A DECOMPOSITION FORMULA FOR EQUIVARIANT STABLE HOMOTOPY CLASSES

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Dedicated to Albrecht Dold and Ed Fadell

Abstract. For any compact Lie group G, we give a decomposition of the group $\{X, Y\}_G^k$ of (unpointed) stable G-homotopy classes as a direct sum of subgroups of fixed orbit types. This is done by interpreting the G-homotopy classes in terms of the generalized fixed-point transfer and making use of conormal maps.

0. Introduction

A description of the homotopy classes, or of the stable homotopy classes of maps between two topological spaces has been a classical question in topology. A variant of the question arises when we assume that a compact Lie group G acts on all spaces involved and that all the maps considered commute with the group action, that is, that the maps are G-equivariant –G-maps for short. Then the corresponding question is to provide a description of the stable G-homotopy classes between G-spaces.

In this paper we give a decomposition of the group of equivariant stable homotopy classes of maps between two G-spaces X and Y, provided that X has trivial G-action (Theorem 1.8). A similar result was proven by Lewis, Jr., May, and McClure in [6, V.10.1] under other assumptions (they consider more general symmetry and their space X is a finite CW-complex) and using rather different methods. Using classical methods in algebraic topology, tom Dieck gives a decomposition of the equivariant homotopy groups in his book [2, II(7.7)]. An

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advantage of our approach is that it gives a short proof showing the geometric interpretation of the maps that form a term of this decomposition, even in the unstable range as in [7]. In particular, we do not need the Adams and Wirthmüller isomorphisms to define the splitting homomorphism. To carry out the decomposition, we use the equivariant fixed-point transfer given by the second author in [9], which is the equivariant generalization of the classical Dold fixed-point transfer [3], and the fixed-point theoretical arguments used in [7]. We intend to make this result clear to nonlinear analysts.

A special case of our main theorem 1.8 yields a decomposition of the G-equivariant 1-stem, that was given using different methods by Kosniowski [5], Hauschild [4], and Balanov-Krawcewicz [1]. This decomposition was also used by us [8] to give a full description of the first G-stem as follows:

$$\pi_1^{G\,\mathrm{st}} = \bigoplus_{\substack{(H)\in \mathrm{Or}_G \\ \dim W(H)\leq 1}} \Pi_1(H)\,,$$

where, if $\dim W(H) = 0$,

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}}\,,$$

and $W(H)_{\mathrm{ab}}$ is the abelianization of $W(H)$, and, if $\dim W(H) = 1$,

$$\Pi_1(H) \cong \begin{cases} \mathbb{Z} & W(H) \text{ is biorientable,} \\ \mathbb{Z}_2 & \text{if } W(H) \text{ is not biorientable.} \end{cases}$$

1. The general decomposition formula

In this section, we use the generalized fixed-point transfer to give a direct sum decomposition of $\{X, Y\}_G^k$. All along the paper, G will denote a compact Lie group. We shall assume that X and Y are metric spaces with a G-action.

Definition 1.1. Let V, W, M, and N denote finite dimensional real G-modules, namely, orthogonal representations of G, and let ρ be the element $[M] - [N] \in \mathrm{RO}(G)$. Then the elements of $\{X, Y\}_G^\rho$ are *stable homotopy classes* represented by equivariant maps of pairs

$$\alpha : (N \times V, N \times V - 0) \times X \longrightarrow (M \times V, M \times V - 0) \times Y\,.$$

Such a map will be *stably homotopic* to another

$$\alpha' : (N \times V', N \times V' - 0) \times X \longrightarrow (M \times V', M \times V' - 0) \times Y\,,$$

if after taking the product of each map with the identity maps of some pairs $(W, W - 0)$ and $(W', W' - 0)$, respectively, they become

G-homotopic, where $V \times W \cong_G V' \times W'$. Denote the class of α by $\{\alpha\}$.

Remark 1.2. Taking the product of X with a pair $(L, L-0)$ for some orthogonal representation L of G amounts to the same as smashing $X^+ = X \sqcup \{*\}$ with the sphere $\mathbb{S}^L$ that is obtained as the one-point compactification of L (which is G-homeomorphic to the unit sphere $S(L \oplus \mathbb{R})$ in the representation $L \oplus \mathbb{R}$, with trivial action on the last coordinate). Thus

$$\begin{aligned}\{X, Y\}^\rho_G &\cong \operatorname*{colim}_V [\mathbb{S}^N \wedge \mathbb{S}^V \wedge X^+, \mathbb{S}^M \wedge \mathbb{S}^V \wedge Y^+]_G \\ &\cong \operatorname*{colim}_V [\mathbb{S}^{N\oplus V} \wedge X^+, \mathbb{S}^{M\oplus V} \wedge Y^+]_G \\ &\cong \operatorname*{colim}_V [X^+, \Omega_{N\oplus V}\mathbb{S}^{M\oplus V} \wedge Y^+]_G\,,\end{aligned}$$

where the colimit of pointed G-homotopy classes is taken over a cofinal system of G-representations V. Observe that this does not coincide with the usual definition, when X is infinite dimensional. For homotopy theoretical purposes, the definition is given by

$$G\text{-}\mathfrak{S}tab^\rho(X, Y) = [X^+, \operatorname*{colim}_V \Omega_{N\oplus V}\mathbb{S}^{M\oplus V} \wedge Y^+]_G$$

with the colimit taken 'inside'. However, for the purposes of nonlinear analysis, our definition seems to be more adequate.

In [9] (see also [10]) one proves that any $\{\alpha\} \in \{X, Y\}^\rho_G$ can be written as a composite

$$\{\alpha\} = \varphi \circ \tau(f)\,, \tag{1.3}$$

where $\tau(f)$ is the equivariant fixed-point transfer of an equivariant fixed-point situation

$$\begin{array}{ccc} N \times E \supset \mathfrak{U} & \xrightarrow{\quad f \quad} & M \times E \\ & {\scriptstyle p\cdot \mathrm{proj}_E}\searrow \quad \swarrow{\scriptstyle p\cdot \mathrm{proj}_E} & \\ & X\,, & \end{array} \tag{1.4}$$

where $E \longrightarrow X$ is a G-ENR_X and the *fixed point set* $\mathrm{Fix}(f) = \{(s, e) \in \mathfrak{U} \mid f(s, e) = (0, e) \in M \times E\}$ lies properly over X, $\rho = [M] - [N] \in \mathrm{RO}(G)$. The transfer is a stable map

$$\tau(f) : (N \times V, N \times V - 0) \times X \longrightarrow (M \times V, M \times V - 0) \times \mathfrak{U}\,,$$

for some orthogonal representation V, and $\varphi : \mathfrak{U} \longrightarrow Y$ is a nonstable equivariant map (by the localization property of the fixed-point transfer, $\mathfrak{U}$ can always be assumed to be a very small open G-neighborhood of the fixed point set $\mathrm{Fix}(f)$; see [10, 4.4]), (the composite is made after suspending φ by taking its product with the identity of $(M \times V, M \times V - 0))$.

We denote by Or_G the set of orbit types of G, that is the set of conjugacy classes (H) of subgroups $H \subset G$. For any G-ENR_X E, where X has trivial G-action, the set of orbit types in E, denoted by $\mathrm{Or}_G(E)$, is always finite.

In what follows, we shall only be concerned with the special case $N = \mathbb{R}^n$, $M = \mathbb{R}^{n+k}$, $k \in \mathbb{Z}$, and we shall assume that X is a space with trivial G-action.

For the statement of the main result of this section we need the following definitions. The first of them was originally given in [7, 5.4].

Definition 1.5. Consider the fixed-point situation (1.4) above. We say that the map $f : \mathcal{U} \longrightarrow \mathbb{R}^{n+k} \times E$ is *conormal* if for every orbit type $(H) \in \mathrm{Or}_G(\mathbb{R}^n \times E) = \mathrm{Or}_G(E)$, there exist an open invariant neighborhood $\mathcal{V}$ of $\mathcal{U}^{(\underline{H})}$ in $\mathcal{U}^{(H)}$ and an equivariant retraction $r : \overline{\mathcal{V}} \longrightarrow \mathcal{U}^{(\underline{H})}$ such that for the restricted map $f^{(H)} = f|_{\mathcal{U}^{(H)}}$ we have

$$f^{(H)}|_{\overline{\mathcal{V}}} = f \circ r : \overline{\mathcal{V}} \longrightarrow \mathbb{R}^{n+k} \times E\,.$$

Here $\mathcal{U}^{(H)}$ consists of the points in $\mathcal{U}$ with isotropy larger than (H) and $\mathcal{U}^{(\underline{H})}$ to those with isotropy **strictly** larger than (H).

Definition 1.6. For any subgroup $H \subset G$, we define the subgroup $\{X, Y\}^k_{(H)}$ of $\{X, Y\}^k_G$ as the subgroup of those classes $\{\alpha\}$ such that $\{\alpha\} = \varphi \circ \tau(f)$, where

(a) f is a conormal map, and
(b) $\mathrm{Fix}(f) \subset \mathcal{U}_{(H)}$, where $\mathcal{U}_{(H)}$ consists of the points in $\mathcal{U}$ with isotropy group conjugate to H.

Remark 1.7. The fact that $\{X, Y\}^k_{(H)}$ is a subgroup of $\{X, Y\}^k_G$ follows easily by observing that both properties (a) and (b) are preserved by the sum of two elements $\{\alpha\} = \varphi \circ \tau(f)$, $\{\beta\} = \psi \circ \tau(g)$, that, by the additivity property of the fixed-point transfer, corresponds to the disjoint union $f + g$ of the fixed-point situations (see [11, 1.17]).

The main result in this section is the following.

Theorem 1.8. *Let X be a space with trivial G-action. Then there is an isomorphism*

$$\{X, Y\}^k_G \cong \bigoplus_{(H)} \{X, Y\}^k_{(H)}\,.$$

For the proof we need some preliminary results. Consider a fixed-point situation as (1.4). First note that it is always possible to provide

$\mathrm{Or}_G(E)$ with an order (H_j), $j = 1, 2, \ldots, l$ such that $(H_i) \subset (H_j)$ implies $j \leq i$. Define $E_i \subset E$ as $\bigcup_{i \leq j} E^{(H_j)}$. These G-subspaces determine a filtration of E such that $E_i - E_{i-1} = E_{(H_i)}$. Let $f_i = f|_{\mathcal{U}_i} : \mathcal{U}_i \longrightarrow \mathbb{R}^{n+k} \times E_i$, where $\mathcal{U}_i = \mathcal{U} \cap (\mathbb{R}^n \times E_i)$.

Proposition 1.9. *For every $i = 1, 2, \ldots, l$ there exists an invariant neighborhood $\mathcal{V}_i$ of E_{i-1} in E_i and an equivariant retraction $r_i : \overline{\mathcal{V}}_i \longrightarrow E_{i-1}$ that is admissibly homotopic to the identity. Thus f_i is admissibly homotopic to $f'_{i-1} = f_{i-1} \circ (\mathrm{id}_{\mathbb{R}^n} \times r_i)$.*

The proof is similar to those of [7, 5.3 and 5.7]. □

Proposition 1.10. *The following hold:*

(a) *f is equivariantly homotopic by an admissible homotopy f_τ to a conormal map $f' = f_1 : V \longrightarrow \mathbb{R}^m \times E$. Moreover, if $A \subset \mathcal{U}$ is a closed G-*ENR *subspace, then this homotopy can be taken relative to A.*

(b) *Furthermore, if f_0 and f_1 are equivariantly homotopic by an admissible homotopy, and each of them is equivariantly homotopic by an admissible homotopy to two conormal maps f'_0, $f'_1 : \mathcal{U} \longrightarrow \mathbb{R}^m \times E$, respectively, then these two conormal maps are equivariantly homotopic by an admissible conormal homotopy.*

The proof is the same as that of [7, 5.7] (see also [11, 2.10 and 2.11] or [12, II.6.8 and III.5.2]). □

We also need a lemma.

Lemma 1.11. *Let $f : \mathcal{U} \longrightarrow \mathbb{R}^{n+k} \times E$ be a fixed-point situation over X such that f is a conormal map and take $(H) \in \mathrm{Or}_G(E)$. Then there is a neighborhood $\mathcal{V}$ of $\mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$ such that $g = f|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{R}^{n+k} \times E$ is a conormal map with $\mathrm{Fix}(g) = \mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$. Denote g by $f_{(H)}$. Consequently,*

$$\tau(f) = \sum_{(H) \in \mathrm{Or}_G(E)} \tau(f_{(H)}) \,. \tag{1.12}$$

Proof. Since f is conormal, the set $F = \mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$ is separated from all other fixed points. Then there is a neighborhood $\mathcal{V}$ of F in $\mathcal{U}$ such that $\mathrm{Fix}(f) \cap \mathcal{V} = F$. Hence $g = f|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{R}^{n+k} \times E$ is a conormal map with the desired properties. By the additivity property of the transfer we obtain the decomposition (1.12). □

We now pass to the proof of Theorem 1.8.

Proof. Any $\{\alpha\} \in \{X,Y\}^k_G$ can be written as the composite (1.3) $\varphi \circ \tau(f)$, where $\tau(f)$ is the equivariant fixed-point transfer of an equivariant fixed-point situation (1.4). By Proposition 1.10 (a), f can be assumed to be a conormal map, and by Lemma 1.11, $\tau(f) = \sum_{(H)\in \mathrm{Or}_G(E)} \tau(f_{(H)})$. Defining $\{\alpha_{(H)}\}$ by $\{\alpha_{(H)}\} = \varphi|_{\mathrm{U}_{(H)}} \circ \tau(f_{(H)})$, we have immediately

$$\{\alpha\} = \sum_{(H)\in \mathrm{Or}_G(E)} \{\alpha_{(H)}\}\,, \tag{1.13}$$

where $\{\alpha_{(H)}\} \in \{X,Y\}^k_{(H)}$. So, by Proposition 1.10 (b), we may define

$$\Phi : \{X,Y\}^k_G \longrightarrow \bigoplus_{(H)} \{X,Y\}^k_{(H)} \quad \text{by} \quad \Phi(\{\alpha\}) = \oplus_{(H)}\{\alpha_{(H)}\}\,.$$

If $(H) \neq (K)$, then $\{X,Y\}^k_{(H)} \cap \{X,Y\}^k_{(K)} = 0$ as easily follows with the same argument used in the proof of [7, 6.2]. Thus we may also define

$$\Psi : \bigoplus_{(H)} \{X,Y\}^k_{(H)} \longrightarrow \{X,Y\}^k_G \quad \text{by} \quad \Psi(\oplus_{(H)}\{\alpha_{(H)}\}) = \sum_{(H)} \{\alpha_{(H)}\}\,.$$

Then Φ and Ψ are inverse isomorphisms. □

Remark 1.14. For any fixed-point situation f (see (1.4)), it is proven in [10, 4.4] that the transfer

$$\tau(f) = \sum_{(H)\in \mathrm{O}(G)} (\tau(f^{(H)}) - \tau(f^{(\underline{H})}))\,,$$

where

$$f^{(H)} = f|_{U^{(H)}} : U^{(H)} \longrightarrow (M \times E)^{(H)}$$

and $U^{(H)} \subset (N \times E)^{(H)} = \mathbb{R}^n \times E^{(H)}$, resp.

$$f^{(\underline{H})} = f|_{U^{(\underline{H})}} : U^{(\underline{H})} \longrightarrow (M \times E)^{(\underline{H})}$$

and $U^{(\underline{H})} \subset (N \times E)^{(\underline{H})} = \mathbb{R}^n \times E^{(\underline{H})}$.

As in (1.3) any $\{\alpha\} = \varphi \circ \tau(f)$ for some fixed-point situation f as above. As in the proof of [10, 4.4], we have

$$\{\alpha^{(H)}\} = \varphi^{(H)} \circ \tau(f^{(H)}) : X \longrightarrow Y^{(H)} \subset Y\,,$$

$$\{\alpha^{(\underline{H})}\} = \varphi^{(\underline{H})} \circ \tau(f^{(\underline{H})}) : X \longrightarrow Y^{(\underline{H})} \subset Y\,.$$

Thus $\{\alpha\} = \sum_{(H)\in \mathrm{O}(G)}(\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\})$. Hence it is each difference $\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\} = \{\alpha_{(H)}\}$; that is, $\alpha_{(H)}$, as given by the conormal map, realizes the (stable) difference $\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\}$ for each orbit type (H) (cf. also [12, III.5]).

References

[1] Z. Balanov, W. Krawcewicz, Remarks on the equivariant degree theory, *Topol. Meth. in Nonlinear Analysis*, **13** (1999), no. 1, 91–103.

[2] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin - New York 1987

[3] A. Dold, The fixed point transfer of fiber-preserving maps, *Math. Z.* **148** (1976), 215–244.

[4] H. Hauschild, Zerspaltung äquivarianter Homotopiemengen, *Math. Ann.* **230** (1977), no. 3, 279–292.

[5] Cz. Kosniowski, Equivariant cohomology and stable cohomotopy, *Math. Ann.* **210** (1974), 83–104.

[6] L. G. Lewis, Jr., J. P. May, M. Steinberger, with contributions of J. E. McClure, *Equivariant Stable Homotopy Theory*, Lecture Notes in Math. **1213**, Springer-Verlag, Berlin, Heidelberg, 1986.

[7] W. Marzantowicz, C. Prieto, The unstable equivariant fixed point index and the equivariant degree, *J. London Math. Soc.* (2) **69** (2004), no. 1, 214–230.

[8] W. Marzantowicz, C. Prieto, Computation of the equivariant 1-stem, *Nonlinear Anal.* **63** (2005), no. 4, 513–524.

[9] C. Prieto, Transfers generate the equivariant stable homotopy category, *Top. and its Appl.* **58**, (1994), 181–191.

[10] C. Prieto, A sum formula for stable equivariant maps, *Nonlinear Anal.* **30 No 6**, (1997), 3475–3480.

[11] C. Prieto, H. Ulrich, Equivariant fixed point index and fixed point transfer in nonzero dimensions, *Trans. Amer. Math. Soc.* **328**, (1991), 731–745.

[12] H. Ulrich, *Fixed Point Theory of Parametrized Equivariant Maps*, Lecture Notes in Math. **1343**, Springer-Verlag, Berlin, Heidelberg, 1988.

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