Braided bialgebras in a generated monoidal Ab-category

Raúl A. Pérez* and Carlos Prieto†

March 15, 2007

Abstract

We start from any small strict monoidal braided Ab-category and extend it to a monoidal nonstrict braided Ab-category which contains braided bialgebras. The objects of the original category turn out to be modules for these bialgebras.

0 Introduction

The notion of bialgebras and Hopf algebras in braided categories was introduced by S. Majid in [4]. He considered a braided monoidal (tensor) category, but in the usual definitions of an algebra, a coalgebra, a bialgebra, and a Hopf algebra he replaced the flip by the braiding in the obvious way. Majid called a bialgebra in a braided category simply a braided bialgebra. We refer to [1] and [2] for the general properties of braided monoidal categories and to [4], [5], and [6] for the definition and results in the theory of braided bialgebras and braided Hopf algebras.

The purpose of this paper is to present a construction in which, starting from a small braided monoidal Ab-category $\mathcal{C}$ and an infinite set S_0, we create a new monoidal braided category $\mathcal{C}^{\mathsf{S}_0}$ that contains the original category $\mathcal{C}$ as a subcategory and, more important, it contains objects with bialgebra structure, in such a way that the objects of the original category $\mathcal{C}$ are modules over these bialgebras. Remember that a category $\mathcal{C}$ is said to be an Ab-*category* (also called *preabelian category*; cf. [7]) if for any pair of objects V, W the set of morphisms $\hom(V,W)$ is an additive abelian group and the composition of morphisms is bilinear. In the context of monoidal Ab-categories we shall assume that the tensor product of morphisms is bilinear. For the construction we proceed as follows. Section 1 is divided into two parts; in the first part, out of any small Ab-category $\mathcal{C}$ and any set S_0, we construct the new category $\mathcal{C}^{\mathsf{S}_0}$, which is also

*Departmento de Matemáticas, Universidad Autónoma de la Ciudad de México, 09790 México, D.F., MEXICO

†Instituto de Matemáticas, Universidad Nacional Autónoma de México, 04510 México, D.F., MEXICO

an Ab-category. Then we assume that $\mathcal{C}$ is strict monoidal and that S_0 is infinite, and so we extend the monoidal structure to $\mathcal{C}^{\mathsf{S}_0}$. However, the extended monoidal structure is not strict, thus we have to work with associative constraints and left and right units. In the second part we show how to extend a braiding and a twist from $\mathcal{C}$ to $\mathcal{C}^{\mathsf{S}_0}$. Since the new category is nonstrict monoidal, we need to define algebras, coalgebras, bialgebras, and modules in this case. This is easily done, if in the categorical definitions of the latter notions we replace the equalities by an equivalence relation in the set of morphisms of $\mathcal{C}^{\mathsf{S}_0}$. Roughly speaking, we declare two morphisms of $\mathcal{C}^{\mathsf{S}_0}$ to be related if their domains and codomains are related by associativity and/or units. This relation obviously agrees with the identity if the category is strict; this is explained in detail at the end of Section 1. We start Section 2 defining algebras, coalgebras, bialgebras, and modules in nonstrict monoidal categories in general, and then we state and prove the main theorem (Theorem 2.1) of this paper. Throughout the proof we use graphical calculus as explained in [1] and [2]. This work is influenced by Yetter's paper [3].

1 The category $\mathcal{C}^{\mathsf{S}_0}$

Let $\mathcal{C}$ be a small Ab-category. We shall denote by $\mathrm{Obj}(\mathcal{C})$ and $\mathcal{H}$ the sets of its objects and morphisms, respectively. We are going to associate to $\mathcal{C}$ a new category $\mathcal{C}^{\mathsf{S}_0}$ as follows. Let us take a fixed set S_0 and consider the set

$$\mathcal{M}(\mathrm{S}_0, \mathrm{Obj}(\mathcal{C})) = \{\ \mathrm{S}_0 \supset \mathrm{S}_f \xrightarrow{f} \mathrm{Obj}(\mathcal{C})\ \},$$

where S_f is any subset of S_0 and f is a set-theoretical function. The *objects* of $\mathcal{C}^{\mathsf{S}_0}$ will be the elements of $\mathcal{M}(\mathrm{S}_0, \mathrm{Obj}(\mathcal{C}))$. Let $f : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$ and $g : \mathrm{S}_g \longrightarrow \mathrm{Obj}(\mathcal{C})$ be two objects. A *morphism* $F : f \longrightarrow g$ will be a two-variable function $F : \mathrm{S}_f \times \mathrm{S}_g \longrightarrow \mathcal{H}$ such that:

(i) $F(x, y) : f(x) \longrightarrow g(y)$, for all $(x, y) \in \mathrm{S}_f \times \mathrm{S}_g$.

(ii) If S_g is infinite, then for each $x \in \mathrm{S}_f$ there exists a finite set $S_x^F \subset \mathrm{S}_g$, such that $F(x, y) = 0$ if $y \in \mathrm{S}_g - S_x^F$.

Let $f : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$, $g : \mathrm{S}_g \longrightarrow \mathrm{Obj}(\mathcal{C})$, and $h : \mathrm{S}_h \longrightarrow \mathrm{Obj}(\mathcal{C})$ be objects, and $F : f \longrightarrow g$, $G : g \longrightarrow h$ be morphisms. Define $G \circ F : f \longrightarrow h$ as the function $G \circ F : \mathrm{S}_f \times \mathrm{S}_h \longrightarrow \mathcal{H}$ given by:

$$(G \circ F)(x, y) = \sum_{z \in \mathrm{S}_g} G(z, y) \circ F(x, z) \tag{1}$$

for $x \in \mathrm{S}_f$ and $y \in \mathrm{S}_h$. This sum is always finite. Indeed, if we write $S_x^F = \{z_1, ..., z_k\}$, then the sum becomes

$$(G \circ F)(x, y) = \sum_{i=1}^{k} G(z_i, y) \circ F(x, z_i) \tag{2}$$

It is clear that the function $G \circ F$ satisfies condition (i). Besides, if $y \notin S^G_{z_1} \cup ... \cup S^G_{z_k}$, then $G(z_i, y) = 0$ for $1 \le i \le k$, so if we choose $S_x^{(G\circ F)} = S^G_{z_1} \cup ... \cup S^G_{z_k}$, then we have $y \in \mathrm{S}_h - S_x^{(G\circ F)}$, thus $(G \circ F)(x, y) = 0$. Therefore $G \circ F$ also satisfies condition (ii).

For any $f : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$ define $\mathrm{Id}_f : f \longrightarrow f$ as the function $\mathrm{Id}_f : \mathrm{S}_f \times \mathrm{S}_f \longrightarrow \mathcal{H}$, given by $\mathrm{Id}_f(x, y) = \delta_{x,y}\mathrm{id}_{f(x)} : f(x) \longrightarrow f(y)$ for $(x, y) \in \mathrm{S}_f \times \mathrm{S}_f$. For $G : f \longrightarrow g$ one has

$$\begin{aligned} (G \circ \mathrm{Id}_f)(x, y) &= \sum_{z\in \mathrm{S}_f} G(z, y) \circ \mathrm{Id}(x, z) \\ &= \sum_{z\in \mathrm{S}_f} G(z, y) \circ \delta_{x,z}\mathrm{id}_{f(x)} \\ &= G(x, y) \end{aligned} \tag{3}$$

Therefore $G \circ \mathrm{Id}_f = G$. Analogously $\mathrm{Id}_g \circ G = G$ for any morphism $G : f \longrightarrow g$.

Furthermore this operation is associative. Indeed, if $F : f \longrightarrow g$, $G : g \longrightarrow h$, and $H : h \longrightarrow i$, then

$$\begin{aligned} ((H \circ G) \circ F)(w, z) &= \sum_{x\in \mathrm{S}_g} (H \circ G)(x, z) \circ F(w, x) \\ &= \sum_{x\in \mathrm{S}_g} \sum_{y\in \mathrm{S}_h} (H(y, z) \circ G(x, y)) \circ F(w, x) \\ &= \sum_{y\in \mathrm{S}_h} H(y, z) \circ (\sum_{x\in \mathrm{S}_g} G(x, y) \circ F(w, x)) \\ &= \sum_{y\in \mathrm{S}_h} H(y, z) \circ (G \circ F)(w, y) \\ &= (H \circ (G \circ F))(w, z) \end{aligned} \tag{4}$$

Hence we have proved that $\mathcal{C}^{\mathsf{S}_0}$ is a category. If for two morphisms $F, G : f \longrightarrow g$ we define the function $(F + G)(x, y) = F(x, y) + G(x, y)$, which trivially satisfies conditions (i) and (ii), we see that $\mathcal{C}^{\mathsf{S}_0}$ is also an Ab-category. The following proposition proves that the direct sum of certain collections of objects in $\mathcal{C}^{\mathsf{S}_0}$ is defined.

Proposition 1.1. *Let $\{f_i : \mathrm{S}_i \longrightarrow \mathrm{Obj}(\mathcal{C})\}_{i\in\mathcal{I}}$ be any collection of functions such that the sets S_i, $i \in \mathcal{I}$, are pairwise disjoint subsets of S_0. Then $(f : \coprod_{i\in\mathcal{I}} \mathrm{S}_i \longrightarrow \mathrm{Obj}(\mathcal{C}), J_i)$, where $f|_{\mathrm{S}_i} = f_i$ and $J_k : \mathrm{S}_k \times \coprod_{i\in\mathcal{I}} \mathrm{S}_i \longrightarrow \mathcal{H}$ is given by $J_k(x, y) = \delta_{xy}\mathrm{id}_x : \mathrm{S}_k \longrightarrow \coprod_{i\in\mathcal{I}} \mathrm{S}_i$, is the coproduct of $\{f_i : \mathrm{S}_i \longrightarrow \mathrm{Obj}(\mathcal{C})\}_{i\in\mathcal{I}}$ in $\mathcal{C}^{\mathsf{S}_0}$.*

Proof. Suppose we are given an object $g : \mathrm{S}_g \longrightarrow \mathrm{Obj}(\mathcal{C})$ and a family of morphisms $T_i : (f_i : \mathrm{S}_i \longrightarrow \mathrm{Obj}(\mathcal{C})) \longrightarrow (g : \mathrm{S}_g \longrightarrow \mathrm{Obj}(\mathcal{C}))$. Define $T : (f : \coprod_{i\in\mathcal{I}} \mathrm{S}_i \longrightarrow \mathrm{Obj}(\mathcal{C})) \longrightarrow (g : \mathrm{S}_g \longrightarrow \mathrm{Obj}(\mathcal{C}))$ to be the function $T(t, y) =$

$T_k(t,y) : f(t) \longrightarrow g(y)$ if $t \in \mathrm{S}_k$. Then if $x \in \mathrm{S}_k$ and $y \in \mathrm{S}_g$, we have

$$\begin{aligned}(T \circ J_k)(x,y) &= \sum_{t \in \coprod \mathrm{S}_i} T(t,x) \circ J_k(x,t) \\ &= \sum_{t \in \coprod \mathrm{S}_i} \delta_{xt} T(t,y) \circ \mathrm{id}_x \\ &= T(x,y) \\ &= T_k(x,y)\end{aligned} \tag{5}$$

The last equality also shows the uniqueness of T. □

In particular we have the following.

Corollary 1.2. *If $\emptyset \neq \mathrm{S}_f \subset \mathrm{S}_0$, then any object $f : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$ is isomorphic to the direct sum of the objects $\{f|_{\{x\}} : \{x\} \longrightarrow \mathrm{Obj}(\mathcal{C})\}_{x \in \mathrm{S}}$.*

Let us suppose now that the category $\mathcal{C}$ is strict monoidal and that the set S_0 is infinite. In what follows we shall endow $\mathcal{C}^{\mathsf{S}_0}$ with a monoidal structure extending the one given in $\mathcal{C}$. However, as we shall see, the structure that we define is not strict in general.

We start by defining the tensor product of objects and morphisms and a unit object. Next we define the associative constraint A, the left and right units L and R, and finally we prove that they satisfy the required conditions.

First, we fix once and for all a bijection $\gamma : \mathrm{S}_0 \times \mathrm{S}_0 \longrightarrow \mathrm{S}_0$. Given two objects $f : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$ and $g : \mathrm{S}_g \longrightarrow \mathrm{Obj}(\mathcal{C})$, define $f \otimes g$ by the following composite

$$f \otimes g : \gamma(\mathrm{S}_f \times \mathrm{S}_g) \xrightarrow{\gamma^{-1}|} \mathrm{S}_f \times \mathrm{S}_g \xrightarrow{f \times g} \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \xrightarrow{\otimes} \mathrm{Obj}(\mathcal{C}) \ . \tag{6}$$

Choose any point $*$ en S_0 and define $\mathrm{I} : \{*\} \longrightarrow \mathrm{Obj}(\mathcal{C})$ by $\mathrm{I}(*) = \mathrm{I} \in \mathrm{Obj}(\mathcal{C})$.

Now, for two morphisms $F : f \longrightarrow f'$, $G : g \longrightarrow g'$, and a point $(z, z') \in \gamma(\mathrm{S}_f \times \mathrm{S}_g) \times \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})$, define $F \otimes G : f \otimes g \longrightarrow f' \otimes g'$ by

$$(F \otimes G)(z,z') := F(x_z, x'_{z'}) \otimes G(y_z, y'_{z'}) : f(x_z) \otimes g(y_z) \longrightarrow f'(x'_{z'}) \otimes g'(y'_{z'}), \tag{7}$$

where $\gamma^{-1}(z) = (x_z, y_z) \in \mathrm{S}_f \times \mathrm{S}_g$ and $\gamma^{-1}(z') = (x'_{z'}, y'_{z'}) \in \mathrm{S}_{f'} \times \mathrm{S}_{g'}$ are the pairs such that $(f \otimes g)(z) = f(x_z) \otimes g(y_z)$ and $(f' \otimes g')(z') = f'(x'_{z'}) \otimes g'(y'_{z'})$.

It is clear that $\gamma(S^F_{x_z} \times S^G_{y_z}) \subset \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})$ is a finite set and that if $z' \in \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'}) - \gamma(S^F_{x_z} \times S^G_{y_z})$, then $\gamma^{-1}(z') \notin S^F_{x_z} \times S^G_{y_z}$. Hence, either $x'_{z'} \notin S^F_{x_z}$ or $y'_{z'} \notin S^G_{y_z}$ and so $(F \otimes G)(z,z') = 0$ if $z' \notin \gamma(S^F_{x_z} \times S^G_{y_z})$.

Before we define the associative constraint A, we shall adopt the following notation. If, for example, $v \in \gamma(\gamma(\mathrm{S}_f \times \mathrm{S}_g) \times \mathrm{S}_h)$, then we write

$$(\gamma^{-1} \times \mathrm{id})\gamma^{-1}(v) = ((x_v, y_v), z_v) \in \mathrm{S}_f \times \mathrm{S}_g \times \mathrm{S}_h.$$

Here, $\gamma(x_v, y_v)$ is the unique element in $\gamma(\mathrm{S}_f \times \mathrm{S}_g) \subset \mathrm{S}_0$ such that $\gamma(\gamma(x_v, y_v), z_v) = v$. In other words, the inner parentheses will indicate the place from left to right of the second γ^{-1} in the composition $(\gamma^{-1}\times \mathrm{id})\gamma^{-1}$. Analogously for $w \in \gamma(\mathrm{S}_f \times \gamma(\mathrm{S}_g \times \mathrm{S}_h))$ we write

$$(\mathrm{id}\times\gamma^{-1})\gamma^{-1}(w) = (x_w, (y_w, z_w)) \in \mathrm{S}_f \times \mathrm{S}_g \times \mathrm{S}_h.$$

When there is no risk of confusion we drop the inner parentheses and simply write $(\gamma^{-1}\times \mathrm{id})\gamma^{-1}(v) = (x_v, y_v, z_v)$ and $(\mathrm{id}\times\gamma^{-1})\gamma^{-1}(w) = (x_w, y_w, z_w)$. In the same way, if for example $v \in \mathrm{S}_{(f\otimes(g\otimes h))\otimes i} = \gamma(\gamma(\mathrm{S}_f \times \gamma(\mathrm{S}_g \times \mathrm{S}_h)) \times \mathrm{S}_i)$, then we write

$$(\mathrm{id}\times\gamma^{-1}\times\mathrm{id})(\gamma^{-1}\times\mathrm{id})\gamma^{-1}(v) = ((x_v, (y_v, z_v)), t_v) \in \mathrm{S}_f \times \mathrm{S}_g \times \mathrm{S}_h \times \mathrm{S}_i,$$

or $(\mathrm{id}\times\gamma^{-1}\times\mathrm{id})(\gamma^{-1}\times\mathrm{id})\gamma^{-1}(v) = (x_v, y_v, z_v, t_v)$, etc.

With this notation, we have

$$\begin{aligned}((F\otimes G)\otimes H)(v,w) &= (F\otimes G)((x,y)_v,(x,y)_w)\otimes H(z_v,z_w)\\ &= F(x_v,x_w)\otimes G(y_v,y_w)\otimes H(z_v,z_w).\end{aligned} \tag{8}$$

Let us define $A_{f,g,h} : (f\otimes g)\otimes h \longrightarrow f\otimes(g\otimes h)$ by

$$A_{f,g,h}(v,w) = \delta^{v,w}_{x;y;z}\mathrm{id}_{f(x_v)\otimes g(y_v)\otimes h(z_v)} : ((f\otimes g)\otimes h)(v) \longrightarrow (f\otimes(g\otimes h))(w)\ , \tag{9}$$

where again, in order to shorten the notation, $\delta^{v,w}_{x;y;z}$ stands for $\delta_{x_v,x_w}\delta_{y_v,y_w}\delta_{z_v,z_w}$. It is easy to see that the inverse of $A_{f,g,h}$ is given by

$$A^{-1}_{f,g,h}(w,v) = \delta^{w,v}_{x;y;z}\mathrm{id}_{f(x_w)\otimes g(y_w)\otimes h(z_w)} : (f\otimes(g\otimes h))(w) \longrightarrow ((f\otimes g)\otimes h)(v)\ . \tag{10}$$

Now we define the right unit $R_f : f\otimes \mathrm{I} \longrightarrow f$. For any object f, the object $f\otimes\mathrm{I}$ is expressed by the composite

$$f\otimes\mathrm{I} : \gamma(\mathrm{S}_f\times\{*\}) \xrightarrow{\gamma^{-1}|} \mathrm{S}_f\times\{*\} \xrightarrow{f\times\mathrm{I}} \mathrm{Obj}(\mathcal{C})\times\mathrm{Obj}(\mathcal{C}) \xrightarrow{\otimes} \mathrm{Obj}(\mathcal{C})\ . \tag{11}$$

For $z\in\gamma(\mathrm{S}_f\times\{*\})$, we write $\gamma^{-1}(z) = (x_z, *) \in \mathrm{S}_f \times *$ and define $R_f : f\otimes\mathrm{I} \longrightarrow f$ by

$$R_f(z,x) = \delta_{x_z,x}\mathrm{id}_{f(x_z)} : (f\otimes\mathrm{I})(z) = f(x_z) \longrightarrow f(x) \tag{12}$$

for $(z,x)\in\gamma(\mathrm{S}_f\times\{*\})\times\mathrm{S}_f$. It is easy to see that R_f is an isomorphism with inverse $R_f^{-1} : f \longrightarrow f\otimes\mathrm{I}$ given by the function

$$R_f^{-1}(x,z) = \delta_{x,x_z}\mathrm{id}_{f(x)} : f(x) \longrightarrow (f\otimes\mathrm{I})(z) = f(x_z)\,. \tag{13}$$

In the same way we define the left unit $L_f : \mathrm{I}\otimes f \longrightarrow f$, that is, if $z\in\gamma(*\times\mathrm{S}_f)$, then we write $\gamma^{-1}(z) = (*, x_z)$ and define

$$L_f(z,x) = \delta_{x_z,x}\mathrm{id}_{f(x_z)} : (\mathrm{I}\otimes f)(z) = f(x_z) \longrightarrow f(x) \tag{14}$$

The inverse of L_f is given by

$$L_f^{-1}(x,z) = \delta_{x,x_z}\mathrm{id}_{f(x)} : f(x) \longrightarrow (\mathrm{id}\otimes f)(z) = f(x_z)\,. \tag{15}$$

Theorem 1.3. *The category $\mathcal{C}^{S_0}$ is a monoidal category with tensor product of objects and morphisms, associative constraint, and right and left units as we have just defined.*

We divide the *proof* into four lemmas.

Lemma 1.4. *If $F : f \longrightarrow f'$, $F' : f' \longrightarrow f''$, $G : g \longrightarrow g'$, and $G' : g' \longrightarrow g''$ are morphisms in $\mathcal{C}^{C_0}$, then*

(i) $(F' \otimes G') \circ (F \otimes G) = (F' \circ F) \otimes (G' \circ G)$ *and*

(ii) $\mathrm{Id}_f \otimes \mathrm{Id}_g = \mathrm{Id}_{f \otimes g}$.

Proof. (i) For $z \in \gamma(\mathrm{S}_f \times \mathrm{S}_g)$ and $z'' \in \gamma(\mathrm{S}_{f''} \times \mathrm{S}_{g''})$ we have

$$\begin{aligned}
((F' \otimes G') \circ (F \otimes G))(z, z'') &= \sum_{z' \in \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})} (F' \otimes G')(z', z'') \circ (F \otimes G)(z, z') \\
&= \sum_{z' \in \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})} (F'(x'_{z'}, x''_{z''}) \circ F(x_z, x'_{z'})) \\
&\quad \otimes (G'(y'_{z'}, y''_{z''}) \circ G(y_z, y'_{z'})) \\
&= (\sum_{x' \in \mathrm{S}_{f'}} F'(x', x''_{z''}) \circ F(x_z, x')) \\
&\quad \otimes (\sum_{y' \in \mathrm{S}_{g'}} G'(y', y''_{z''}) \circ G(y_z, y')) \\
&= (F' \circ F)(x_z, x''_{z''}) \otimes (G' \circ G)(y_z, y''_{z''}) \\
&= ((F' \circ F) \otimes (G' \circ G))(z, z'')
\end{aligned} \tag{16}$$

The third equality follows from the fact that γ establishes a bijection between $\mathrm{S}_{f'} \times \mathrm{S}_{g'}$ and $\gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})$.

(ii) For $z, z' \in \gamma(\mathrm{S}_f \times \mathrm{S}_g)$ we have

$$\begin{aligned}
(\mathrm{Id}_f \otimes \mathrm{Id}_g)(z, z') &= \mathrm{Id}_f(x_z, x_{z'}) \otimes \mathrm{Id}_g(y_z, y_{z'}) \\
&= \delta_{x_z, x_{z'}} \mathrm{id}_{f(x_z)} \otimes \delta_{y_z, y_{z'}} \mathrm{id}_{g(y_z)} \\
&= \delta_{z, z'} \mathrm{id}_{f(x_z)} \otimes \mathrm{id}_{g(y_z)} \\
&= \delta_{z, z'} \mathrm{id}_{f(x_z) \otimes g(y_z)} \\
&= \delta_{z, z'} \mathrm{id}_{(f \otimes g)(z)} \\
&= \mathrm{Id}_{f \otimes g}(z, z')
\end{aligned} \tag{17}$$

$\square$

Lemma 1.5. *The associative constraint A defined above is a natural isomorphism that satisfies the Pentagonal Axiom.*

Proof. We already saw that A is an isomorphism. To show that it is natural, we have, on the one hand,

$$
\begin{aligned}
((F\otimes(G\otimes H))\circ A_{f,g,h})(v,w') &= \sum_{w\in\gamma(\mathrm{S}_f\times\gamma(\mathrm{S}_g\times\mathrm{S}_h))} (F\otimes(G\otimes H))(w,w')\circ \\
&\qquad \circ A_{f,g,h}(v,w) \\
&= \sum_{w\in\gamma(\mathrm{S}_f\times\gamma(\mathrm{S}_g\times\mathrm{S}_h))} (F(x_w,x_{w'})\otimes G(y_w,y_{w'})\otimes \\
&\qquad \otimes H(z_w,z_{w'}))\circ \delta^{v,w}_{x;y;z}\mathrm{id}_{f(x_v)\otimes g(y_v)\otimes h(z_v)} \\
&= F(x_v,x_{w'})\otimes G(y_v,y_{w'})\otimes H(z_v,z_{w'})\,.
\end{aligned} \tag{18}
$$

On the other hand, we have

$$
\begin{aligned}
(A_{f',g',h'}\circ((F\otimes G)\otimes H))(v,w') &= \sum_{v'\in\gamma(\gamma(\mathrm{S}_{f'}\times\mathrm{S}_{g'})\times\mathrm{S}_{h'})} A_{f',g',h'}(v',w')\circ((F\otimes G)\otimes \\
&\quad H)(v,v') \\
&= \sum_{v'\in\gamma(\gamma(\mathrm{S}_{f'}\times\mathrm{S}_{g'})\times\mathrm{S}_{h'})} \delta^{v',w'}_{x';y';z'}\mathrm{id}_{f'(x'_{v'})\otimes g'(y'_{v'})\otimes h'(z'_{v'})}\circ \\
&\quad \circ(F(x_v,x'_{v'})\otimes G(y_v,y'_{v'})\otimes H(z_v,z'_{v'})) \\
&= F(x_v,x_{w'})\otimes G(y_v,y_{w'})\otimes H(z_v,z_{w'}).
\end{aligned} \tag{19}
$$

Therefore, $A_{f',g',h'}\circ((F\otimes G)\otimes H)=(F\otimes(G\otimes H))\circ A_{f,g,h}$, so A is natural.

Let us prove now that A satisfies the Pentagonal Axiom. Set $M(s,w)=((\mathrm{id}_f\otimes A_{g,h,i})\circ A_{f,g\otimes h,i}\circ(A_{f,g,h}\otimes\mathrm{id}_i))(s,w)$. For $s\in\gamma(\gamma(\gamma(\mathrm{S}_f\times\mathrm{S}_g)\times\mathrm{S}_h)\times\mathrm{S}_i)$ and $w\in\gamma(\mathrm{S}_f\times\gamma(\mathrm{S}_g\times\gamma(\mathrm{S}_h\times\mathrm{S}_i)))$, we have

$$
\begin{aligned}
M(s,w) &= \sum_{\substack{u\in\mathrm{S}_{f\otimes((g\otimes h)\otimes i)}\\ v\in\mathrm{S}_{f\otimes((g\otimes h)\otimes i)}}} (\mathrm{id}_f\otimes A_{g,h,i})(v,w)\circ A_{f,g\otimes h,i}(u,v)\circ(A_{f,g,h}\otimes\mathrm{id}_i)(s,u) \\
&= \sum_{\substack{u\in\mathrm{S}_{f\otimes((g\otimes h)\otimes i)}\\ v\in\mathrm{S}_{f\otimes((g\otimes h)\otimes i)}}} (\delta_{x_v,x_w}\mathrm{id}_{f(x_v)}\otimes\delta^{v,w}_{y;z;t}\mathrm{id}_{g(y_v)\otimes h(z_v)\otimes i(t_v)})\circ\delta^{u,v}_{x;y;z;t} \\
&\quad \mathrm{id}_{f(x_u)\otimes g(y_u)\otimes h(z_u)\otimes i(t_u)}\circ(\delta^{s,u}_{x;y;z}\mathrm{id}_{f(x_s)\otimes g(y_s)\otimes h(z_s)}\otimes\delta_{t_s,t_u}\mathrm{id}_{i(t_s)}) \\
&= \delta^{s,w}_{x;y;z;t}\mathrm{id}_{f(x_s)\otimes g(y_s)\otimes h(z_s)\otimes i(t_s)}.
\end{aligned} \tag{20}
$$

Set $N(s,w) = (A_{f,g,h\otimes i} \circ A_{f\otimes g,h,i})(s,w)$. Then,

$$\begin{aligned}
N(s,w) &= \sum_{r\in \mathrm{S}_{(f\otimes g)\otimes(h\otimes i)}} (A_{f,g,h\otimes i})(r,w) \circ (A_{f\otimes g,h,i})(s,r) \\
&= \sum_{r\in \mathrm{S}_{(f\otimes g)\otimes(h\otimes i)}} \delta^{r,w}_{x;y;z;t} \mathrm{id}_{f(x_r)\otimes g(y_r)\otimes h(z_r)\otimes i(t_r)} \circ \delta^{s,r}_{x;y;z;t} \mathrm{id}_{f(x_s)\otimes g(y_s)\otimes h(z_s)\otimes i(t_s)} \\
&= \delta^{s,w}_{x;y;z;t} \mathrm{id}_{f(x_s)\otimes g(y_s)\otimes h(z_s)\otimes i(t_s)}.
\end{aligned} \tag{21}$$

Thus $M(s,w) = N(s,w)$ and so, A satisfies the Pentagonal Axiom. □

Lemma 1.6. *The right unit R and the left unit L are natural isomorphisms.*

Proof. We already saw that R_f is an isomorphism. For $z \in \gamma(\mathrm{S}_f \times *)$ and $x' \in \mathrm{S}_{f'}$ we have

$$\begin{aligned}
(F \circ R_f)(z,x') &= \sum_{x\in \mathrm{S}_f} F(x,x') \circ R_f(z,x) \\
&= F(x,x') \circ \delta_{x_z,x'} \mathrm{id}_{f(x_z)} \\
&= F(x_z,x').
\end{aligned} \tag{22}$$

On the other hand

$$\begin{aligned}
(R_{f'} \circ (F \otimes \mathrm{Id_I}))(z,x') &= \sum_{z'\in\gamma(\mathrm{S}_{f'}\times *)} R_{f'}(z',x') \circ (F\otimes \mathrm{Id_I})(z,z') \\
&= \delta_{x'_{z'},x'} \mathrm{id}_{f(x'_{z'})} \circ (F(x_z,x'_{z'}) \otimes \mathrm{Id_I}(*,*)) \\
&= \delta_{x'_{z'},x'} \mathrm{id}_{f(x'_{z'})} \circ F(x_z,x'_{z'}) \\
&= F(x_z,x')\,.
\end{aligned} \tag{23}$$

The proof for L is analogous. □

Lemma 1.7. *The morphisms A, R and L satisfy the Triangular Axiom.*

Proof. Set $P(v,w) = ((\mathrm{Id}_f \otimes L_g) \circ A_{f,\mathrm{I},g})(v,w)$. Then

$$\begin{aligned}
P(v,w) &= \sum_{u\in S_{f\otimes(\mathrm{I}\otimes g)}} (\mathrm{Id}_f \otimes L_g)(u,w) \circ A_{f,\mathrm{I},g}(v,u) \\
&= (\delta_{x_u,x_w} \mathrm{id}_{f(x_u)} \otimes \delta_{y_u,y_w} \mathrm{id}_{g(y_u)}) \circ \delta^{v,u}_{x;y} \mathrm{id}_{f(x_v)\otimes \mathrm{I}(*)\otimes g(y_v)} \\
&= \delta_{x_v,x_w} \mathrm{id}_{f(x_v)} \otimes \delta_{y_v,y_w} \mathrm{id}_{g(y_v)} \\
&= (R_f \otimes \mathrm{Id}_g)(v,w).
\end{aligned} \tag{24}$$

So $(\mathrm{Id}_f \otimes L_g) \circ A_{f,\mathrm{I},g} = R_f \otimes \mathrm{Id}_g$. □

These four lemmas finish the proof of 1.3

Proposition 1.8. *The category* $\mathcal{C}^{\mathsf{S}_0}$ *has a full subcategory, which is tensor equivalent to* $\mathcal{C}$.

Proof. Recall that a tensor functor is a triple $(F, \varphi_0, \varphi_2)$, where F is a functor, φ_0 is an isomorphism from I to $F(\mathrm{I})$, and $\varphi_2(U, V) : F(U) \otimes F(V) \longrightarrow F(U \otimes V)$ is a family of natural isomorphisms compatible with the associative constraint and the left and right units (see [1, p.287]). Define a functor $J : \mathcal{C} \longrightarrow \mathcal{C}^{\mathsf{S}_0}$, by choosing for any object V in $\mathcal{C}$ any point $x_V \in \mathrm{S}_0$ and a function $f_V : \{x_V\} \longrightarrow \mathrm{Obj}(\mathcal{C})$, given by $f_V(x_V) = V$. Then we define $J(V) = f_V$. To any morphism $\alpha : V \longrightarrow W$ we assign the function $F_\alpha(x_V, x_W) = \alpha : f_V(x_V) = V \longrightarrow f_W(x_W) = W$ and then define $J(\alpha) = F_\alpha$. For the unit object I of $\mathcal{C}$ we choose the fixed point $*$ as before, so that $J(\mathrm{I}) = \mathrm{I} \in \mathrm{Obj}(\mathcal{C}^{\mathsf{S}_0})$. For U, V objects of $\mathcal{C}$, define $\varphi_2(U, V) : J(U) \otimes J(V) = f_U \otimes f_V \longrightarrow J(U \otimes V) = f_{U \otimes V}$, as follows. If $\gamma^{-1}(\{x_U\} \times \{x_V\}) = \{x'_{U,V}\}$, then $(f_U \otimes f_V)(x'_{U,V}) = U \otimes V$ and $f_{U \otimes V}(x_{U \otimes V}) = U \otimes V$, then take $\varphi_2(U, V)(x'_{U,V}, (x_{U \otimes V})) = \mathrm{id}_{U \otimes V}$. The morphisms φ_0 and φ_2 are identities, so that the functor J is strict, and it is straightforward to prove that they satisfy the required compatibility conditions. □

1.1 Extending the braiding and the twist

Let us now assume that the category $\mathcal{C}$ is braided with braiding c. For $v \in \gamma(\mathrm{S}_f \times \mathrm{S}_g)$ and $w \in \gamma(\mathrm{S}_g \times \mathrm{S}_f)$, define $\mathrm{C}_{f,g}(v, w)$ by

$$\begin{aligned} \mathrm{C}_{f,g}(v, w) = \delta^{v,w}_{x;y} \mathrm{c}_{f(x_v), g(y_v)} : (f \otimes g)(v) = f(x_v) \otimes g(y_v) &\longrightarrow g(y_w) \otimes f(x_w) \\ &= (g \otimes f)(w)\,. \end{aligned} \tag{25}$$

It is clear that $\mathrm{C}_{f,g}$ is invertible with inverse given by $\mathrm{C}^{-1}_{f,g}(w, v) = \delta^{w,v}_{x;y} \mathrm{c}^{-1}_{f(x_w), g(y_w)}$.

Proposition 1.9. *The family* C *of isomorphisms* $\mathrm{C}_{f,g}$ *is a braiding in the category* $\mathcal{C}^{\mathsf{S}_0}$.

Proof. We have to prove that C is natural and satisfies the Hexagonal Axiom. For $F : f \longrightarrow f'$ and $G : g \longrightarrow g'$ we have, on the one hand

$$\begin{aligned} ((G \otimes F) \circ \mathrm{C}_{f,g})(v, w') &= \sum_{w \in \gamma(\mathrm{S}_g \otimes \mathrm{S}_f)} (G \otimes F)(w, w') \circ \mathrm{C}_{f,g}(v, w) \\ &= \sum_{w \in \gamma(\mathrm{S}_g \otimes \mathrm{S}_f)} (G(y_w, y_{w'}) \otimes F(x_w, x_{w'})) \circ \delta^{v,w}_{x;y} \mathrm{c}_{f(x_v), g(y_v)} \\ &= (G(y_v, y_{w'}) \otimes F(x_v, x_{w'})) \circ \mathrm{c}_{f(x_v), g(y_v)}\,. \end{aligned} \tag{26}$$

On the other hand,

$$\begin{aligned}
\mathrm{C}_{f',g'} \circ (F \otimes G)(v,w') &= \sum_{v' \in \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})} \mathrm{C}_{f',g'}(v',w') \circ (F \otimes G)(v,v') \\
&= \sum_{v' \in \gamma(\mathrm{S}_{f'} \times \mathrm{S}_{g'})} \delta^{v',w'}_{x;y} \mathrm{c}_{f'(x'_{v'}),g'(y'_{v'})} \circ F(x_v,x'_{v'}) \otimes G(y_v,y_{v'}) \\
&= \mathrm{c}_{f'(x'_{w'}),g'(y'_{w'})} \circ (F(x_v,x'_{w'}) \otimes G(y_v,y_{w'})).
\end{aligned} \tag{27}$$

Both sums are equal since c is a braiding in $\mathcal{C}$ and therefore it is natural. Thus C is natural. We now show the commutativity of one of the diagrams of the Hexagonal Axiom. Put $M(w,w') = (A_{g,h,f} \circ \mathrm{C}_{f,g\otimes h} \circ A_{f,g,h})(w,w')$. Then

$$\begin{aligned}
M(w,w') &= \sum_{\substack{u \in \mathrm{S}_{(f\otimes g)\otimes h} \\ v \in \mathrm{S}_{f \otimes (g \otimes h)}}} A_{f,g,h}(u,w') \circ \mathrm{C}_{f,g\otimes h}(v,u) \circ A_{f,g,h}(w,v) \\
&= \sum_{\substack{u \in \mathrm{S}_{(f\otimes g)\otimes h} \\ v \in \mathrm{S}_{f \otimes (g \otimes h)}}} \delta^{u,w'}_{x;y;z} \mathrm{id}_{f(x_u)\otimes g(y_u) \otimes h(z_u)} \circ \delta^{v,u}_{x;y;z} \mathrm{c}_{f(x_v),g(y_v)\otimes h(z_v)} \\
&\quad \circ \delta^{w,v}_{x;y;z} \mathrm{id}_{f(x_w)\otimes g(y_w)\otimes h(z_w)} \\
&= \delta^{w,w'}_{x;y;z} \mathrm{c}_{f(x_w),g(y_w)\otimes h(z_w)}.
\end{aligned} \tag{28}$$

Set $N(w,w') = ((\mathrm{Id}_g \otimes \mathrm{C}_{f,h}) \circ A_{g,f,h} \circ (\mathrm{C}_{f,g} \otimes \mathrm{Id}_h))(w,w')$. Then

$$\begin{aligned}
N(w,w') &= \sum_{\substack{u \in \mathrm{S}_{g \otimes (f\otimes h)} \\ v \in \mathrm{S}_{(g \otimes f) \otimes h}}} (\mathrm{Id}_g \otimes \mathrm{C}_{f,h})(u,w') \circ A_{g,f,h}(v,u) \circ (\mathrm{C}_{f,g} \otimes \mathrm{Id}_h)(w,v) \\
&= \sum_{\substack{u \in \mathrm{S}_{g \otimes (f\otimes h)} \\ v \in \mathrm{S}_{(g \otimes f) \otimes h}}} (\delta_{y_u,y_{w'}} \mathrm{id}_{g(y_u)} \otimes \delta^{u,w'}_{x;z} \mathrm{c}_{f(x_u),h(z_u)}) \circ \delta^{v,u}_{x;y;z} \mathrm{id}_{g(y_v)\otimes f(x_v) \otimes h(z_v)} \\
&\quad \circ (\delta^{w,v}_{x;y} \mathrm{c}_{f(x_w),g(y_w)} \otimes \delta_{z_w,z_v} \mathrm{id}_{h(z_w)}) \\
&= \delta^{w,w'}_{x;y;z} (\mathrm{id}_{g(y_w)} \otimes \mathrm{c}_{f(x_w),h(z_w)}) \circ (\mathrm{c}_{f(x_w),g(y_w)} \otimes \mathrm{id}_{h(z_w)}).
\end{aligned} \tag{29}$$

Again, since c is a strict braiding in $\mathcal{C}$, we have the equality $M(w,w') = N(w,w')$. The commutativity of the other hexagon is proved analogously. □

In the same way, if the category $\mathcal{C}$ has a twist, then we can easily prove the following assertion.

Proposition 1.10. *Let θ be a twist for the the category $\mathcal{C}$. Then the category $\mathcal{C}^{S_0}$ has a twist $\Theta_f : f \longrightarrow f$ given by*

$$\Theta_f(x,y) = \delta_{x,y} \theta_{f(x)} : f(x) \longrightarrow f(y) \tag{30}$$

for any object f in $\mathcal{C}^{S_0}$. □

However, it is not possible to extend a duality from $\mathcal{C}$ to $\mathcal{C}^{\mathsf{S}_0}$. Although we have for any $f : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$ a canonical candidate for $f^* : \mathrm{S}_f \longrightarrow \mathrm{Obj}(\mathcal{C})$, namely the function f^* defined by $f^*(x) = (f(x))^*$ as well as a canonical candidate for the evaluation $D_f : f^* \otimes f \longrightarrow \mathrm{I}$, given by $D_f(v, \{*\}) = \delta_{x_v^*, x_v} \mathrm{d}_{f(x_v)} : f^*(x_v^*) \otimes f(x_v) \longrightarrow \mathrm{I}(*) = \mathrm{I}$, where $\gamma^{-1}(v) = (x_v^*, x_v) \in \mathrm{S}_f \times \mathrm{S}_f$, this is not the case for the coevaluation. Indeed, the canonical extension $B_f : \mathrm{I} \longrightarrow f \otimes f^*$ given by $B_f(*, v) = \delta_{x_v^*, x_v} \mathrm{b}_{f(x_v)} : \mathrm{I} \longrightarrow f(x_v) \otimes f^*(x_v^*)$ is not a morphism in $\mathcal{C}^{\mathsf{S}_0}$ if S_f is infinite, since condition (ii) of page 2 does not hold.

Nevertheless, if we consider the full subcategory $\mathcal{C}_\sharp^{\mathsf{S}_0}$ which as objects has functions f with finite domain S_f, then it is possible to extend the duality according to the given formulas. It is easy to see that the inclusion functor $J : \mathcal{C} \longrightarrow \mathcal{C}^{\mathsf{S}_0}$ factors through $\mathcal{C}_\sharp^{\mathsf{S}_0}$, i.e.,

$$J : \mathcal{C} \hookrightarrow \mathcal{C}_\sharp^{\mathsf{S}_0} \hookrightarrow \mathcal{C}^{\mathsf{S}_0} \tag{31}$$

The following assertion is also easy to prove.

Proposition 1.11. *If the category $\mathcal{C}$ is a ribbon category, then the extended structure in $\mathcal{C}_\sharp^{\mathsf{S}_0}$ is pivotal braided (but nonstrict in general, so it is not ribbon).* □

Remark 1.12. In order to simplify the next computations, we shall adopt the following notation. Let $\mathcal{A}$ be the set of isomorphisms of $\mathcal{C}^{\mathsf{S}_0}$ generated by the set $(\mathrm{Id}_\chi, A^{\pm 1}_{\kappa,\lambda,\mu}, R_\zeta, L_\varsigma)$ under tensor products and compositions, where χ, κ, λ, μ, ζ, and ς are any objects in $\mathcal{C}^{\mathsf{S}_0}$. In other words, $\mathcal{A}$ is the set of isomorphisms that relate different objects by associativity and units. If F and G are morphisms in $\mathcal{C}^{\mathsf{S}_0}$, we shall write $F \doteq G$ if $G = X \circ F \circ Y$, where X and Y are elements of $\mathcal{A}$. For example, $F \doteq G$ if the following diagram commutes.

$$\begin{tikzcd}
(((f_1 \otimes f_2) \otimes f_3) \otimes f_4) \otimes f_5 \arrow[r, "F"] \arrow[d, "A_{f_1\otimes f_2, f_3, f_4} \otimes \mathrm{id}_{f_5}"'] & (g_1 \otimes g_2) \otimes g_3 \arrow[ddd, "A_{g_1,g_2,g_3}"] \\
((f_1 \otimes f_2) \otimes (f_3 \otimes f_4)) \otimes f_5 \arrow[d, "A_{f_1\otimes f_2, f_3\otimes f_4, f_5}"'] & \\
(f_1 \otimes f_2) \otimes ((f_3 \otimes f_4) \otimes f_5) \arrow[d, "\mathrm{Id}_{f_1\otimes f_2} \otimes A^{-1}_{f_3,f_4,f_5}"'] & \\
(f_1 \otimes f_2) \otimes (f_3 \otimes (f_4 \otimes f_5)) \arrow[r, "G"] & g_1 \otimes (g_2 \otimes g_3)
\end{tikzcd}$$

The relation $\doteq$ is an equivalence relation in the set of morphisms of $\mathcal{C}^{\mathsf{S}_0}$ which is compatible with composition and tensor product in the sense that if $F \doteq G$ and $F' \doteq G'$ then $F' \circ F \doteq G' \circ G$, if the compositions are defined, and $F \otimes F' \doteq G \otimes G'$. Indeed, for the composition, suppose $A \circ F \circ B = G$ and that $C \circ F' \circ D = G'$, for elements A, B, C, and D in $\mathcal{A}$. Then $G' \circ G = C \circ F' \circ D \circ A \circ F \circ B$. The morphism $D \circ A$ is an endomorphism of the domain $s(F')$ of F' which is equal to the codomain $t(F)$ of F and is an element of $\mathcal{A}$. Mac Lane's coherence theorem states that this element has to be the identity morphism $\mathrm{Id}_{s(F')}$. Hence $G' \circ G = C \circ F' \circ F \circ B$. The tensor part follows from the identity $(A \circ F \circ B) \otimes (C \circ F' \circ D) = (A \otimes C) \circ (F \otimes F') \circ (B \otimes D)$. In what follows we shall use this notation without further comments.

2 Bialgebras in $\mathcal{C}^{\mathsf{S}_0}$

Let $\mathbf{V}$ be a monoidal category. We say that an object A of $\mathbf{V}$ is an *algebra in* $\mathbf{V}$, if there exist morphisms $\mu : A \otimes A \longrightarrow A$ and $\eta : \mathrm{I} \longrightarrow A$ such that

$$\mu(\mu \otimes \mathrm{id}_A) \doteq \mu(\mathrm{id}_A \otimes \mu)\,, \tag{32}$$

$$\mu(\eta \otimes \mathrm{id}_A) \doteq \mathrm{id}_A \doteq \mu(\mathrm{id}_A \otimes \eta). \tag{33}$$

Dually, we say that C is a *coalgebra in* $\mathbf{V}$, if there exist morphisms $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow \mathrm{I}$ such that

$$(\Delta \otimes \mathrm{id}_C)\Delta \doteq (\mathrm{id}_C \otimes \Delta)\Delta\,, \tag{34}$$

$$(\varepsilon \otimes \mathrm{id}_C)\Delta \doteq \mathrm{id}_C \doteq (\mathrm{id}_C \otimes \varepsilon)\Delta. \tag{35}$$

If H is an algebra, then the product in $H \otimes H$ is defined by the following composite

$$\begin{aligned} \widehat{\mu} : (H \otimes H) \otimes (H \otimes H) &\xrightarrow{A^{-1}_{H \otimes H, H, H}} ((H \otimes H) \otimes H) \otimes H \xrightarrow{A_{H,H,H} \otimes \mathrm{id}_H} \\ (H \otimes (H \otimes H)) \otimes H &\xrightarrow{\mathrm{id}_H \otimes \mathrm{c}_{H,H} \otimes \mathrm{id}_H} (H \otimes (H \otimes H)) \otimes H \xrightarrow{A^{-1}_{H,H,H} \otimes \mathrm{id}_H} \\ ((H \otimes H) \otimes H) \otimes H &\xrightarrow{A_{(H \otimes H), H, H}} (H \otimes H) \otimes (H \otimes H) \xrightarrow{\mu \otimes \mu} \\ &\longrightarrow H \otimes H\,. \end{aligned} \tag{36}$$

We say that H is a *bialgebra in* $\mathbf{V}$, if $\widehat{\mu}(\Delta \otimes \Delta) \doteq \Delta\mu$ and $\varepsilon\mu = \varepsilon \otimes \varepsilon$.
If A is an algebra, an object V is an A-module, if there exists a morphism $T : A \otimes V \longrightarrow V$, such that $T(\mu \otimes \mathrm{id}_V) \doteq T(\mathrm{id}_A \otimes T)$ and $T(\eta \otimes \mathrm{id}_V) \doteq \mathrm{id}_V$. Note that if the category is strict monoidal, the latter are the concepts of algebra, coalgebra, bialgebra and module in strict braided monoidal categories.

We are going to find bialgebras in $\mathcal{C}^{\mathsf{S}_0}$, when $\mathcal{C}$ is a braided strict monoidal category with left duality.

Let $h : \mathrm{S}_h \longrightarrow \mathrm{Obj}(\mathcal{C})$ be an injective function such that $h(\mathrm{S}_h) \subset \mathrm{Obj}(\mathcal{C})$ is closed under $\otimes$, that is, for any pair $(x, y) \in \mathrm{S}_h \times \mathrm{S}_h$, there exists a unique

$z \in \mathrm{S}_h$ such that $h(x) \otimes h(y) = h(z)$ and suppose $\mathrm{I} \in h(\mathrm{S}_h)$. For example, we can take a set S_0 with the same cardinality as $\mathrm{Obj}(\mathcal{C})$ and $h : \mathrm{S}_0 \longrightarrow \mathrm{Obj}(\mathcal{C})$ to be any bijection, if $\mathrm{Obj}(\mathcal{C})$ is an infinite set.

Set $\Delta_h = \{(x,x) \mid x \in \mathrm{S}_h\} \subset \mathrm{S}_h \times \mathrm{S}_h$ and let $\overline{\mathrm{h}}$ be the object defined by the composite

$$\overline{\mathrm{h}} : \gamma(\Delta_h) \xrightarrow{\gamma^{-1}} \Delta_h \xrightarrow{h^* \times h} \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \xrightarrow{\otimes} \mathrm{Obj}(\mathcal{C}) \tag{37}$$

where $h^*(x) := (h(x))^*$. That is, $\overline{\mathrm{h}}$ is defined by the relation $\overline{\mathrm{h}}(\gamma(x,x)) = h^*(x) \otimes h(x)$, for $\gamma(x,x) \in \mathrm{S}_{\overline{\mathrm{h}}} = \gamma(\Delta_h)$.

The main theorem in this section is the following.

Theorem 2.1. *The object $\overline{\mathrm{h}}$ is a bialgebra in $\mathcal{C}^{\mathcal{S}_0}$ and the objects of $\mathcal{C}$, considered as a subcategory of $\mathcal{C}^{\mathcal{S}_0}$, are $\overline{\mathrm{h}}$-modules.*

To prove it, we shall establish two previous lemmas. Let $\chi : \mathrm{S}_h \times \mathrm{S}_h \longrightarrow \mathrm{S}_h$ be the function defined by the relation $h(\chi(x,y)) = h(x) \otimes h(y)$.

Lemma 2.2. *The function χ satisfies $\chi(\chi(x,y),z) = \chi(x,\chi(y,z))$.*

Proof.

$$h(\chi(\chi(x,y),z)) = h(\chi(x,y)) \otimes h(z) = h(x) \otimes h(y) \otimes h(z) =$$
$$= h(x) \otimes h(\chi(y,z)) = h(\chi(x,\chi(y,z)))\,.$$

Thus $\chi(\chi(x,y),z) = \chi(x,\chi(y,z))$. □

In the following lemma we use letters $..., x, y, z$ to denote objects of $\mathbf{V}$. Let x, y be objects of $\mathbf{V}$. Recall that there exists an isomorphism $\gamma_{x,y} : y^* \otimes x^* \longrightarrow (x \otimes y)^*$ given by

$$\gamma_{x,y} = (\mathrm{d}_y \otimes \mathrm{id}_{(x\otimes y)^*})(\mathrm{id}_{y^*} \otimes \mathrm{d}_x \otimes \mathrm{id}_{y\otimes(x\otimes y)^*})(\mathrm{id}_{y^*\otimes x^*} \otimes \mathrm{b}_{x\otimes y})\,. \tag{38}$$

Now define the isomorphism $\Gamma_{x,y} : y^* \otimes y \otimes x^* \otimes x \longrightarrow (x \otimes y)^* \otimes (x \otimes y)$ by the composite

$$\Gamma_{x,y} : y^* \otimes y \otimes x^* \otimes x \xrightarrow{\mathrm{id}_{y^*} \otimes \mathrm{c}_{y,x^*} \otimes \mathrm{id}_x} y^* \otimes x^* \otimes y \otimes x \xrightarrow{\gamma_{x,y} \otimes \mathrm{c}_{y,x}} (x \otimes y)^* \otimes (x \otimes y)\,.$$

Lemma 2.3. *The isomorphisms $\Gamma_{x,y}$ satisfy the relation*

$$\Gamma_{x,y\otimes z}(\Gamma_{y,z} \otimes \mathrm{id}_{x^*\otimes x}) = \Gamma_{x\otimes y,z}(\mathrm{id}_{z^*\otimes z} \otimes \Gamma_{x,y}).$$

That is, if x, y and z are objects of $\boldsymbol{V}$ then the following diagram commutes

$$\begin{tikzcd}
z^* \otimes z \otimes y^* \otimes y \otimes x^* \otimes x \arrow[r, "\Gamma_{y,z} \otimes \mathrm{id}_{x^*\otimes x}"] \arrow[d, "\mathrm{id}_{z^*\otimes z} \otimes \Gamma_{x,y}"'] & (y \otimes x)^* \otimes (y \otimes x) \otimes x^* \otimes x \arrow[d, "\Gamma_{x,y\otimes z}"] \\
z^* \otimes z \otimes (x \otimes y)^* \otimes (x \otimes y) \arrow[r, "\Gamma_{x\otimes y,z}"] & (x \otimes y \otimes z)^* \otimes (x \otimes y \otimes z)\,.
\end{tikzcd}$$

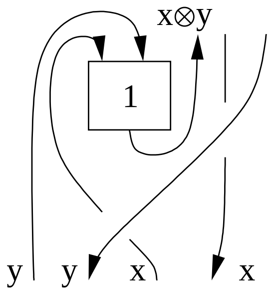

Figure 1: The morphism $\Gamma_{x,y}$

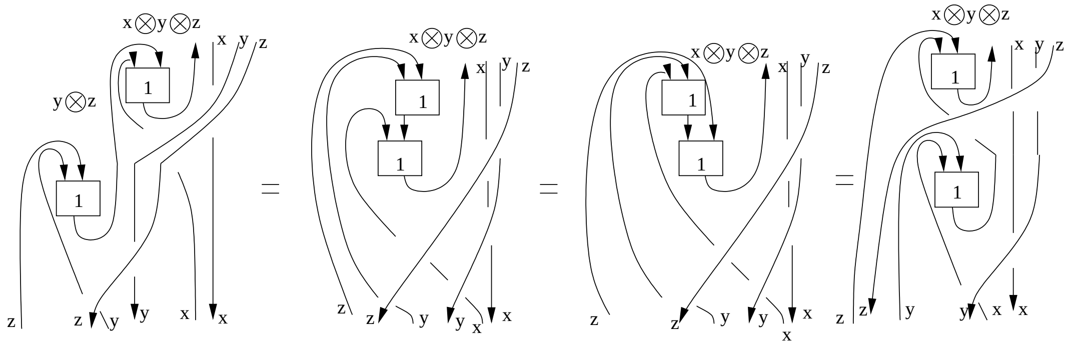

Figure 2: $\Gamma_{x,y\otimes z}(\Gamma_{y,z}\otimes \mathrm{id}_{x^*\otimes x}) = \Gamma_{x\otimes y,z}(\mathrm{id}_{z^*\otimes z}\otimes \Gamma_{x,y})$

Proof. We prove it by using graphical calculus. In Figure 1 the morphism $\Gamma_{x,y}$ is represented. Figure 2 proves the Lemma. The left and right diagrams represent the morphisms $\Gamma_{x,y\otimes z}(\Gamma_{y,z}\otimes \mathrm{id}_{x^*\otimes x})$ and $\Gamma_{x\otimes y,z}(\mathrm{id}_{z^*\otimes z}\otimes \Gamma_{x,y})$, respectively. □

Proof of Theorem 2.1. Define $\mu : \overline{\mathrm{h}}\otimes\overline{\mathrm{h}} \longrightarrow \overline{\mathrm{h}}$ by

$$\begin{aligned}\mu(v,\gamma(z,z)) : (\overline{\mathrm{h}}\otimes\overline{\mathrm{h}})(v) &= h^*(x_v)\otimes h(x_v)\otimes h^*(y_v)\otimes h(y_v)\\ &\xrightarrow{\delta_{z,\chi(y_v,x_v)}\Gamma_{h(y_v),h(x_v)}} \overline{\mathrm{h}}(\gamma(z,z)) = h^*(z)\otimes h(z)\end{aligned} \tag{39}$$

Since there is a unique $x_0 \in \mathrm{S}_h$ such that $h(x_0) = \mathrm{I} \in \mathrm{Obj}(\mathcal{C})$, we can define $\eta : \mathrm{I} \longrightarrow \overline{\mathrm{h}}$ by

$$\eta(*,\gamma(y,y)) = \delta_{x_0,y}\mathrm{id}_{\mathrm{I}} : \mathrm{I} = h^*(x_0)\otimes h(x_0) \longrightarrow h^*(y)\otimes h(y)\,.$$

We have to prove now that $\mu(\mu\otimes \mathrm{Id}_{\overline{\mathrm{h}}}) \doteq \mu(\mathrm{Id}_{\overline{\mathrm{h}}}\otimes\mu)$ and $\mu(\eta\otimes\mathrm{Id}_{\overline{\mathrm{h}}}) = \mathrm{Id}_{\overline{\mathrm{h}}} = \mu(\mathrm{Id}_{\overline{\mathrm{h}}}\otimes\eta)$.
Set $S = \mu(\mu\otimes\mathrm{Id}_{\overline{\mathrm{h}}})(w,\gamma(t,t))$ and $R = \mu(\mathrm{Id}_{\overline{\mathrm{h}}}\otimes\mu)(w',\gamma(t,t))$. We have on the

one hand

$$
\begin{aligned}
S &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \mu(v, \gamma(t,t)) \circ (\mu \otimes \mathrm{Id}_{\overline{\mathrm{h}}})(w, v) \\
&= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \delta_{t, \chi(y_v, x_v)} \Gamma_{h(y_v), h(x_v)} \circ (\delta_{x_v, \chi(y_w, x_w)} \Gamma_{h(y_w), h(x_w)} \otimes \delta_{z_w, y_v} \mathrm{id}_{h^*(z_w) \otimes h(z_w)}) \\
&= \delta_{t, \chi(z_w, \chi(y_w, x_w))} \Gamma_{h(z_w), h(\chi(y_w, x_w))} \circ (\Gamma_{h(y_w), h(x_w)} \otimes \mathrm{id}_{h^*(z_w) \otimes h(z_w)}) \\
&= \delta_{t, \chi(z_w, \chi(y_w, x_w))} \Gamma_{h(z_w), h(y_w) \otimes h(x_w)} \circ (\Gamma_{h(y_w), h(x_w)} \otimes \mathrm{id}_{h^*(z_w) \otimes h(z_w)})
\end{aligned} \tag{40}
$$

On the other hand we have

$$
\begin{aligned}
R &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \mu(v, \gamma(t,t)) \circ (\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \mu)(w', v) \\
&= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \delta_{t, \chi(y_v, x_v)} \Gamma_{h(y_v), h(x_v)} \circ (\delta_{x_{w'}, x_v} \mathrm{id}_{h^*(x_{w'}) \otimes h(x_{w'})} \otimes \delta_{y_v, \chi(z_{w'}, y_{w'})} \Gamma_{h(z_{w'}), h(y_{w'})}) \\
&= \delta_{t, \chi(\chi(z_{w'}, y_{w'}), x_{w'})} \Gamma_{h(\chi(z_{w'}, y_{w'})), h(x_v)} \circ (\mathrm{id}_{h^*(x_{w'}) \otimes h(x_{w'})} \otimes \Gamma_{h(z_{w'}), h(y_{w'})})) \\
&= \delta_{t, \chi(\chi(z_{w'}, y_{w'}), x_{w'})} \Gamma_{h(z_{w'}) \otimes h(y_{w'}), h(x_v)} \circ (\mathrm{id}_{h^*(x_{w'}) \otimes h(x_{w'})} \otimes \Gamma_{h(z_{w'}), h(y_{w'})}))
\end{aligned} \tag{41}
$$

According to Lemma 2.2, we have $\chi(\chi(z_{w'}, y_{w'}), x_{w'}) = \chi(z_{w'}, \chi(y_{w'}, x_{w'}))$. From this and Lemma 2.3, it is easy to see that $R \circ A_{\overline{\mathrm{h}}, \overline{\mathrm{h}}, \overline{\mathrm{h}}} = S$ so $R \doteq S$.

We shall prove now that $\mu(\eta \otimes \mathrm{Id}_{\overline{\mathrm{h}}}) \doteq \mathrm{Id}_{\overline{\mathrm{h}}}$. Set $J = \mu(\eta \otimes \mathrm{Id}_{\overline{\mathrm{h}}})(u, \gamma(z,z))$. From $h(\chi(x_u, x_0)) = h(x_u) \otimes h(x_0) = h(x_u) \otimes \mathrm{I} = h(x_u)$ we deduce that $\chi(x_u, x_0) = x_u$ and since $\Gamma_{a, \mathrm{I}} = \mathrm{id}_{h^*(a) \otimes h(a)}$ for any object a of $\mathcal{C}$, we have

$$
\begin{aligned}
J &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \mu(v, \gamma(z,z)) \circ (\eta \otimes \mathrm{Id}_{\overline{\mathrm{h}}})(u, v) \\
&= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \delta_{z, \chi(y_v, x_v)} \Gamma_{h(y_v), h(x_v)} \circ (\eta(*, \gamma(x_v, x_v)) \otimes \mathrm{Id}_{\overline{\mathrm{h}}}(\gamma(x_u, x_u), \gamma(y_v, y_v))) \\
&= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}}} \delta_{z, \chi(y_v, x_v)} \Gamma_{h(y_v), h(x_v)} \circ (\delta_{x_0, x_v} \mathrm{id}_{\mathrm{I}} \otimes \delta_{x_u, y_v} \mathrm{id}_{h^*(x_u) \otimes h(x_u)}) \\
&= \delta_{z, \chi(x_u, x_0)} \Gamma_{h(x_u), h(x_0)} \\
&= \delta_{z, x_u} \Gamma_{h(x_u), \mathrm{I}} \\
&= \delta_{z, x_u} \mathrm{id}_{h^*(x_u) \otimes h(x_u)} \\
&= \mathrm{Id}_{\overline{\mathrm{h}}}(u, \gamma(z,z))
\end{aligned} \tag{42}
$$

The relation $\mu(\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \eta) \doteq \mathrm{Id}_{\overline{\mathrm{h}}}$ is proved in a similar way.

We have thus shown that $(\overline{\mathrm{h}}, \mu, \eta)$ is an algebra in $\mathcal{C}^{\mathsf{S}_0}$. Define now $\Delta : \overline{\mathrm{h}} \longrightarrow \overline{\mathrm{h}} \otimes \overline{\mathrm{h}}$ to be the function

$$
\Delta(\gamma(x,x), v) : h^*(x) \otimes h(x) \longrightarrow h^*(y_v) \otimes h(y_v) \otimes h^*(z_v) \otimes h(z_v)
$$

given by the following composite

$$\delta_{x,y_v}\delta_{x,z_v}\mathrm{id}_{h^*(x)} \otimes \mathrm{b}_{h(x)} \otimes \mathrm{id}_{h(x)} : h^*(x) \otimes h(x) \longrightarrow$$
$$\longrightarrow h^*(y_v) \otimes h(y_v) \otimes h^*(z_v) \otimes h(z_v)$$

and define $\varepsilon : \overline{\mathrm{h}} \longrightarrow \mathrm{I}$ as the function given by

$$\varepsilon(\gamma(x,x),*) = \mathrm{d}_{h(x)} : h^*(x) \otimes h(x) \longrightarrow \mathrm{I}\,.$$

We are going to prove that $(\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \Delta)\Delta \doteq (\Delta \otimes \mathrm{Id}_{\overline{\mathrm{h}}}\Delta)$ and $(\varepsilon \otimes \mathrm{Id}_{\overline{\mathrm{h}}})\Delta = \mathrm{Id}_{\overline{\mathrm{h}}} = (\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \varepsilon)$. Set $L = (\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \Delta)\Delta(\gamma(t,t),w)$. Then

$$\begin{aligned} L &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}}\otimes\overline{\mathrm{h}}}} (\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \Delta)(v,w') \circ \Delta(\gamma(t,t),v) \\ &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}}\otimes\overline{\mathrm{h}}}} (\delta_{x_v,x_w}\mathrm{id}_{h^*(x_v)\otimes h(x_v)} \otimes \delta_{y_v,y_{w'}}\delta_{y_v,z_{w'}}\mathrm{id}_{h^*(y_v)} \otimes \mathrm{b}_{h(y_v)} \otimes \mathrm{id}_{h(y_v)})\circ \\ &\quad (\delta_{t,x_v}\delta_{t,y_v}\mathrm{id}_{h^*(t)} \otimes \mathrm{b}_{h(t)} \otimes \mathrm{id}_{h(t)}) \\ &= \delta_{t,x_v}\delta_{t,y_v}\delta_{t,z_v}(\mathrm{id}_{h^*(t)\otimes h(t)} \otimes \mathrm{id}_{h^*(t)} \otimes \mathrm{b}_{h(t)} \otimes \mathrm{id}_{h(t)}) \circ (\mathrm{id}_{h^*(t)} \otimes \mathrm{b}_{h(t)} \otimes \mathrm{id}_{h(t)}) \end{aligned} \tag{43}$$

Set $R = (\Delta \otimes \mathrm{Id}_{\overline{\mathrm{h}}}\Delta)(\gamma(t,t),w)$. Then

$$\begin{aligned} R &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}}}\otimes\mathrm{S}_{\overline{\mathrm{h}}}} (\Delta \otimes \mathrm{Id}_{\overline{\mathrm{h}}})(v,w) \circ \Delta(\gamma(t,t),v) \\ &= \sum_{v \in \mathrm{S}_{\overline{\mathrm{h}}}\otimes\mathrm{S}_{\overline{\mathrm{h}}}} \delta_{x_v,x_w}\delta_{x_v,y_w}(\mathrm{id}_{h^*(x_v)} \otimes \mathrm{b}_{h(x_v)} \otimes \mathrm{id}_{h(x_v)} \otimes \delta_{y_v,z_w}\mathrm{id}_{h^*(y_v)\otimes h(y_v)})\circ \\ &\quad (\delta_{t,x_v}\delta_{t,y_v}\mathrm{id}_{h^*(t)} \otimes \mathrm{b}_{h(t)} \otimes \mathrm{id}_{h(t)}) \\ &= \delta_{t,x_v}\delta_{t,y_v}\delta_{t,z_v}(\mathrm{id}_{h^*(t)} \otimes \mathrm{b}_{h(t)} \otimes \mathrm{id}_{h(t)} \otimes \mathrm{id}_{h^*(t)\otimes h(t)}) \circ (\mathrm{id}_{h^*(t)} \otimes \mathrm{b}_{h(t)} \otimes \mathrm{id}_{h(t)}) \end{aligned} \tag{44}$$

Taking $x = h(t)$, Figure 3 shows that R and L are equal up to associativity, that is $A_{\overline{\mathrm{h}},\overline{\mathrm{h}},\overline{\mathrm{h}}}(w,w') \circ R = L$. Thus $L \doteq R$.

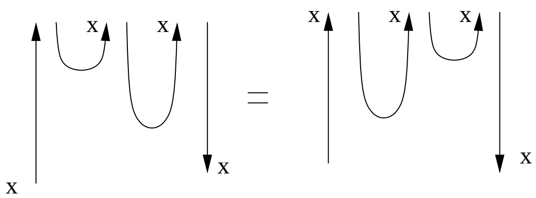

Figure 3: $(\mathrm{id}_{h_i^*} \otimes \mathrm{b}_{h_i} \otimes \mathrm{id}_{h_i} \otimes \mathrm{id}_{h_i^*\otimes h_i})(\mathrm{id}_{h_i^*} \otimes \mathrm{b}_{h_i} \otimes \mathrm{id}_{h_i}) = (\mathrm{id}_{h_i^*\otimes h_i} \otimes \mathrm{id}_{h_i^*} \otimes \mathrm{b}_{h_i} \otimes \mathrm{id}_{h_i})(\mathrm{id}_{h_i^*} \otimes \mathrm{b}_{h_i} \otimes \mathrm{id}_{h_i})$

Next we prove $(\varepsilon \otimes \mathrm{Id}_{\overline{\mathrm{h}}})\Delta = \mathrm{Id}_{\overline{\mathrm{h}}}$. Set $J = (\varepsilon \otimes \mathrm{Id}_{\overline{\mathrm{h}}})\Delta(\gamma(x,x), \gamma(y,y))$. Then

$$\begin{aligned} J &= \sum_{x \in \mathrm{S}_{\overline{\mathrm{h}}} \otimes \mathrm{S}_{\overline{\mathrm{h}}}} (\varepsilon \otimes (\mathrm{Id}_{\overline{\mathrm{h}}}))(v, \gamma(y,y)) \circ \Delta(\gamma(x,x), v) \\ &= \sum_{x \in \mathrm{S}_{\overline{\mathrm{h}}} \otimes \mathrm{S}_{\overline{\mathrm{h}}}} (\varepsilon(\gamma(x_v, x_v), *) \otimes \mathrm{Id}_{\overline{\mathrm{h}}}(\gamma(y_v, y_v), \gamma(y,y)))\circ \\ &\quad (\delta_{x,x_v}\delta_{x,y_v}\mathrm{id}_{h^*(x)} \otimes \mathrm{b}_{h(x)} \otimes \mathrm{id}_{h(x)}) \\ &= (\mathrm{d}_{h(x_v)} \otimes \delta_{y_v,y}\mathrm{id}_{h^*(y_v)\otimes h(y_v)}) \circ (\delta_{x,x_v}\delta_{x,y_v}\mathrm{id}_{h^*(x)} \otimes \mathrm{b}_{h(x)} \otimes \mathrm{id}_{h(x)}) \\ &= \delta_{x,y}(\mathrm{d}_{h(x)} \otimes \mathrm{id}_{h^*(x)\otimes h(x)}) \circ (\mathrm{id}_{h^*(x)} \otimes \mathrm{b}_{h(x)} \otimes \mathrm{id}_{h(x)}) \end{aligned} \tag{45}$$

From the definition of left duality we get $(\mathrm{d}_{h(x)} \otimes \mathrm{id}_{h^*(x)})(\mathrm{id}_{h^*(x)} \otimes \mathrm{b}_{h(x)}) = \mathrm{id}_{h^*(x)}$, so $J = \delta_{x,y}\mathrm{id}_{h^*(x)\otimes h(x)} = \mathrm{Id}_{\overline{\mathrm{h}}}(\gamma(x,x), \gamma(y,y))$.

The relation $\mathrm{Id}_{\overline{\mathrm{h}}} = (\mathrm{Id}_{\overline{\mathrm{h}}} \otimes \varepsilon)$ is proved in a similar way, and with this we have showed $(\overline{\mathrm{h}}, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{C}^{\mathsf{S}_0}$.

It is enough to prove now that Δ and ε are algebra morphisms. For Δ we have to show that the diagram

$$\begin{tikzcd} \overline{\mathrm{h}} \otimes \overline{\mathrm{h}} \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"'] & (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \otimes (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \arrow[d, "\hat{\mu}"] \\ \overline{\mathrm{h}} \arrow[r, "\Delta"] & \overline{\mathrm{h}} \otimes \overline{\mathrm{h}} \end{tikzcd}$$

commutes up to the relation $\doteq$, where $\hat{\mu}$ is the product in $\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}$ and it is defined, as in (36), by the composite

$$\begin{tikzcd} \hat{\mu} : (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \otimes (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \arrow[r, "A^{-1}_{\overline{\mathrm{h}}\otimes\overline{\mathrm{h}},\overline{\mathrm{h}},\overline{\mathrm{h}}}"] & ((\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \otimes \overline{\mathrm{h}}) \otimes \overline{\mathrm{h}} \arrow[d, "(\mathrm{Id}_{\overline{\mathrm{h}}}\otimes \mathrm{C}_{\overline{\mathrm{h}},\overline{\mathrm{h}}}\otimes \mathrm{Id}_{\overline{\mathrm{h}}})A^{-1}_{\overline{\mathrm{h}},\overline{\mathrm{h}},\overline{\mathrm{h}}}"] \\ & (\overline{\mathrm{h}} \otimes (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}})) \otimes \overline{\mathrm{h}} \arrow[d, "(\mu\otimes\mu)A_{\overline{\mathrm{h}}\otimes\overline{\mathrm{h}},\overline{\mathrm{h}},\overline{\mathrm{h}}}(A^{-1}_{\overline{\mathrm{h}},\overline{\mathrm{h}},\overline{\mathrm{h}}}\otimes \mathrm{id}_{\overline{\mathrm{h}}})"] \\ & \overline{\mathrm{h}} \otimes \overline{\mathrm{h}}\,. \end{tikzcd}$$

The morphism $\mathrm{Id}_{\overline{\mathrm{h}}}\otimes \mathrm{C}_{\overline{\mathrm{h}},\overline{\mathrm{h}}}\otimes \mathrm{Id}_{\overline{\mathrm{h}}}(v,w) : (\overline{\mathrm{h}}\otimes(\overline{\mathrm{h}}\otimes\overline{\mathrm{h}})\otimes\overline{\mathrm{h}})(v) \longrightarrow (\overline{\mathrm{h}}\otimes(\overline{\mathrm{h}}\otimes\overline{\mathrm{h}})\otimes\overline{\mathrm{h}})(w)$ is related to $F_w(v,w) : (\overline{\mathrm{h}} \otimes (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \otimes \overline{\mathrm{h}})(v) \longrightarrow (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}}) \otimes (\overline{\mathrm{h}} \otimes \overline{\mathrm{h}})(w)$, which is represented by the following vertical arrow

$$\begin{tikzcd} h(x_v)^* \otimes h(x_v) \otimes h(y_v)^* \otimes h(y_v) \otimes h(z_v)^* \otimes h(z_v) \otimes h(t_v)^* \otimes h(t_v) \arrow[d, "F_w(v,w)=\delta_x^{v,w}\mathrm{id}_{h(x_v)^*\otimes h(x_v)}\otimes\delta_{y;z}^{v,w}\mathrm{c}_{h(y_v)^*\otimes h(y_v),h(z_v)^*\otimes h(z_v)}\otimes\delta_t^{v,w}\mathrm{id}_{h(t_v)^*\otimes h(t_v)}"'] \\ h(x_w)^* \otimes h(x_w) \otimes h(z_w)^* \otimes h(z_w) \otimes h(y_w)^* \otimes h(y_w) \otimes h(t_w)^* \otimes h(t_w) \end{tikzcd}$$

since their codomains are related by associativity.

The morphism $(\mu\otimes\mu)(w,u) : (\overline{\mathrm{h}}\otimes\overline{\mathrm{h}})\otimes(\overline{\mathrm{h}}\otimes\overline{\mathrm{h}}) \longrightarrow \overline{\mathrm{h}}\otimes\overline{\mathrm{h}}$, is represented by the following vertical arrow

$$\begin{array}{c} h(x_w)^*\otimes h(x_w)\otimes h(z_w)^*\otimes h(z_w)\otimes h(y_w)^*\otimes h(y_w)\otimes h(t_w)^*\otimes h(t_w) \\ \Big\downarrow{\scriptstyle \delta_{x_u,\chi(z_w,x_w)}\Gamma_{h(z_w),h(x_w)}\otimes\delta_{y_u,\chi(t_w,y_w)}\Gamma_{h(t_w),h(y_w)}} \\ h(x_u)^*\otimes h(x_u)\otimes h(y_u)^*\otimes h(y_u) \end{array}$$

It is not difficult to see that $\hat{\mu} \doteq \sum_w (\mu\otimes\mu)\circ F_w$ and that this last morphism turns out to be equal to

$$\begin{array}{c} h(x_v)^*\otimes h(x_v)\otimes h(y_v)^*\otimes h(y_v)\otimes h(z_v)^*\otimes h(z_v)\otimes h(t_v)^*\otimes h(t_v) \\ \Big\downarrow{\scriptstyle \delta_{x_u,\chi(z_v,x_v)}\delta_{y_u,\chi(t_v,y_v)}(\Gamma_{h(z_v),h(x_v)}\otimes\Gamma_{h(t_v),h(y_v)})(\mathrm{id}_{h(x_v)^*\otimes h(x_v)}\otimes\mathrm{c}_{h(y_v)^*\otimes h(y_v),h(z_v)^*\otimes h(z_v)}\otimes\mathrm{id}_{h(t_v)^*\otimes h(t_v)})} \\ h(x_u)^*\otimes h(x_u)\otimes h(y_u)^*\otimes h(y_u) \end{array}$$

Hence $\hat{\mu}(\Delta\otimes\Delta) \doteq \sum_v G_v\circ(\Delta\otimes\Delta)$, where G_v is the last vertical arrow. But $(\Delta\otimes\Delta)(p,v) : (\overline{\mathrm{h}}\otimes\overline{\mathrm{h}})(p) \longrightarrow ((\overline{\mathrm{h}}\otimes\overline{\mathrm{h}})\otimes(\overline{\mathrm{h}}\otimes\overline{\mathrm{h}}))(v)$ is given by

$$(\Delta\otimes\Delta)(p,v) = \delta_{x_p,x_v}\delta_{x_p,y_v}\delta_{y_p,z_v}\delta_{y_p,t_v}(\mathrm{id}_{h(x_p)^*}\otimes\mathrm{b}_{h(x_p)}\otimes\mathrm{id}_{h(x_p)})(\mathrm{id}_{h(y_p)^*}\otimes\mathrm{b}_{h(y_p)}\otimes\mathrm{id}_{h(y_p)})$$

so the sum yields

$$\begin{aligned} M = {} & \delta_{x_u,\chi(y_p,x_p)}\delta_{y_u,\chi(y_p,x_p)}(\Gamma_{y_p,x_p}\otimes\Gamma_{y_p,x_p})(\mathrm{id}_{h(x_p)^*\otimes h(x_p)}\otimes\mathrm{c}_{h(x_p)^*\otimes h(x_p),h(y_p)^*\otimes h(y_p)} \\ & \otimes\mathrm{id}_{h(y_p)^*\otimes h(y_p)})(\mathrm{id}_{h(x_p)^*}\otimes\mathrm{b}_{h(x_p)}\otimes\mathrm{id}_{h(x_p)})(\mathrm{id}_{h(y_p)^*}\otimes\mathrm{b}_{h(y_p)}\otimes\mathrm{id}_{h(y_p)}) \end{aligned}$$

On the other hand, $(\mu\Delta)(p,u)$ is the sum over v of the following composite

$$\begin{array}{ccc} h(x_p)^*\otimes h(x_p)\otimes h(y_p)^*\otimes h(y_p) & \xrightarrow{\delta_{x_v,\chi(y_p,x_p)}\Gamma_{h(y_p),h(x_p)}} & h(x_v)^*\otimes h(x_v) \\ & & \Big\downarrow{\scriptstyle \delta_{x_v,x_u}\delta_{x_v,y_u}(\mathrm{id}_{h(x_v)^*}\otimes\mathrm{b}_{h(x_v)}\otimes\mathrm{id}_{h(x_v)})} \\ & & h(x_u)^*\otimes h(x_u)\otimes h(y_u)^*\otimes h(y_u) \end{array}$$

which is equal to

$$(\mu\Delta)(p,u) = \delta_{x_u,\chi(y_p,x_p)}\delta_{y_u,\chi(y_p,x_p)}(\mathrm{id}^*_{h(\chi(y_p,x_p))}\otimes\mathrm{b}_{h(\chi(y_p,x_p))}\otimes\mathrm{id}_{h(\chi(y_p,x_p))})\Gamma_{h(y_p),h(x_p)}$$

In Figure 4, taking $y = x_p$ and $x = y_p$, the picture on the left side represents M, while that on the right side represents $(\mu\Delta)(p,u)$. Hence both are equal and then $\mu\Delta \doteq \hat{\mu}(\Delta\otimes\Delta)$. Finally, we have to prove that ε is an algebra morphism, that is, we have to prove that the diagram

$$\begin{tikzcd} \overline{\mathrm{h}}\otimes\overline{\mathrm{h}} \arrow[d, "\mu"'] \arrow[dr, "\varepsilon\otimes\varepsilon"] & \\ \overline{\mathrm{h}} \arrow[r, "\varepsilon"] & \mathrm{I} \end{tikzcd}$$

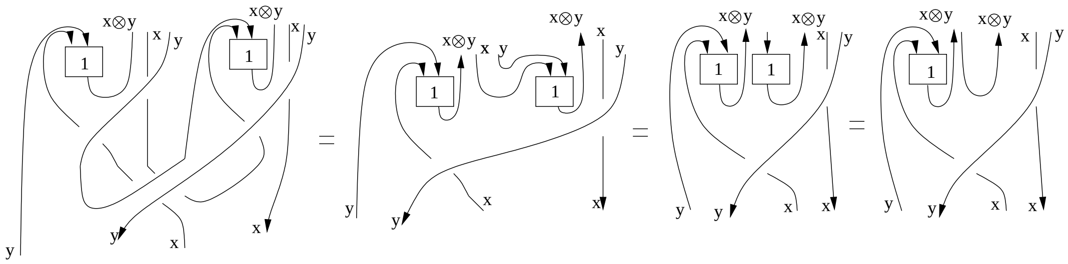

Figure 4: $(\hat{\mu}(\Delta \otimes \Delta))(p,u) \doteq (\mu\Delta)(p,u)$

commutes. We have

$$\begin{aligned}(\varepsilon\mu)(u,*) &= \sum_w \varepsilon(\gamma(x_w, x_w), *) \circ \mu(u,w) \\ &= \sum_w \mathrm{d}_{h(x_w)} \circ (\delta_{x_w,\chi(y_u,x_u)} \Gamma_{h(y_u),h(x_u)}) \\ &= \mathrm{d}_{h(\chi(y_u,x_u))} \Gamma_{h(y_u),h(x_u)} \\ &= \mathrm{d}_{h(y_u)\otimes h(x_u)} \Gamma_{h(y_u),h(x_u)}\end{aligned}$$

On the other hand, $(\varepsilon \otimes \varepsilon)(u,*) = \mathrm{d}_{h(x_u)\otimes h(y_u)}$.

Figure 5, taking $x = y_u$ and $y = x_u$ as before, shows that these two morphisms are equal. Therefore $(\overline{\mathrm{h}}, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra.

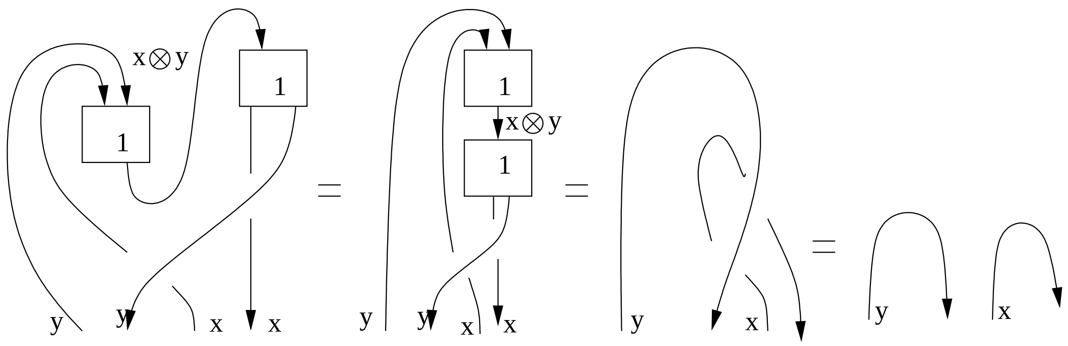

Figure 5: $\mathrm{d}_{h(y_u)\otimes h(x_u)} \Gamma_{h(y_u),h(x_u)} = \mathrm{d}_{h(x_u)\otimes h(y_u)}$

We shall now define the action of $\overline{\mathrm{h}}$ on the objects of $\mathcal{C}$. Take the point x_0 of S_h such that $h(x_0) = \mathrm{I}$ and define $j_V(x_0) = V$, for each V object of $\mathcal{C}$. Define $T : \overline{\mathrm{h}} \otimes j_V \longrightarrow j_V$, by

$$T(\gamma(\gamma(x,x),x_0),x_0) = \mathrm{d}_{h(x)} \otimes \mathrm{id}_V : h^*(x) \otimes h(x) \otimes V \longrightarrow V$$

where $x \in S_h$. It is not difficult to see that T is indeed an action as we defined it before. The proof of that is similar (although easier and shorter) to the previous proofs and we omitted it.

References

[1] Kassel, C. *Quantum Groups*, Springer-Verlag, New York 1995.

[2] Turaev, V. G., *Quantum Invariants of Knots and* 3*-Manifolds* de Gruyter Studies in Math., vol. 18, Walter de Gruyter, Berlin, 1994.

[3] Yetter, D. N., Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge Philos. Soc.* **108** (1990), no. 2, 261–290.

[4] Majid, S., Algebras and Hopf algebras in braided categories, *Marcel Dekker L. N. in Math.***158**(1994), 55–105.

[5] Takeuchi, M., Finite Hopf algebras in braided tensor categories, *J. Pure Appl. Algebra* 138(1999), 59–82.

[6] Takeuchi,M., Survey of braided Hopf algebras, in *New trends in Hopf Algebra Theory* (La Falda, 1999), Contemp. Mathematics **267** (2000), 301–324.

[7] Mac Lane, S., *Categories for the Working Mathematician*, Spinger-Verlag, New York-Heidelberg-Berlin 1971.