L}$, $H_q^G(X;\overline{L})$.

The main theorem of the paper is the following.

Theorem 1.2 *Let X be a pointed G-space of the same homotopy type of a G-CW-complex. Then there is an isomorphism $\widetilde{H}_q^G(X;\overline{L}) \longrightarrow \pi_q(F^G(X,L))$ given by sending a homology class $[u]$ represented by a cycle $u = \sum l_\sigma \sigma$ all of whose faces are zero, to the map $\overline{u} : (\Delta^q, \dot{\Delta}^q) \longrightarrow (F^G(X,L),*)$ such that $\overline{u}(t) = \sum l_\sigma \sigma(t)$.*

In particular, when G is the trivial group and $L = \mathbb{Z}$, this will give a new proof of the classical Dold-Thom theorem, since $\mathrm{SP}^\infty X \approx F(X,\mathbb{N}) \simeq F(X,\mathbb{Z})$.

Definition 1.3 Define the (reduced) *equivariant homology theory* $\widetilde{\mathbb{H}}_*^G$ *with coefficients in the coefficient system* $\overline{L}$ by

$$\widetilde{\mathbb{H}}_q^G(X;\overline{L}) = \pi_q(F^G(X,L))\,.$$

As usual, $\mathbb{H}_q^G(X;\overline{L}) = \widetilde{\mathbb{H}}_q^G(X^+;\overline{L})$, where $X^+ = X \sqcup \{*\}$ has the obvious extended action.

We devote the next section to the proof of Theorem 1.2. Meanwhile, we make some further considerations on our equivariant homology groups.

We shall now prove that our equivariant homology theory $\mathbb{H}_*^G$ is additive. To prove that, we need the following concept. Let X_α, $\alpha \in \Lambda$, be a family of pointed G-spaces and let L be a $\mathbb{Z}[G]$-module. Then we have (algebraically) the direct sum $F = \bigoplus_{\alpha\in\Lambda} F^G(X_\alpha, L)$. In order to furnish it with a convenient topology, take

$$F^n = \{(u_\alpha) \in F \mid \#\{\alpha \in \Lambda \mid u_\alpha \neq 0\} \leq n\}\,.$$

Then obviously $F^n \subset F^{n+1}$, and $\bigcup_n F^n = F$. For each n, there is a surjection

$$\coprod_{\alpha_1,\ldots,\alpha_n\in\Lambda} (X_{\alpha_1} \times L) \times \cdots \times (X_{\alpha_n} \times L) \twoheadrightarrow F^n\,.$$

Furnish F^n with the identification topology and F with the topology of the union of the F^ns. One clearly has the following.

Lemma 1.4 *There is an isomorphism of topological groups*

$$\bigoplus_{\alpha\in\Lambda} F^G(X_\alpha, L) \cong F^G(\bigvee_{\alpha\in\Lambda} X_\alpha, L))$$

induced by the inclusions $X_\alpha \hookrightarrow \bigvee X_\alpha$.

Proof: The inverse is given by the restrictions $F^G(\bigvee X_\alpha, L) \longrightarrow F^G(X_\alpha, L)$, $u \mapsto u|_{X_\alpha}$. ■

Since obviously $\pi_q(\bigoplus_\alpha F^G(X_\alpha, L)) \cong \bigoplus_\alpha \pi_q(F^G(X_\alpha, L))$, as a consequence, we have the next.

Theorem 1.5 *There is an isomorphism* $\widetilde{\mathbb{H}}_q^G(\bigvee_\alpha X_\alpha;\overline{L}) \cong \bigoplus_\alpha \widetilde{\mathbb{H}}_q^G(X_\alpha;\overline{L})$. ■

A more general case is as follows.

Definition 1.6 A G-space X is said to be G-0-*connected* (or G-*path connected*), if given any two points $x, y \in X$, then there exists a G-*path* $(\sigma, g) : x \simeq_G y$, that is, an element $g \in G$ and an ordinary path σ from x to gy. The relation $\simeq_G$ is clearly an equivalence relation, and the equivalence classes are called the G-*path components* of X (see [20]).

Assume that a G-space X is locally 0-connected, then, since every G-path component X_α of X is a topological sum of ordinary path components, there is a decomposition $X = \coprod_\alpha X_\alpha$. Given that $(\coprod_\alpha X_\alpha)^+ = \bigvee_\alpha X_\alpha^+$, a consequence of the additivity (1.4) is the following.

Corollary 1.7 *Let X be a locally 0-connected G-space. There is an isomorphism $\mathbb{H}_q^G(X;\overline{L}) \cong \bigoplus_\alpha \mathbb{H}_q^G(X_\alpha;\overline{L})$, where the G-spaces X_α denote the G-path components of X.* ■

2 Proof of the main theorem

In this section we prove Theorem 1.2. We use the techniques of simplicial sets for this. As already mentioned, in particular, this will provide a new proof of the classical Dold-Thom theorem [7].

We denote by Δ the category whose objects are the sets $\overline{n} = \{0, 1, 2, \dots, n\}$ and whose morphisms $f \in \Delta(\overline{m}, \overline{n})$ are monotonic functions $f : \overline{m} \longrightarrow \overline{n}$. Recall that a *simplicial set* is a contravariant functor $K : \Delta \longrightarrow \mathcal{S}\mathrm{et}$; we denote the set $K(\overline{n})$ simply by K_n. Let $\Delta[q]$ be the simplicial set $\Delta[q]_n = \Delta(-, \overline{q})$. We write $|K|$ for the *geometrical realization* given by

$$|K| = \bigsqcup_n (K_n \times \Delta^n)/\!\sim\,,$$

where $\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ i = 0, 1, 2, \dots, n,\ t_0 + t_1 + \cdots + t_n = 1\}$ is the standard n-simplex, and the equivalence relation is given by $(f^K(\sigma), t) \sim (\sigma, f_\#(t))$, $\sigma \in K_n$, $t \in \Delta^m$. Here $f_\#$ denotes the map affinely induced by f in the standard simplices. Denote the elements of $|K|$ by $[\sigma, t]$, $\sigma \in K_n$ and $t \in \Delta^n$.

We say that a simplicial set K is *pointed*, if it is provided with a morphism (natural transformation) $\Delta[0] \longrightarrow K$. This means that each set K_n has a base point and that for each $f : \overline{m} \longrightarrow \overline{n}$, the induced function $f^K : K_n \longrightarrow K_m$ is base-point preserving.

Definition 2.1 Given a pointed simplicial set K and an abelian group L, we define the simplicial abelian group $F(K, L)$ by $F(K, L)_n = F(K_n, L)$ (as mentioned in Section 1, where K_n has the discrete topology). The homomorphism induced by $f : \overline{m} \longrightarrow \overline{n}$ is $f_*^K : F(K_n, L) \longrightarrow F(K_m, L)$.

The proof of the following uses results of Milnor (see [17]).

Lemma 2.2 *The geometric realization $|F(K,L)|$ is an abelian topological group such that $[v,t]+[v',t]=[v+v',t]$.*

Proof: Consider the projections $p_i : F(K,L)\times F(K,L) \longrightarrow F(K,L)$, $i=1,2$, and the induced maps $|p_i| : |F(K,L)\times F(K,L)| \longrightarrow |F(K,L)|$, and define $\eta : |F(K,L)\times F(K,L)| \longrightarrow |F(K,L)|\times|F(K,L)|$ by

$$\begin{aligned}\eta[(v,v'),t] &= (|p_1|[(v,v'),t],|p_2|[(v,v'),t])\\ &= ([p_1(v,v'),t],[p_2(v,v'),t])\\ &= ([v,t],[v',t])\,.\end{aligned}$$

By [17, 14.3], η is a homeomorphism. The group structure $+$ in $|F(K,L)|$ is then given by the diagram

$$\begin{tikzcd} {|F(K,L)|\times|F(K,L)|} \arrow[r, "\eta^{-1}"] \arrow[dr, "+"'] & {|F(K,L)\times F(K,L)|} \arrow[d, "|\mu|"] \\ & {|F(K,L)|}\,, \end{tikzcd}$$

where $\mu : F(K,L)\times F(K,L) \longrightarrow F(K,L)$ is the simplicial group structure. ■

Proposition 2.3 *The topological groups $F(|K|,L)|$ and $|F(K,L)|$ are naturally isomorphic.*

Proof: Take

$$\varphi : F(|K|,L) \longrightarrow |F(K,L)|$$

given by

$$\varphi(u) = \sum_{[\sigma,t]\in|K|}[u[\sigma,t]\sigma,t]\,,$$

where $u : |K| \longrightarrow L$, $\sigma\in K_n$, and $t\in\Delta^n$ (thus $u[\sigma,t]\sigma\in F(K_n,L)$). Using Lemma 2.2 one shows that φ is a homomorphism. Thus we only need to check that $\varphi|_{F_k(|K|,L)}$ is continuous. Consider the diagram

$$\begin{tikzcd} (L\times|K|)^k \arrow[r, dashed] \arrow[d] & {|F(K,L)|^k} \arrow[d, "\mathrm{sum}"] \\ F_k(|K|,L) \arrow[r, "\varphi|_{F_k(|K|,L)}"'] & {|F(K,L)|}\,, \end{tikzcd}$$

where the map on the top is the product of the maps given by the next diagram.

$$\begin{tikzcd} L\times(\bigsqcup_n(K_n\times\Delta^n) \arrow[r, dashed] \arrow[d] & \bigsqcup_n(F(K_n,L)\times\Delta^n \arrow[d] \\ L\times|K| \arrow[r] & F(|K|,L)\,, \end{tikzcd}$$

where the top map is given by $(l,\sigma,t)\mapsto(l\sigma,t)$.

To see that φ is an isomorphism of topological groups, we define its inverse $\psi : |F(K,L)| \longrightarrow F(|K|,L)$ as follows. Take $v\in F(K_n,L)$; then $\psi[v,t] = \sum_{\sigma\in K_n} v(\sigma[\sigma,t]$. To see that ψ is well defined, take $v=\sum_{i=1}^r l_i\sigma_i$; then $f_*^K(v) = \sum_{i=1}^r l_i f^K(\sigma_i)$. Thus

$$\psi[f_*^K(v),t] = \sum_{i=1}^r [f^K(\sigma_i),t] = \sum_{i=1}^r [\sigma_i, f_\#(t)] = \psi[v,f_\#(t)]\,.$$

To see that ψ is continuous, consider the diagram

$$\begin{tikzcd}
\coprod_n (F(K_n,L)\times\Delta^n \arrow[r, dashed] \arrow[d] & \mathrm{SP}^\infty F(|K|,L) \arrow[d, "\mathrm{sum}"] \\
|F(K,L)| \arrow[r, "\psi"'] & F(|K|,L)\,,
\end{tikzcd}$$

where the top arrow given by $(\sum_i l_i,\sigma_i),t)\mapsto\langle l_1[\sigma_1,t],\dots\rangle$ is obviously continuous.

Moreover, ψ is a homomorphism. Namely, given $[v,t],[v',t']\in|F(K,L)|$, by Lemma 2.2, there exist unique elements, w,w',t'' such that $[v,t]=[w,t'']$, $[v',t']=[w',t'']$. Thus

$$\begin{aligned}
\psi([v,t]+[v',t'] &= \psi([w,t'']+[w',t'']) \\
&= \psi[w+w',t''] \\
&= \sum_\sigma (w+w')(\sigma)[\sigma,t''] \\
&= \psi[w,t'']+\psi[w',t''] = \psi[v,t]+\psi[v',t']\,.
\end{aligned}$$

In generators, we have that $\psi\varphi(l[\sigma,t]) = \psi[l\sigma,t] = l[\sigma,t]$, thus $\psi\circ\varphi$ is the identity. On the other hand, $\varphi\psi[v,t] = \varphi(\sum_{\sigma\in K_n} v(\sigma[\sigma,t]) = \sum_{\sigma\in K_n}[v(\sigma)\sigma,t] = [\sum_{\sigma\in K_n} v(\sigma)\sigma,t] = [v,t]$, where the next to the last equality follows by Lemma 2.2. ■

Definition 2.4 Let G be a finite group. A *(pointed) G-simplicial set* is a (pointed) simplicial set K such that G acts on each K_n and the action of every $g\in G$ determines a (pointed) isomorphism of K. In other words, it is a functor $K:\Delta\longrightarrow G\text{-}\mathcal{S}\mathrm{et}_*$.

Given a pointed G-simplicial set K and a $\mathbb{Z}[G]$-module L, then $F(K,L)$ inherits an action of G, as also do $|K|$, $F(|K|,L)$, and $|F(K,L)|$. By the naturality of the isomorphism of Proposition 2.3 we obtain the following.

Corollary 2.5 *Let K be a G-simplicial set. Then the topological groups $F(|K|, L)$ and $|F(K, L)|$ are G-isomorphic.* ■

Let K be a G-simplicial set and $H \subset G$ be a subgroup. Define K^H as the simplicial set such that $(K^H)_n = (K_n)^H$ (write this set as K_n^H). Then K^H is a simplicial subset of K and one easily verifies that $|K^H| = |K|^H$ and $|F^H(K, L)| = |F(K, L)|^H$. Thus we have the following.

Corollary 2.6 *Let K be a G-simplicial set. Then the topological groups $F^H(|K|, L)$, $|F(K, L)|^H$, and $|F^H(K, L)|$ are isomorphic.* ■

Let $\mathcal{A}$ be a simplicial abelian group. Recall that the q-homotopy group of $\mathcal{A}$ is defined by

$$\pi_q(\mathcal{A}) = H_q(N(\mathcal{A}), \widetilde{\partial}) ,$$

where $N(\mathcal{A})_q = \mathcal{A}_q \cap \ker d_0 \cap \cdots \cap \ker d_{q-1}$ and $\widetilde{\partial}_q = (-1)^q d_q$; here d_i is the ith face operator of $\mathcal{A}$. On the other hand, $\mathcal{A}$ can be seen as a chain complex, with $\partial : \mathcal{A}_q \longrightarrow \mathcal{A}_{q-1}$ given by $\sum_{i=0}^{q} (-1)^i d_i$.

We have the following result (cf. [17, 22.1]).

Proposition 2.7 *The canonical inclusion of chain complexes $N(\mathcal{A}) \hookrightarrow \mathcal{A}$ induces an isomorphism in homology.*

Proof: The chain complex $\mathcal{A}$ is filtered by chain complexes $\mathcal{A}^p$, where

$$\mathcal{A}_q^p = \{u \in \mathcal{A}_q \mid d_i(u) = 0, \ \ 0 \leq i < \min\{q, p\}\} .$$

The canonical inclusion $i^p : \mathcal{A}^{p+1} \hookrightarrow \mathcal{A}^p$ is a chain homotopy equivalence with inverse $r^p : \mathcal{A}^p \longrightarrow \mathcal{A}^{p+1}$ given by $r^p(u) = u - s_p d_p(u)$, where s_p is the pth degeneracy operator of $\mathcal{A}$. Obviously, $r^p \circ i^p = 1_{\mathcal{A}^{p+1}}$; conversely, $i^p \circ r^p$ is chain homotopic to $1_{\mathcal{A}^p}$ via the chain homotopy $h^p : \mathcal{A}_q^p \longrightarrow \mathcal{A}_{q+1}^p$ given by

$$h^p(u) = \begin{cases} 0 & \text{if } q < p \\ (-1)^p s_p(u) & \text{if } q \geq p \, . \end{cases}$$

■

Let G be a finite group and X a pointed G-space and let $\mathcal{S}(X)$ denote the *singular simplicial set* given for each q by

$$\mathcal{S}_q(X) = \{\sigma : \Delta^q \longrightarrow X \mid \sigma \text{ is a map}\} .$$

Then, in fact, $\mathcal{S}(X)$ is a pointed simplicial G-set with the usual simplicial structure. On the other hand, let $\mathcal{T}_q^G(X)$ be the *G-singular set* given by

$$\mathcal{T}_q^G(X) = \{T : \Delta^q \times G/H \longrightarrow X \mid T \text{ is an equivariant map and } H \subset G\},$$

$q \in \mathbb{N}$ (here Δ^q has trivial G-action).

Let L be a $\mathbb{Z}[G]$-module. Define $\widehat{F}(\mathcal{T}_q^G(X), L) = \{v : \mathcal{T}_q^G(X)^+ \longrightarrow L \mid v \in F(\mathcal{T}_q^G(X)^+, L),\ v(T) \in L^H \text{ if } T : \Delta^q \times G/H \longrightarrow X\}$. One easily sees that these groups are exactly Illman's groups $\widehat{C}_q^G(X; \overline{L})$ ([13, Def. 3.3]). As Illman does, we may declare that the generator lT is related to $l'T'$ if there exists a G-function $\alpha : G/H \longrightarrow G/H'$ such that the following diagram commutes

$$\begin{tikzcd} \Delta^q \times G/H \arrow[rr, "\mathrm{id} \times \alpha"] \arrow[dr, "T"'] & & \Delta^q \times G/H' \arrow[dl, "T'"] \\ & X & \end{tikzcd}$$

and $l' = \alpha_*(l) \in L^{H'}$ (see 3.2 3. below). Divide the group $\widehat{F}(\mathcal{T}_q^G(X), L)$ by the subgroup generated by the differences $lT - l'T'$ where either lT is related to $l'T'$ or $l'T'$ is related to lT, as well as by all elements lT such that $T : \Delta^q \times G/H \longrightarrow X$ is constant with value the base point x_0, to obtain the group $F'(\mathcal{T}_q^G(X), L)$. The following is clear.

Proposition 2.8 *The simplicial group $F^G(\mathcal{S}(X), L)$ and the graded group $F'(\mathcal{T}^G(X), L)$ are chain complexes.* ∎

In fact, the chain complex $F'(\mathcal{T}^G(X), L)$ is identical to Illman's chain complex $S^G(X, x_0; \overline{L})$ (cf. [13, p. 15]). Then we have the following.

Theorem 2.9 *The chain complexes $F'(\mathcal{T}^G(X), L)$ and $F^G(\mathcal{S}(X), L)$ are isomorphic.*

The proof of this theorem requires some preparation.

Let S be a pointed G-set with G-fixed base point σ_0. For each $\sigma \in S$, let $\mu_\sigma : L^{G_\sigma} \longrightarrow F^G(S, L)$ be given by $\mu_\sigma(l) = \sum_{i=1}^n (g_i l)(g_i \sigma)$, where $\{[g_1], \dots [g_n]\} = G/G_\sigma$. Then $\mu_{\sigma_0} = 0$ and $\mu_\sigma = \mu_{g\sigma} \circ \lambda_g$, where $\lambda_g(l) = gl$.

We have the following universal property of $F^G(S, L)$.

Proposition 2.10 *If A is an abelian group and there is a family of homomorphisms $\gamma_\sigma : L^{G_\sigma} \longrightarrow A$ satisfying $\gamma_{\sigma_0} = 0$ and $\gamma_\sigma = \gamma_{g\sigma} \circ \lambda_g$, then there is a unique $\gamma : F^G(S, L) \longrightarrow A$ such that $\gamma \circ \mu_\sigma = \gamma_\sigma$. Thus we have*

$$\begin{tikzcd} L^{G_\sigma} \arrow[r, "\mu_\sigma"] \arrow[dr, "\gamma_\sigma"'] & F^G(S,L) \arrow[d, dashed, "\gamma"] \\ & A\,. \end{tikzcd}$$

Proof: Take $u \in F^G(S, L)$. Write u as a sum $\sum_{i=1}^n l_i\sigma_i$, where no two of the elements σ_i are equal. Select one of them, for instance $\sigma = \sigma_1$. Since $u(g\sigma_1) = gl_1$, $g\sigma_1$ has to be some other σ_i. So the full orbit of σ appears in the sum, namely $\sum_{i=1}^k (g_i l)(g_i\sigma)$, where $\{[g_1], \ldots, [g_k]\} = G/G_\sigma$ is a subsum of u. Consider now $u' = u - \sum_{i=1}^k (g_i l)(g_i\sigma) \in F^G(S, L)$ and repeat the process until getting zero. Thus we rewrite $u = \sum_j \mu_{\sigma_j}(l_j)$ and define $\gamma(u) = \sum_j \gamma_{\sigma_j}(l_j)$. This is obviously well defined and has the desired property.

Observe that since u is G-equivariant, the sum $\sum_{i=1}^k (g_i l)(g_i\sigma)$ can be presented by different pairs (l, σ), namely, $\sum_{i=1}^k (g_i l)(g_i\sigma) = \sum_{i=1}^k (g_i gl)(g_i g\sigma)$, where $\{[g_1'], \ldots, [g_k']\} = G/G_{g\sigma}$ is also presented by the pair $(gl, g\sigma)$, $g \in G$. But since $\gamma_\sigma = \gamma_{g\sigma} \circ \lambda_g$, we have that $\gamma_{g\sigma_j}(gl_j) = \mu_{\sigma_j}(l_j)$, so the value of $\gamma(u)$ does not change. ∎

Let X be a pointed G-space. Then the groups $F'(\mathcal{T}_q^G(X), L)$ have the same universal property for $S = \mathcal{S}_q(X)$. Namely, take the homomorphisms $\nu_\sigma : L^{G_\sigma} \longrightarrow F'(\mathcal{T}_q^G(X), L)$ given by $\nu_\sigma(l) = [lT_\sigma]$, where $T_\sigma : \Delta^q \times G/G_\sigma \longrightarrow X$ is defined by $T_\sigma(t, [g]) = g\sigma(t)$. Then $\nu_{\sigma_0} = 0$ and $\nu_\sigma = \nu_{g\sigma} \circ \lambda_g$. Then the universal property is given by the following.

Proposition 2.11 *If A is an abelian group and there is a family of homomorphisms $\gamma_\sigma : L^{G_\sigma} \longrightarrow A$ satisfying $\gamma_{\sigma_0} = 0$ and $\gamma_\sigma = \gamma_{g\sigma} \circ \lambda_g$, then there is a unique $\gamma : F'(\mathcal{T}^G(X), L) \longrightarrow A$ such that $\gamma \circ \nu_\sigma = \gamma_\sigma$. Thus we have*

$$\begin{tikzcd} L^{G_\sigma} \arrow[r, "\nu_\sigma"] \arrow[dr, "\gamma_\sigma"'] & F'(\mathcal{T}_q^G(X),L) \arrow[d, dashed, "\gamma"] \\ & A\,. \end{tikzcd}$$

Proof: Given a G-map $T : \Delta^q \times G/H \longrightarrow X$, define $\sigma_T : \Delta^q \longrightarrow X$ by $\sigma_T(t) = T(t, [e])$, where $e \in G$ is the neutral element.

Define $\gamma : F'(\mathcal{T}^G(X), L) \longrightarrow A$ by $\gamma[lT] = \gamma_{\sigma_T}(\sum_{i=1}^n g_i l)$, where $G_{\sigma_T}/H = \{[g_1]_H, \ldots, [g_n]_H\}$. This is well defined; if lT is related to $l'T'$, we analyze two cases.

We assume first that $H \subset H'$ and that $\alpha : G/H \longrightarrow G/H'$ is the quotient map. Then $T' \circ (\mathrm{id} \times \alpha) = T$, thus $T(t, [g]_H) = T'(t, [g]_{H'})$, and $l' = \alpha_*(l) = \sum_{j=1}^r h'_j l$, where $H'/H = \{[h'_1]_H, \dots, [h'_r]_H\}$. Notice that therefore $\sigma_{T'} = \sigma_T$, so that we have $H \subset H' \subset G_{\sigma_T}$. Let $G_{\sigma_T}/H' = \{[g'_1]_{H'}, \dots, [g'_m]_{H'}\}$. Therefore, $G_{\sigma_T}/H = g'_1(H'/H) \cup \cdots \cup g'_m(H'/H)$. Hence

$$\gamma[lT] = \gamma_{\sigma_T}\left(\sum_{j=1}^r (g'_1 h'_j l) + \cdots + \sum_{j=1}^r (g'_m h'_j l)\right)$$

On the other hand,

$$\gamma[l'T'] = \gamma_{\sigma_T}\left(\sum_{i=1}^m (g'_i l')\right),$$

but $l' = \sum_{i=1}^r h'_i l$. Thus $\gamma[lT] = \gamma[l'T']$.

Now assume that $H' = g_0^{-1} H g_0$ and $\alpha = \rho_{g_0} : G/H \longrightarrow G/H'$ is given by right translation with g_0. Then $\alpha_* : L^H \longrightarrow L^{H'}$ is given by left translation with g_0^{-1}, namely $\alpha_*(l) = g_0^{-1} l$. Hence $l' = g_0^{-1} l$ and $\sigma_{T'} = g_0^{-1}\sigma_T$. Moreover, $G_{\sigma_{T'}} = g_0^{-1} G_{\sigma_T} g_0$. Thus, if $G_{\sigma_T}/H = \{[g_1]_H, \dots, [g_n]_H\}$, then $G_{\sigma_{T'}}/H' = \{[g_0^{-1} g_1 g_0]_{H'}, \dots, [g_0^{-1} g_n g_0]_{H'}\}$, and so

$$\begin{aligned}
\gamma[l'T'] &= \gamma_{\sigma_{T'}}\left(\sum_{i=1}^n g_0^{-1} g_i g_0 l'\right) = \gamma_{g_0^{-1}\sigma_T}\left(\sum_{i=1}^n g_0^{-1} g_i l\right) \\
&= \gamma_{\sigma_T} g_0 \left(\sum_{i=1}^n g_0^{-1} g_i l\right) = \gamma_{\sigma_T}\left(\sum_{i=1}^n g_i l\right) = \gamma[lT].
\end{aligned}$$

Obviously, γ has the desired properties. ■

Proof of Theorem 2.9: The isomorphism $F'(\mathcal{T}_q^G(X), L) \longrightarrow F^G(\mathcal{S}_q(X), L)$ in the previous corollary, as provided by the universal property, is given by

$$[lT] \longmapsto \sum_{i=1}^n (g_i l)(g_i \sigma_T),$$

where $T : \Delta^q \times G/H \longrightarrow X$, $l \in L^H$, and $G/H = \{[g_1]_H, \dots, [g_n]_H\}$. One easily verifies that this is a chain map. ■

The following proposition is a generalization to the equivariant case of a theorem of Milnor (see [17, 16.6]).

Proposition 2.12 *Let X be a pointed G-space of the same homotopy type of a G-CW-complex. Then $\rho : |\mathcal{S}(X)| \longrightarrow X$ given by $\rho[\sigma, t] = \sigma(t)$ is a G-homotopy equivalence.*

Proof: Let $H \subset G$ be any subgroup. Note first, as already mentioned before, that the identity induces a homeomorphism $|K^H| \approx |K|^H$ for any simplicial G-set K. On the other hand, one also has a canonical isomorphism of simplicial sets $\mathcal{S}(X^H) \cong \mathcal{S}(X)^H$. We have that Milnor's map $\rho : |\mathcal{S}(X)| \longrightarrow X$, being natural, is a G-map. On the other hand, again by the naturality, it restricts to $\rho_H : |\mathcal{S}(X^H)| \longrightarrow X^H$, which by a theorem of Milnor is a homotopy equivalence. Therefore, one has the following commutative triangle

$$\begin{tikzcd} |\mathcal{S}(X)|^H \arrow[r, "\rho^H"] & X^H \\ |\mathcal{S}(X^H)| \arrow[u, "\approx"] \arrow[ur, "\rho_H"'] & \end{tikzcd}$$

where the vertical arrow is a homeomorphism, as mentioned above. Hence, ρ^H is a homotopy equivalence for every $H \subset G$. By a result of Bredon [4, II(5.5)], then ρ is a G-homotopy equivalence. ∎

Proof of Theorem 1.2: We shall give an isomorphism

$$\widetilde{H}^G(X; \overline{L}) \cong H_q(F^G(\mathcal{S}_q(X), L)) \longrightarrow \pi_q(F^G(X, L)) = \widetilde{\mathbb{H}}_q^G(X; \overline{L}).$$

Here the left-hand side is the Bredon-Illman (reduced) homology of X, and the first isomorphism follows from the natural isomorphism of Theorem 2.9.

To construct the arrow, we shall give several isomorphisms as depicted in the following diagram, where $H \subset G$ is any subgroup.

$$\begin{tikzcd} H_q(F^H(\mathcal{S}(X), L)) \arrow[d, dashed] & \pi_q(F^H(\mathcal{S}(X), L)) \arrow[l, "i_*", "\cong"'] \arrow[r, "\Psi", "\cong"'] & \pi_q(\mathcal{S}(|F^H(\mathcal{S}(X), L)|)) \arrow[d, "\cong"', "\Phi"] \\ \pi_q(F^H(X, L)) & \pi_q(F^H(|\mathcal{S}(X)|, L)) \arrow[l, "\cong", "\rho_*"'] & \pi_q(|F^H(\mathcal{S}(X), L)|) \arrow[l, "\cong", "\psi_*"'] \end{tikzcd}$$

By Proposition 2.7, i_* is an isomorphism. In particular, this shows that every cycle in $\widetilde{H}^G(X; \overline{L})$ is represented by a chain u, all of whose faces are zero. We shall call this a *special chain*.

The homomorphism Ψ, which is given by $\Psi(u)[t] = [u, t]$, where u is a special q-chain and $t \in \Delta^q$, is an isomorphism, as follows from [17, 16.6].

In order to define Φ, we must express $\Psi(u)$ as a map $\gamma : (\Delta[q], \dot{\Delta}[q]) \longrightarrow (\mathcal{S}|F^H(\mathcal{S}(X), L)|, *)$. By the Yoneda lemma, γ is the unique map such that $\gamma(\delta_q) = \Psi(u)$, where $\delta_q = \mathrm{id} : \overline{q} \longrightarrow \overline{q}$. The homomorphism Φ, defined by $\Phi[\gamma][f, s] = \gamma(f)(s)$, for $f \in \Delta[q]_n$ and $s \in \Delta^n$, is given by the adjunction between the realization functor and the singular complex functor (see [17, 16.1]).

That ψ_* is an isomorphism follows from Proposition 2.3. Finally, the homomorphism ρ_* is an isomorphism by 2.12.

Chasing along this diagram and using the homeomorphism $|\Delta[q]| \longrightarrow \Delta^q$ given by $[f, t] \mapsto f_\#(t)$, one obtains that the isomorphism maps a homology class $[u] \in H_q(F^H(\mathcal{S}(X), L))$ represented by a special chain $u = \sum_\sigma l_\sigma \sigma$, to the map $\overline{u} : (\Delta^q, \dot{\Delta}^q) \longrightarrow (F^H(X, L), *)$ given by $\overline{u}(t) = \sum l_\sigma \sigma(t)$. ■

3 Equivariant homology and other coefficient systems

In this section we recall the general concept of a (covariant) coefficient system for a group G and give explicit examples of ordinary equivariant homology theories with particular systems as coefficients.

Definition 3.1 Let G be a finite group. A *covariant coefficient system* for G, M, is a covariant functor from the category of homogeneous sets G/H, H a subgroup of G, and G-functions $\alpha : G/H \longrightarrow G/K$, to the category of abelian groups. We denote the induced homomorphisms by $\alpha_* : M(G/H) \longrightarrow M(G/K)$. There is, of course, a category $\mathrm{Coeffsys}_G$ of covariant coefficient systems for G.

Examples 3.2 Let L be a $\mathbb{Z}[G]$-module. We have the following associated covariant coefficient systems.

1. To start with, consider the *constant* coefficient system, denoted again by L, with value the group L given by $L(G/H) = L$ for all $H \subset G$ and for $\alpha : G/H \longrightarrow G/K$, $H \subset K \subset G$, by $\alpha_* = 1_L$. This is realized by the theory
$$h_q^G(X; L) = H_q(X/G; L)\,,$$
where the second term is singular homology with coefficients in L.
2. Another useful example is the following. Let $\underline{L}$ be given by defining $\underline{L}(G/H) = L_H$, the quotient of L by the subgroup generated by the elements of the form $l - h \cdot l$, $l \in L$, $h \in H$, and for $a : G/H \longrightarrow G/K$, $a_* : L_H \longrightarrow L_K$ is the quotient homomorphism. We call it the system of

coinvariants of L. If, in particular, L has trivial action, then this coefficient system coincides with the constant system given in 1. The coefficient system $\underline{L}$ is realized by a theory constructed by Eilenberg and Steenrod [8] (see also [6]) as follows. Consider the usual singular complex $S_*(X)$ on which G acts (here G acts on X on the right). Take the complex $S_*(X) \otimes_{\mathbb{Z}[G]} L$. Define the *classical ordinary equivariant homology theory* by

$$\mathcal{H}_q^G(X;\underline{L}) = H_q(S_*(X) \otimes_{\mathbb{Z}[G]} L)\,.$$

If $X = G/H$, then $\mathcal{H}_q^G(G/H;\underline{L}) = 0$ if $q > 0$ and $\mathcal{H}_0^G(G/H;\underline{L}) = S_0(G/H) \otimes_{\mathbb{Z}[G]} L = \mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]} L \cong L_H$.

3. The other example that we consider is $\overline{L}$, that we call the system of *invariants* of L.[†] It is defined as follows. Take $\overline{L}(G/H) = L^H = \{l \in L \mid h \cdot l = l \text{ for all } h \in H\}$. For $H \subset K \subset G$ and $\alpha : G/H \longrightarrow G/K$ the quotient function, take $\alpha_*(l) = \sum_{k\in[K/H]} kl$, where $[K/H]$ denotes a set of representatives in K of the cosets of H in K. For any equivariant map $\alpha : G/H \longrightarrow G/K$ one has that $H \subset {}^aK = aKa^{-1}$ for some $a \in G$, and so α is the composite of the quotient map $G/H \longrightarrow G/{}^aK$ and the obvious bijection $G/{}^aK \approx G/K$. Hence we define $a_* : L^H \longrightarrow L^K$ as the composite of $L^H \longrightarrow L^{{}^aK}$ given by $l \mapsto \sum_{k\in[{}^aK/H]} kl$ followed by the isomorphism $L^{{}^aK} \longrightarrow L^K$ given by $l \mapsto a^{-1}l$. As shown in the main theorem, our theory

$$\mathbb{H}_q^G(X;\overline{L}) = \pi_q(F^G(X,L))$$

realizes $\overline{L}$. Observe that if L has trivial G-action, then $\overline{L}$ defines the *semiconstant* coefficient system, namely, $\overline{L}(G/H) = L$ for all $H \subset G$ and for $\alpha : G/H \longrightarrow G/K$, define $\alpha_* : L \longrightarrow L$ by $\alpha_*(l) = (|K|/|H|)l$.

As shown in [12], the system of invariants is right adjoint to the forgetful functor from the category of coefficient systems to the category of $\mathbb{Z}[G]$-modules. Similarly, one can show that the system of coinvariants is left adjoint to the same forgetful functor. One has the following.

Proposition 3.3 *There are natural bijections*

$$\mathrm{Coeffsys}_G(\underline{L}, M) \cong \mathrm{Hom}_G(L, M(G))\,,$$
$$\mathrm{Coeffsys}_G(M, \overline{L}) \cong \mathrm{Hom}_G(M(G), L)\,.$$

∎

[†]Notice that the system of invariants of L is denoted by $\underline{L}$ in [16] and [21].

As shown by Illman [13], given any (covariant) coefficient system, there is an ordinary equivariant homology theory $H^G_*(-;M)$ with M as coefficients. Moreover, given any natural transformation $\mu : M \longrightarrow N$ of coefficient systems, there is an extension of it to a natural transformation $\widehat{\mu} : H^G_*(-;M) \longrightarrow H^G_*(-;N)$ of (ordinary) homology theories. On the other hand, Willson [22] constructs for any equivariant homology theory h^G a (natural) spectral sequence such that $E^2_{pq} \cong H^G_p(X;h^G_q(\cdot))$, where $h^G_q(\cdot)$ is the coefficient system given by $G/H \mapsto h^G_q(G/H)$, that converges to $h^G_{p+q}(X)$. Clearly, if h^G is ordinary, then this spectral sequence collapses. This shows that if $\mu : h^G \longrightarrow k^G$ is a natural transformation of ordinary homology theories such that it induces the zero transformation in coefficients $h^G(\cdot) \longrightarrow k^G(\cdot)$, then μ itself has to be zero. We thus have the following.

Proposition 3.4 *There is an isomorphism between natural transformations of coefficient systems and natural transformations of ordinary homology theories. More precisely, we have*

$$\mathrm{Coeffsys}_G(M,N) \cong \mathrm{Nat}(H^G_*(-;M), H^G_*(-;N))\,.$$

■

By this and the adjunction result 3.3 above, we have the following.

Theorem 3.5 *There are isomorphisms*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(L, h^G_0(G)) \cong \mathrm{Nat}(\mathcal{H}^G_*(-;\underline{L}), h^G_*)\,,$$
$$\mathrm{Hom}_{\mathbb{Z}[G]}(h^G_0(G), L) \cong \mathrm{Nat}(h^G_*, \mathbb{H}^G_*(-;\overline{L}))\,,$$

where h^G_* represents any ordinary equivariant homology theory, $\mathcal{H}^G_*(-;\underline{L})$ the classical equivariant homology theory, and $\mathbb{H}^G_*(-;\overline{L})$ the homology theory that we defined in 1.3. ■

4 Some computations of the equivariant homology groups $\mathbb{H}^G_*(-;\overline{L})$

In this section, we compute some equivariant homology groups, especially in dimension 0, for the theory defined in 1.3.

Theorem 4.1 *Let X be a 0-connected space with a free G-action, and let L be a $\mathbb{Z}[G]$-module. Then*

$$\mathbb{H}^G_0(X;\overline{L}) \cong L_G\,,$$

where $L_G = L/\langle l - g\cdot l\rangle$ is the quotient of L divided by the subgroup generated by the elements $l - g\cdot l$, $l \in L$, $g \in G$.

Proof: First observe that if the G-action on L is trivial, then the transfer of $p : X \longrightarrow X/G$ yields an isomorphism $t_p : F(X/G^+, L) \xrightarrow{\cong} F^G(X^+, L)$ (see [2, Thm. 6.2]), that implies the result.

For the general case, let $\{x_i\}$ be a set of representatives in X of the orbits (in X/G), and define

$$\varepsilon^G : F^G(X^+, L) \longrightarrow L_G \quad \text{by} \quad \varepsilon^G(u) = \sum_i \overline{u(x_i)}\,,$$

where $\overline{l} \in L_G$ denotes the class of $l \in L$. One easily verifies that ε^G is independent of the choice of the representatives x_i. We shall show below that ε^G is continuous, but for the time being, we assume it is.

Let $\alpha : L \longrightarrow F^G(X^+, L)$ be given by $\alpha(l) = \widetilde{lx_0}$, where

$$\widetilde{lx_0}(x) = \begin{cases} g\cdot l & \text{if } x = gx_0, \\ 0 & \text{if } x \notin \text{orbit}(x_0), \end{cases}$$

and x_0 is one of the representatives of the orbits taken above. We then have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d] & F^G(X^+, L) \arrow[d] \\ L_G \arrow[r, dashed, "\overline{\alpha}"'] & \pi_0(F^G(X^+, L))\,. \end{tikzcd}$$

Namely, if we take $h \in G$, we have to prove that both $\widetilde{lx_0}$ and $\widetilde{(hl)x_0}$ lie in the same path component of $F^G(X^+, L)$. Clearly, $\widetilde{(hl)x_0} = l\widetilde{(h^{-1}x_0)}$. Let $\sigma : I \longrightarrow X$ be a path from x_0 to $h^{-1}x_0$, and define $\widetilde{\sigma} : I \longrightarrow F^G(X^+, L)$ by $\widetilde{\sigma}(t) = \widetilde{l\sigma(t)}$. Then $\widetilde{\sigma}(0) = \widetilde{lx_0}$, and $\widetilde{\sigma}(1) = l\widetilde{(h^{-1}x_0)} = \widetilde{(hl)x_0}$.

In order to prove that $\widetilde{\sigma}$ is continuous, note first that $\widetilde{lx} = \sum_{g\in G}(g\cdot l)(gx)$, where

$$(lx)(x') = \begin{cases} l & \text{if } x' = x, \\ 0 & \text{if } x' \neq x, \end{cases}$$

and $t \mapsto (g\cdot l)(g\sigma(t))$ is continuous because the action of G on X is continuous. Thus, $\widetilde{\sigma}$ is continuous and $\overline{\alpha}$ is well defined.

Consider $\varepsilon^G\overline{\alpha}(\overline{l}) = \varepsilon^G[\widetilde{lx_0}] = \overline{l}$. Thus $\varepsilon^G \circ \overline{\alpha} = 1_{L_G}$ and hence $\overline{\alpha}$ is injective.

To see that $\overline{\alpha}$ is also surjective, take any $u \in F^G(X^+, L)$ and call $l_i = u(x_i)$ for each representative x_i of the orbits in X. Then $u = \sum_i \widetilde{l_i x_i}$ and hence $\{\widetilde{lx_i}\}$, $l \in L$, is a set of generators of $F^G(X^+, L)$. Since $\overline{\alpha}$ is obviously a homomorphism, it is enough to show that every generator $[\widetilde{lx_i}]$ of $\pi_0(F^G(X^+, L))$ is in the image of $\overline{\alpha}$. Let now $\sigma : I \longrightarrow X$ be a path from x_0 to x_i, then $\widetilde{\sigma}(t) = \widetilde{l\sigma(t)}$ is a path from $\alpha(l) = \widetilde{lx_0}$ to $\widetilde{lx_i}$.

To finish the proof we have to show that ε^G is indeed continuous. Let $\pi : L \longrightarrow L_G$ be the quotient homomorphism and call $q = p_*|_{F^G(X^+,L)} : F^G(X^+, L) \longrightarrow F(X/G^+, L)$. Take $u \in F^G(X^+, L)$; then

$$\begin{aligned}
(\pi_* q(u))[x] &= \pi\left(\sum_{x' \in p^{-1}[x]} u(x')\right) \\
&= \pi\left(\sum_{g \in G} u(gx)\right) \\
&= \pi\left(\sum_{g \in G} g \cdot u(x)\right) \\
&= \sum_{g \in G} \pi\left(g \cdot u(x)\right) \\
&= |G|\overline{u(x)},
\end{aligned}$$

hence the image of $\pi_* \circ q$ lies in $F(X/G^+, |G|L_G)$ where one can divide by $|G|$. Call $\gamma_* : F(X/G^+, |G|L_G) \longrightarrow F(X/G^+, L_G)$ the homomorphism given by dividing the values of the elements by $|G|$. Then ε^G is the composite

$$F^G(X^+, L) \xrightarrow{\gamma_* \circ \pi_* \circ q} F(X/G^+, L_G) \xrightarrow{\varepsilon} L_G\ ,$$

where $\varepsilon : F(X/G^+, L_G) \longrightarrow L_G$ is the *augmentation* given by

$$\varepsilon(v) = \sum_{\overline{x} \in X/G} v(\overline{x})\,.$$

Since all maps in the composite are continuous, so is ε^G too. ■

Theorem 4.2 *Let X be a G-0-connected G-space with a single orbit type and let L have trivial G-action. Then $\mathbb{H}_0^G(X; \overline{L}) \cong L$.*

Proof: The result follows from the same arguments of 4.1, but, on the one hand, since L has trivial G-action, G-paths are good enough, and on the other, instead of dividing by the order of G, we divide by the order of any orbit. ■

As a consequence of 1.7 and 4.2, we have the next result.

Corollary 4.3 *Let X be a locally path-connected G-space and let X_α, $\alpha \in \Lambda$, be the G-path components of X such that each G-path component has a single orbit type. Then*

$$\mathbb{H}_0^G(X;\overline{L}) \cong \bigoplus_{\alpha\in\Lambda} L_\alpha\,, \qquad L_\alpha = L \quad \forall\alpha\,.$$

■

Remark 4.4 Call $\pi_0^G(Y)$ the set of G-path connected components of the G-space Y. There is a canonical function $\pi_0^G(Y) \longrightarrow \pi_0(Y/G)$ which is obviously surjective. It is also injective. To see this one has to apply the Covering Homotopy Theorem for orbit maps $q : Y \longrightarrow Y/G$ of Palais (see [5, II.7.3]). Namely, let x and y be two points in Y such that $q(x)$ and $q(y)$ are connected in Y/G by a path ω. Taking $X = G$ in Palais' theorem (in Bredon's notation), then ω can be seen as a homotopy $F' : X/G \times I \longrightarrow Y/G$. Taking $f : X \longrightarrow Y$ to be given by $f(g) = gx$, we have an equivariant map, so that the assumptions of the theorem are fulfilled (then $f' : X/G \longrightarrow Y/G$ chooses $q(x)$). Thus there exists an equivariant homotopy $F : X \times I \longrightarrow Y$ starting at F and covering F', that is, by restricting F to $\{e\}\times I$ we have a path $\widetilde{\omega}$ in Y starting at x and covering ω. Hence, $q\widetilde{\omega}(1) = \omega(1) = q(y)$, and so $\widetilde{\omega}(1) = gy$ for some $g \in G$. This means that x and y are in the same G-path component of Y.

By the previous remark, it is the trivial G-homology theory h_*^G given by $h_*^G(X;L) = H_*(X/G;L)$, the one that has the property that if X is locally 0-connected, then $h_0^G(X;L) \cong \oplus_{\pi_0^G(X)} L$, L some abelian group with no G-action. That is, the trivial zero-G-homology groups measure the G-path-connectedness. In what follows we analyze the groups $\mathbb{H}_0^G(X;\overline{L})$ in some special cases.

Let X be a G-space with (at least) one fixed point x_0. The inclusion $i : \mathbb{S}^0 \hookrightarrow X^+$ that sends one point to $+$ and the other to x_0 is an equivariant embedding whose image is a retract with the obvious retraction $r : X^+ \longrightarrow \mathbb{S}^0$. Thus we have that $r_* : F^G(X^+, L) \longrightarrow F(\mathbb{S}^0, L) = L$ is a split epimorphism. Thus $F^G(X^+, L) \cong L \times \ker r_*$. Thus we have the following.

Proposition 4.5 *Let X be a G-space with a fixed point x_0 under the G-action, and let L have a trivial G-action. Then $\mathbb{H}_0^G(X;\overline{L}) \cong L \times \pi_0(\ker r_*)$.*

■

In what follows we analyze the 0-connectedness of $\ker r_*$. In case that the G-0-connected G-space X has one fixed point x_0 (thus it is 0-connected) and G acts freely in the complement, let $\{x_i\}$ be a set of representatives of the orbits of X and let $\sigma_i : I \longrightarrow X$ be a path from x_0 to x_i for each i (σ_0 is the constant path). Assume that $u \in \ker r_*$; this means that $r_*(u) = \sum_{x\in X} u(x) = 0$. For each $x \in X$, $x = gx_i$ for some i and some unique $g \in G$. Let $\sigma_x : I \longrightarrow X$ be given by $\sigma_x(t) = g\sigma_i(t)$ and define $\widetilde{\sigma} : I \longrightarrow F^G(X^+, L)$ by $\widetilde{\sigma}(t) = \sum_{x\in X} u(x)\sigma_x(t)$. Then $\widetilde{\sigma}(t)$ is G-invariant. Namely, $\sum_{gx\in X} u(gx)\sigma_{gx}(t) = \sum_{x\in X} u(x)g\sigma_x(t)$, since $u(gx) = u(x)$ and $\sigma_{gx}(t) = g\sigma_x(t)$. On the other hand, $\widetilde{\sigma}(0) = \sum_{x\in X} u(x)x_0 = 0$, $\widetilde{\sigma}(1) = \sum_{x\in X} u(x)x = u$; moreover, for any $t \in I$, $r_*(\widetilde{\sigma}(t)) = \sum_{y\in X}(\sum_{x\in X} u(x)\sigma_x(t))(y) = \sum_{x\in X} u(x) = 0$ and hence $\widetilde{\sigma}(t) \in \ker r_*$ for every t. Hence we have the following.

Proposition 4.6 *Let X be G-0-connected G-space with one fixed point x_0 and such that G acts freely in the complement. Then $\ker(r_*)$ is 0-connected and so $H_0^G(X; L) \cong L$ if L has trivial G-action.* ■

It is quite straightforward to verify that the previous proof holds also if the G-0-connected G-space X has exactly one fixed point x_0 and for any other point x the isotropy group G_x is a fixed subgroup $H \subset G$. Thus we have the following too.

Proposition 4.7 *Let X be G-0-connected G-space with one fixed point x_0 and such that $G_x = H \subset G$ for all $x \neq x_0$, for some fixed subgroup H. Then $\ker(r_*)$ is 0-connected and so $H_0^G(X; L) \cong L$ if L has trivial G-action.* ■

The following proposition deals with the transfer for the ramified covering map $X \longrightarrow X/G$ studied in [2] and it will be useful below. It generalizes Theorem 6.2 therein.

Proposition 4.8 *Let G act on a space X such that the fixed point set X^G coincides with the base point $*$ and the action in the complement of X^G is free. Then the transfer for $p : X \longrightarrow X/G$ induces an isomorphism*

$$\tau : \widetilde{H}_n(X/G) \longrightarrow \widetilde{H}_n^G(X)\,.$$

Its inverse is given by $\frac{1}{|G|} \cdot p_*$.

Proof: In the McCord topological groups $F(X/G,\mathbb{Z})$ and $F^G(X,\mathbb{Z})$ every element is zero in the base point. Thus in the diagram

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "u"'] & & X/G \arrow[dl, "\overline{u}"] \\ & \mathbb{Z} & \end{tikzcd}$$

where u is G-invariant, u exists if and only if $\overline{u}$ exists. Since off the fixed point $*$ the multiplicity function of the $|G|$-fold ramified covering map $p : X \longrightarrow X/G$ is equal 1 (except in $*$, where it is $|G|$, but $u(*) = 0 = \overline{u}(*)$), the transfer $\tau : F(X/G,\mathbb{Z}) \longrightarrow F^G(X,\mathbb{Z})$ is given by $\tau(\overline{u}) = p \circ \overline{u}$ and thus it is an isomorphism. Since $\tau p_*(u)(\overline{x}) \in |G|\mathbb{Z}$, we can divide by $|G|$, which yields a continuous homomorphism. Thus the inverse is given by $u \mapsto \frac{1}{|G|} \cdot \overline{u}$. ■

The following theorem shows, in particular, that there is nontrivial information contained in the equivariant homology groups of dimension 0.

Theorem 4.9 *Let X be a G-CW-complex such that the fixed point set X^G is finite, say $X^G = \{x_1, x_2, , \dots, x_k\}$, and the action on the complement of X^G is free. Then*

$$\widetilde{\mathbb{H}}_0^G(X) \cong \bigoplus_{k-1} \mathbb{Z}_{|G|} \quad \text{and} \quad \tau : \widetilde{H}_n(X/G) \xrightarrow{\cong} \widetilde{\mathbb{H}}_n^G(X)\,, \quad \text{if } n \geq 1\,,$$

where $\mathbb{Z}_{|G|}$ is the cyclic group of order $|G|$.

Proof: Let $\widetilde{X}$ denote the G-space obtained by collapsing X^G in X to one (fixed) point, and consider the diagram of cofiber sequences of G-maps

$$\begin{tikzcd} X^G \arrow[r, hook] \arrow[d, equal] & X \arrow[r, two heads] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "p"] \\ X^G \arrow[r, hook] & X/G \arrow[r, two heads] & \widetilde{X}/G \end{tikzcd}$$

Applying equivariant homology in dimensions 0 and 1 we obtain

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\mathbb{H}}_1^G(X) \arrow[r] \arrow[d, shift right, "p_*"'] & \widetilde{\mathbb{H}}_1^G(\widetilde{X}) \arrow[r] \arrow[d, shift right, "p_*"'] & \bigoplus_{k-1} \mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{H}}_0^G(X) \arrow[r] & 0 \\ 0 \arrow[r] & \widetilde{H}_1(X/G) \arrow[r] \arrow[u, shift right, "\tau \cong"'] & \widetilde{H}_1(\widetilde{X}/G) \arrow[r] \arrow[u, shift right, "\tau \cong"'] & \bigoplus_{k-1} \mathbb{Z} \arrow[r] & 0 & \end{tikzcd}$$

By 4.8, the transfer τ in the middle is an isomorphism and so, by the five lemma, the transfer τ on the left is an isomorphism too, thus the second assertion in the case $n = 1$ follows. Moreover, the last epimorphism on the bottom right splits. Hence, the diagram transforms into the following.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathbb{H}}_1^G(X) \arrow[r] \arrow[d, "p_*"'] & \widetilde{H}_1(X/G) \oplus \bigoplus_{k-1} \mathbb{Z} \arrow[r] \arrow[d, "|G|\cdot 1"] & \bigoplus_{k-1} \mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{H}}_0^G(X) \arrow[r] & 0 \\
0 \arrow[r] & \widetilde{H}_1(X/G) \arrow[r] & \widetilde{H}_1(X/G) \oplus \bigoplus_{k-1} \mathbb{Z} \arrow[r] & \bigoplus_{k-1} \mathbb{Z} \arrow[r] & 0 &
\end{tikzcd}$$

From this, the first assertion follows easily. For the second if $n > 1$, apply equivariant homology in dimension n, to obtain, by the pullback property of the transfer and [3, Theorem 2.11],

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathbb{H}}_n^G(X) \arrow[r, "\cong"] \arrow[d, shift right, "p_*"'] & \widetilde{\mathbb{H}}_n^G(\widetilde{X}) \arrow[r] \arrow[d, shift right, "p_*"'] & 0 \\
0 \arrow[r] & \widetilde{H}_n(X/G) \arrow[r, "\cong"] \arrow[u, shift right, "\tau"'] & \widetilde{H}_n(\widetilde{X}/G) \arrow[r] \arrow[u, shift right, "\tau\cong"'] & 0
\end{tikzcd}$$

From here the second part follows. ∎

The following example is an interesting application of the previous theorem.

Example 4.10 Let $G = \mathbb{Z}_2$ act on $\mathbb{S}^1$ by complex conjugation. Then $\mathbb{S}^0 = \{-1, 1\} \subset \mathbb{S}^1$ is the fixed point set of the action and we have

$$\widetilde{H}_0^{\mathbb{Z}_2}(\mathbb{S}^1) \cong \mathbb{Z}_2 \quad \text{and} \quad \widetilde{H}_n^{\mathbb{Z}_2}(\mathbb{S}^1) = 0 \quad \text{if} \quad n \geq 1\,.$$

More generally than the previous example we have the following.

Example 4.11 Let $G = \mathbb{Z}_2$ act on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, with $n > 1$, by changing the sign of the last coordinate. Then $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ is the fixed point set of this action and it is connected, and we have the G-cofiber sequence

$$\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n \twoheadrightarrow \mathbb{S}^n \vee \mathbb{S}^n\,,$$

where $\mathbb{Z}_2$ acts on the wedge by interchanging the summands. Passing this sequence to the orbit spaces we have a commutative diagram

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook] \arrow[d] & \mathbb{S}^n \arrow[r, two heads] \arrow[d] & \mathbb{S}^n \vee \mathbb{S}^n \arrow[d] \\
\mathbb{S}^{n-1} \arrow[r, hook] & \mathbb{B}^n \arrow[r, two heads] & \mathbb{S}^n\,,
\end{tikzcd}$$

that yields a commutative diagram of homology groups with exact sequences on top and bottom

$$\begin{tikzcd}
\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{k+1}(\mathbb{S}^n \vee \mathbb{S}^n) \arrow[r] \arrow[d] & \widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{k}(\mathbb{S}^{n-1}) \arrow[r] \arrow[d] & \widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{k}(\mathbb{S}^{n}) \arrow[r] \arrow[d] & \widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{k}(\mathbb{S}^n \vee \mathbb{S}^n) \arrow[d] \\
\widetilde{H}_{k+1}(\mathbb{S}^n) \arrow[r] & \widetilde{H}_{k}(\mathbb{S}^{n-1}) \arrow[r] & \widetilde{H}_{k}(\mathbb{B}^{n}) \arrow[r] & \widetilde{H}_{k}(\mathbb{S}^{n}) .
\end{tikzcd}$$

In the case $k = n-1$, the diagram translates into

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "?"] \arrow[d, "\cdot 2"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{n-1}(\mathbb{S}^{n}) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z} \arrow[r, "\cong"'] & \mathbb{Z} \arrow[r] & 0 .
\end{tikzcd}$$

In order to verify that, indeed, the vertical arrow on the left-hand side is multiplication by 2, it is enough to note that $F^{\mathbb{Z}_2}(\mathbb{S}^n \vee \mathbb{S}^n, \mathbb{Z}) \subset F(\mathbb{S}^n \vee \mathbb{S}^n, \mathbb{Z}) \cong F(\mathbb{S}^n, \mathbb{Z}) \oplus F(\mathbb{S}^n, \mathbb{Z})$ corresponds to the diagonal, that is, $F^{\mathbb{Z}_2}(\mathbb{S}^n \vee \mathbb{S}^n, \mathbb{Z}) \cong F(\mathbb{S}^n, \mathbb{Z})$. Since the orbit map induces the sum of the factors in $F(\mathbb{S}^n, \mathbb{Z}) \oplus F(\mathbb{S}^n, \mathbb{Z}) \longrightarrow F(\mathbb{S}^n, \mathbb{Z})$, its restriction to the diagonal (modulo the obvious isomorphism) yields multiplication by 2. On the other hand, $\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{n-1}(\mathbb{S}^n \vee \mathbb{S}^n) \cong \widetilde{H}_{n-1}(\mathbb{S}^n) = 0$. Hence the commutativity of the diagram implies that the question mark is also multiplication by ± 2 and so

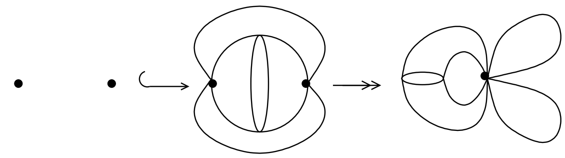
$$\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{n-1}(\mathbb{S}^n) \cong \mathbb{Z}_2 .$$

In the case $k = n$, the (extended) top row becomes $0 \longrightarrow 0 \longrightarrow \widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{n}(\mathbb{S}^n) \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$, hence $\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{n}(\mathbb{S}^n) = 0$. Moreover, in the case $0 \leq k \leq n-2$ or $k > n$, the top row of the diagram converts into $0 \longrightarrow 0 \longrightarrow \widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{k}(\mathbb{S}^n) \longrightarrow 0$, thus

$$\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{k}(\mathbb{S}^n) = 0 \quad \text{if} \quad k \neq n-1 .$$

The following two are other examples of a computation of $\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{0}(X)$ and $\widetilde{\mathbb{H}}^{\mathbb{Z}_2}_{1}(X)$.

Example 4.12 Let X consist of a 2-sphere on which $\mathbb{Z}/2$ acts rotating it 180° around its (horizontal) axis. Glue to its poles two symmetric arcs, which are exchanged by the group. This $\mathbb{Z}_2$-space has two fixed points (the poles) and the action is free elsewhere. Call $\widetilde{X}$ the quotient of X by identifying the poles. See the figure.



The equivariant cofiber sequence $\mathbb{S}^0 \hookrightarrow X \twoheadrightarrow \widetilde{X}$ and that of its orbit spaces $\mathbb{S}^0 \hookrightarrow X/\mathbb{Z}_2 \twoheadrightarrow \widetilde{X}/\mathbb{Z}_2$ yield a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathbb{H}}_1^{\mathbb{Z}_2}(X) \arrow[r] \arrow[d] & \widetilde{\mathbb{H}}_1^{\mathbb{Z}_2}(\widetilde{X}) \arrow[r] \arrow[d, shift right] & \widetilde{\mathbb{H}}_0^{\mathbb{Z}_2}(\mathbb{S}^0) \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{H}}_0^{\mathbb{Z}_2}(X) \arrow[r] & 0 \\
0 \arrow[r] & \widetilde{H}_1(X/\mathbb{Z}_2) \arrow[r] & \widetilde{H}_1(\widetilde{X}/\mathbb{Z}_2) \arrow[r] \arrow[u, bend right, shift right, "\tau \cong"'] & \widetilde{H}_0(\mathbb{S}^0) \arrow[r] & 0\,. &
\end{tikzcd}$$

This diagram translates into

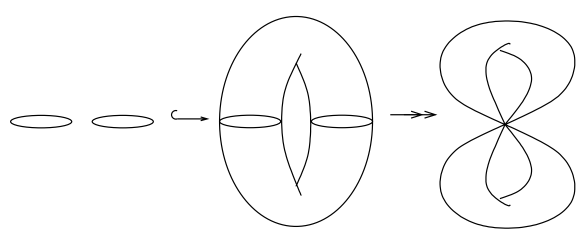
$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathbb{H}}_1^{\mathbb{Z}_2}(X) \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{H}}_0^{\mathbb{Z}_2}(X) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\Delta"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0\,. &
\end{tikzcd}$$

(One easily shows that $\widetilde{H}_1(X/\mathbb{Z}_2) \cong \mathbb{Z} \oplus \mathbb{Z}$.) Thus we conclude that

$$\widetilde{\mathbb{H}}_1^{\mathbb{Z}_2}(X) \cong \mathbb{Z} \quad \text{and} \quad \widetilde{\mathbb{H}}_0^{\mathbb{Z}_2}(X) \cong \mathbb{Z}_2\,.$$

By 4.9 we may easily see that $\widetilde{\mathbb{H}}_2^{\mathbb{Z}_2}(X) \cong \widetilde{\mathbb{H}}_2^{\mathbb{Z}_2}(\widetilde{X}) \cong \mathbb{Z}$, and $\widetilde{\mathbb{H}}_n^{\mathbb{Z}_2}(X) \cong \widetilde{\mathbb{H}}_n^{\mathbb{Z}_2}(\widetilde{X}) = 0$, $n > 2$.

EXAMPLE 4.13 Set $T = \mathbb{S}^1 \times \mathbb{S}^1$ and let $\mathbb{Z}_2$ act on T by $(-1) \cdot (\zeta, \eta) = (\overline{\zeta}, \eta)$. Then $T^{\mathbb{Z}_2} = \{-1, 1\} \times \mathbb{S}^1 = \mathbb{S}^1 \sqcup \mathbb{S}^1$. Moreover, $T/T^{\mathbb{Z}_2} \approx \widetilde{X} \vee \widetilde{X}$, where $\widetilde{X}$ is the result of identifying in a 2-sphere the two poles in one point, and $\mathbb{Z}_2$ acts interchanging the summands (see figure).



Using similar techniques as in the previous examples, one can prove that

$$\widetilde{\mathbb{H}}_1^{\mathbb{Z}_2}(T) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad \text{and} \quad \widetilde{\mathbb{H}}_0^{\mathbb{Z}_2}(T) \cong \mathbb{Z}_2\,.$$

Once more these groups differ from the corresponding nonequivariant groups of the orbit spaces.

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