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Topology and its Applications 154 (2007) 2826–2848
Topology and its Applications
www.elsevier.com/locate/topol

Equivariant homotopical homology with coefficients in a Mackey functor

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Received 8 August 2006; received in revised form 8 June 2007; accepted 8 June 2007

Abstract

Let M be a Mackey functor for a finite group G. In this paper, generalizing the Dold–Thom construction, we construct an ordinary equivariant homotopical homology theory $\mathbb{H}_*^G(-;M)$ with coefficients in M, whose values on the category of finite G-sets realize the bifunctor M, both covariantly and contravariantly. Furthermore, we extend the contravariant functor to define a transfer in the theory $\mathbb{H}_*^G(-;M)$ for G-equivariant covering maps. This transfer is given by a continuous homomorphism between topological abelian groups.

We prove a formula for the composite of the transfer and the projection of a G-equivariant covering map and characterize those Mackey functors M for which that formula has an expression analogous to the classical one.

MSC: 55N91; 55P91; 14F43

Keywords: Equivariant homology; Homotopy groups; Mackey functors; Topological abelian groups

0. Introduction

Let M be a Mackey functor for a finite group G. By forgetting the contravariant part of M, we obtain a covariant coefficient system M_* defined on the category $\mathcal{O}(G)$ of canonical orbits. In [6], Illman constructed an ordinary G-equivariant homology theory $H^G(-;M_*)$, whose coefficients are isomorphic to M_*. If X is a G-space, then $H^G(X;M_*)$ is obtained by taking the homology of a chain complex associated to X and M_*. In this paper we shall consider both the covariant functor M_* and the contravariant functor M^* associated to M. For each (pointed) G-space X we construct a topological abelian group $F^G(X,M)$ and we define an ordinary G-equivariant homology theory by taking the homotopy groups of $F^G(X,M)$. More precisely, we define

$$\mathbb{H}_q^G(X;M)=\pi_q\big(F^G(X^+,M)\big),$$

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1 This author was partially supported by PAPIIT grant IN105106-3 and by CONACYT grant 43724.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.06.005

where $X^+ = X \sqcup \{*\}$. If S is a finite G-set, then we have a natural equivalence of bifunctors $\mathbb{H}_0^G(S; M) \cong M(S)$. This approach to equivariant homology has had many applications in algebraic geometry. See for instance [14,15].

When G is the trivial group, $\mathbb{H}_*^G(-; M)$ is singular homology with coefficients in the group M, and our statement is the classical Dold–Thom theorem [5], which was extended to the equivariant case when the coefficient group is a G-module L by Lima-Filho [8] (when $L = \mathbb{Z}$ with trivial G-action) and by dos Santos [14] (when L is any G-module). Both the original result and these equivariant generalizations were proved by showing that the homotopical definition satisfies the axioms of an ordinary or an equivariant homology theory, and then using a uniqueness theorem for homology theories.

In this paper, we prove the equivariant Dold–Thom theorem when the coefficient group is an arbitrary Mackey functor M, by giving an explicit isomorphism $H_q^G(X; M_*) \to \pi_q(F^G(X^+, M))$ for all q and any X of the homotopy type of a G-CW-complex. Our approach provides a new proof even for the classical Dold–Thom theorem.

Let $\alpha : S \to T$ be a G-function between finite G-sets. Then we have, on the one hand, as expected, an induced homomorphism $\alpha_* : \mathbb{H}^G(S, M) \to \mathbb{H}^G(T, M)$, which corresponds to $M_*(\alpha) : M(S) \to M(T)$ under the isomorphism mentioned above. Moreover, on the other hand, we have a transfer homomorphism $\tau_\alpha : \mathbb{H}^G(T, M) \to \mathbb{H}^G(S, M)$, which corresponds to $M^*(\alpha) : M(T) \to M(S)$ under the isomorphism. Furthermore, this transfer has a topological counterpart, namely, let $p : E \to X$ be a G-equivariant covering map with finite fibers, then there is a continuous homomorphism $t_p^G : F^G(X^+, M) \to F^G(E^+, M)$ which induces a transfer homomorphism $\tau_p^G : \mathbb{H}_q^G(X; M) \to \mathbb{H}_q^G(E; M)$.

When G is the trivial group and M is just an abelian group A, the group $F^G(X, M)$ coincides with McCord's group $B(A, X)$ [12], but their topologies are different (see Example 4.11).

The theory $\mathbb{H}_q^G(-; M)$ can also be described as the homology of a certain chain complex, as shown in Section 2.

The results in this paper generalize what we did in [1], where we studied the case of the Mackey functor $M = M_L$, where L is a G-module and $M_L(G/H) = L^H$.

A different construction of ordinary equivariant homology theory with coefficients in a Mackey functor using equivariant stable homotopy theory was given by Lewis, May, and McClure in [7].

The paper is organized in five sections. In Section 1 we establish the properties of our group functors $F^G(-, M)$ in the category of pointed G-sets.

In Section 2 we generalize the construction of the function groups to simplicial G-sets, namely we associate to each simplicial pointed G-set K a simplicial abelian group $F^G(K, M)$ and study its properties.

In Section 3 we topologize the abelian groups $F^G(X, M)$, when X is a topological space. In order to make these groups into homotopy functors, we defined a topology which is not the obvious generalization of the topology on $F^G(X, L)$, where L is a G-module (as in [1,8,12,14]). When we take coefficients in an arbitrary Mackey functor M, certain discontinuities of maps appear with the obvious topology. The correct topology is defined using the simplicial abelian groups introduced in the previous section. In the case of L, we get two different topological groups $F^G(X, L)$ and $F^G(X; M_L)$, which are equal as groups, but only homotopy equivalent as topological spaces (see Example 4.11). We also construct the transfer for G-equivariant covering maps and prove the first main Theorem 3.16.

Section 4 is devoted to the proof of the second main Theorem 4.1. Then we give an explicit isomorphism $H_q^G(X; M_*) \to \mathbb{H}_q^G(X, M)$ for all q and any space X of the homotopy type of a G-CW-complex, where M_* is the covariant part of the Mackey functor M. This result is equivalent to the second main theorem. The isomorphism is constructed in two steps as follows. Let $\mathcal{S}_*(X)$ be the singular simplicial set of X. Since X has a G-action, so does $\mathcal{S}_*(X)$, i.e., it is a simplicial G-set. Then we have a simplicial abelian group $F^G(\mathcal{S}_*(X), M)$, which determines an associated chain complex. Using the theory of simplicial sets, we give an isomorphism between the homology of this chain complex $H_*(F^G(\mathcal{S}_*(X), M))$ and $\pi_*(F^G(X, M))$. Then we show that both the chain complex $F^G(\mathcal{S}_*(X), M)$ and Illman's chain complex, which defines $H_*^G(X; M_*)$, have the same universal property (Propositions 1.6 and 4.6) so that they are canonically isomorphic.

Finally, in Section 5 we extend the results of the previous section to Mackey functors with values in R-modules and prove a formula for the composite of the transfer and the projection of a G-equivariant covering map. Furthermore, we characterize those theories $\mathbb{H}^G(-; M)$, for which that formula has an expression analogous to the classical one.

1. Equivariant function groups with coefficients in a Mackey functor

Recall that a *Mackey functor* (see [4] or [16], for instance) consists of two functors, one covariant and one contravariant, both with the same object function $M : G\text{-}\mathcal{S}\mathrm{et}_{\mathrm{fin}} \to \mathcal{A}\mathrm{b}$. If $\alpha : S \to T$ is a G-function between G-sets, we denote the covariant part in morphisms by $M_*(\alpha): M(S) \to M(T)$ and the contravariant part by $M^*(\alpha): M(T) \to M(S)$. The functor has to be additive in the sense that the two embeddings $S \hookrightarrow S \sqcup T \hookleftarrow T$ into the disjoint union of G-sets define an isomorphism $M(S \sqcup T) \cong M(S) \oplus M(T)$ and if one has a pullback diagram of G-sets

$$\begin{tikzcd} U \arrow[r, "\tilde\beta"] \arrow[d, "\tilde\alpha"'] & S \arrow[d, "\alpha"] \\ T \arrow[r, "\beta"'] & V, \end{tikzcd} \tag{1.1}$$

then $M_*(\tilde\beta) \circ M^*(\tilde\alpha) = M^*(\alpha) \circ M_*(\beta)$ (see [4] for details).

By the additivity property, the Mackey functor M is determined by its restriction $M: \mathcal{O}(G) \to \mathcal{A}b$, where $\mathcal{O}(G)$ is the full subcategory of G-orbits G/H, $H \subset G$. A particular role will be played by the G-function $R_{g^{-1}}: G/H \to G/gHg^{-1}$, given by *right translation* by $g^{-1} \in G$, namely

$$R_{g^{-1}}(aH) = aHg^{-1} = ag^{-1}\big(gHg^{-1}\big).$$

We shall often denote aH by $[a]_H$. Observe that if S is a G-set and $x \in S$, then the canonical bijection $G/G_x \to G/G_{gx}$ is precisely $R_{g^{-1}}$. Here G_x denotes the *isotropy subgroup* of x, namely, the maximal subgroup of G that leaves x fixed.

Definition 1.2. Let M be a Mackey functor and let X be any pointed G-set (where the base point x_0 remains fixed under the action of G). Define $\widehat{M} = \bigcup_{H \subset G} M(G/H)$ and consider the set

$$F(X, M) = \big\{u: X \to \widehat{M} \mid u(x) \in M(G/G_x), u(x_0) = 0, \text{ and } u(x) = 0 \text{ for almost all } x \in X\big\}.$$

One can write the elements $u \in F(X, M)$ as $u = \sum_{x \in X} l_x x$, where $l_x = u(x) \in M(G/G_x)$ (the sum is obviously finite). $F(X, M)$ is again a G-set with the left action of G on $F(X, M)$ given by

$$(g \cdot u)(x) = M_*(R_{g^{-1}})\big(u\big(g^{-1}x\big)\big).$$

The G-set $F(X, M)$ is indeed an abelian group with the sum $u + v$ for $u, v \in F(X, M)$ given by $(u + v)(x) = u(x) + v(x) \in M(G/G_x)$. We shall denote by $F^G(X, M)$ the subgroup of fixed points of $F(X, M)$ under the action of G.

Definition 1.3. For each $x \in X$, there is a homomorphism $\gamma_x: M(G/G_x) \to F(X, M)$ given by $\gamma_x(l) = lx$.

The group $F(X, M)$ is characterized by the following universal property.

Proposition 1.4. *Let A be an abelian group and for each $x \in X$ let $\varphi_x: M(G/G_x) \to A$ be a homomorphism, such that $\varphi_{x_0} = 0$, where $x_0 \in X$ is the base point. Then there exists a unique homomorphism $\varphi: F(X, M) \to A$ such that $\varphi \circ \gamma_x = \varphi_x$. In a diagram*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x"] \arrow[dr, "\varphi_x"'] & F(X, M) \arrow[d, dotted, "\varphi"] \\ & A. \end{tikzcd}$$

Proof. Take an element $u = \sum_{x \in X} l_x x \in F(X, M)$, where $l_x \in M(G/G_x)$. Define $\varphi(u) = \sum_{x \in X} \varphi_x(l_x)$. Clearly, since the sum is finite, this is well defined and is unique. $\square$

Definition 1.5. For each $x \in X$, let $\gamma_x^G: M(G/G_x) \to F^G(X, M)$ be given by $\gamma_x^G(l) = \sum_{i=1}^k M_*(R_{g_i^{-1}})(l) g_i x$, where $\{[g_1], \dots [g_k]\} = G/G_x$. Clearly $\gamma_{x_0}^G = 0$ and $\gamma_{gx}^G = \gamma_x^G \circ M_*(R_g)$.

We have the following universal property of $F^G(X, M)$ with respect to the G-set X.

Proposition 1.6. *If A is an abelian group and there is a family of homomorphisms $\varphi_x : M(G/G_x) \to A$, one for each $x \in X$, satisfying $\varphi_{x_0} = 0$ and $\varphi_{gx} = \varphi_x \circ M_*(R_g)$, then there is a unique homomorphism $\varphi : F^G(X, M) \to A$ such that $\varphi \circ \gamma_x^G = \varphi_x$. Thus we have*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[dr, "\varphi_x"'] & F^G(X, M) \arrow[d, dotted, "\varphi"] \\ & A. \end{tikzcd}$$

Proof. Let $X/G = \{[x_\alpha] \mid \alpha \in \Lambda\}$ be the set of orbits of X, and take $u \in F^G(X, M)$. Define φ by

$$\varphi(u) = \sum_{\alpha \in \Lambda} \varphi_{x_\alpha}\bigl(u(x_\alpha)\bigr).$$

Using the property of the homomorphisms φ_x, one can easily check that φ is well defined.

Clearly, any element $u \in F^G(X, M)$ can be written as $u = \sum_{\alpha \in \Lambda} \gamma_{x_\alpha}^G(u(x_\alpha))$. Using this fact, one can show that φ is unique. The commutativity of the triangles follows easily from the definition of φ. $\square$

Definition 1.7. Let $f : X \to Y$ be a pointed G-function between pointed G-sets. For each $x \in X$, we denote by $\hat{f}_x : G/G_x \to G/G_{f(x)}$ the induced quotient function. We define a family of homomorphisms $f_x : M(G/G_x) \to F^G(Y, M)$ by $f_x = \gamma_{f(x)}^G \circ M_*(\hat{f}_x)$ so that

$$f_x(l) = \sum_{i=1}^{m} M_*\bigl(R_{g_i^{-1}} \circ \hat{f}_x\bigr)(l) g_i f(x),$$

where $G/G_{f(x)} = \{[g_1], \ldots, [g_m]\}$. Since $f(x_0) = y_0$ and $\gamma_{y_0}^G = 0$, then $f_{x_0} = 0$. Since $\gamma_{gf(x)}^G = \gamma_{f(x)}^G \circ M_*(R_g)$ and clearly $R_g \circ \hat{f}_{gx} = \hat{f}_x \circ R_g$, then $f_{gx} = f_x \circ M_*(R_g)$. Therefore, by the universal property 1.6, this family determines a homomorphism

$$f_*^G : F^G(X, M) \longrightarrow F^G(Y, M).$$

Therefore, $X \mapsto F^G(X, M)$ is a covariant functor from the category G-$\mathcal{S}\mathrm{et}_*$ of pointed G-sets to the category $\mathcal{A}\mathrm{b}$ of abelian groups.

Remark 1.8. Note that the previous definition of the functor $F^G(X, M)$ can be equally given for any covariant coefficient system M.

Proposition 1.9. *Given a Mackey functor M, the restriction of the functor $X \mapsto F^G(X^+, M)$ to the category $\mathcal{O}(G)$ is naturally isomorphic to the covariant part M_* of the Mackey functor M.*

Proof. First observe that the mapping $F^G(G/H^+, M) \to M(G/H)$ given by $u \mapsto u([e]_H)$, where $[e]_H$ is the coset $eH = H \in G/H$, $e \in G$ the neutral element, is an isomorphism. Take $H \subset K \subset G$ and let $\alpha : G/H \to G/K$ be the quotient function and $R_{g^{-1}} : G/H \to G/gHg^{-1}$ be the right translation. We now need to show that the following diagrams commute:

$$\begin{tikzcd} F^G(G/H^+, M) \arrow[r, "\cong"] \arrow[d, "\alpha_*^G"'] & M(G/H) \arrow[d, "M_*(\alpha)"] \\ F^G(G/K^+, M) \arrow[r, "\cong"] & M(G/K). \end{tikzcd} \qquad \begin{tikzcd} F^G(G/H^+, M) \arrow[r, "\cong"] \arrow[d, "(R_{g^{-1}})_*^G"'] & M(G/H) \arrow[d, "M_*(R_{g^{-1}})"] \\ F^G(G/gHg^{-1+}, M) \arrow[r, "\cong"] & M(G/gHg^{-1}). \end{tikzcd}$$

To see that the diagram on the left commutes, just observe that the inverse of the isomorphism on the top is given precisely by $\gamma_H^G : M(G/H) \to F^G(G/H^+, M)$, as defined in 1.5, where $H \in G/H^+$ denotes the coset of the identity element in G, while the inverse of the isomorphism on the bottom is given correspondingly by γ_K^G. By definition of α_*^G, the diagram

$$\begin{tikzcd} M(G/H) \arrow[r, "\gamma_H^G"] \arrow[d, "M_*(\hat{\alpha}_H)"'] & F^G(G/H^+, M) \arrow[d, "\alpha_*^G"] \\ M(G/K) \arrow[r, "\gamma_K^G"'] & F^G(G/K^+, M) \end{tikzcd}$$

commutes, and clearly $\hat{\alpha}_H = \alpha$.

To see that the diagram on the right commutes, notice that since $M_*(\widehat{R_{g^{-1}}}_{[e]_H}) = \mathrm{Id}$, then by 1.7 $(R_{g^{-1}})_*^G \circ \gamma_{[e]_H}^G = \gamma_{[g^{-1}]_{gHg^{-1}}}^G$. Now by 1.6, $\gamma_{[g^{-1}]_{gHg^{-1}}}^G = \gamma_{[e]_{gHg^{-1}}}^G \circ M_*(R_{g^{-1}})$. Therefore, $(R_{g^{-1}})_*^G \circ \gamma_{[e]_H}^G = \gamma_{[e]_{gHg^{-1}}}^G \circ M_*(R_{g^{-1}})$, thus the diagram commutes. □

Definition 1.10. Let M be a Mackey functor and $p : E \to X$ a G-function between G-sets with finite fibers. We define the *transfer* of p,

$$t_p : F(X^+, M) \to F(E^+, M) \quad \text{by} \quad t_p(u)(\sigma) = M^*(\hat{p}_\sigma)\big(u\big(p(\sigma)\big)\big)$$

(and $t_p(u)(*) = 0$). One can easily check that $t_p(u) \in F(E^+, M)$. This transfer, in the generators $\gamma_x(l) = lx \in F(X, M)$ is given by

$$t_p\big(\gamma_x(l)\big) = \sum_{i=1}^{n} \gamma_{a_i} M^*(\hat{p}_{a_i})(l),$$

where $p^{-1}(x) = \{a_1, \ldots, a_n\}$.

To see that this function is G-equivariant with respect to the action defined in 1.2, we have on the one hand

$$t_p(g \cdot u)(a) = M^*(\hat{p}_a)\big(g \cdot u\big(p(a)\big)\big) = M^*(\hat{p}_a) M_*(R_{g^{-1}})\big(u\big(g^{-1} p(a)\big)\big),$$

while on the other hand we have

$$\big(g \cdot t_p(u)\big)(a) = M_*(R_{g^{-1}})\big(t_p(u)(g^{-1}a)\big) = M_*(R_{g^{-1}}) M^*(\hat{p}_{g^{-1}a})\big(u\big(p(g^{-1}a)\big)\big).$$

Both terms are equal, since $M^*(\hat{p}_a) \circ M_*(R_{g^{-1}}) = M_*(R_{g^{-1}}) \circ M^*(\hat{p}_{g^{-1}a})$, and this follows from the fact that the square

$$\begin{tikzcd} G/G_{g^{-1}a} \arrow[r, "R_{g^{-1}}", "\cong"'] \arrow[d, "\hat{p}_{g^{-1}a}"'] & G/G_a \arrow[d, "\hat{p}_a"] \\ G/G_{g^{-1}p(a)} \arrow[r, "\cong", "R_{g^{-1}}"'] & G/G_{p(a)} \end{tikzcd}$$

is clearly a pullback diagram. Thus, by restriction, t_p induces also a *transfer*

$$t_p^G : F^G(X^+, M) \to F^G(E^+, M).$$

The isotropy group G_x acts on $p^{-1}(x) = \{a_1, \ldots, a_n\}$, and the inclusion $j : p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $\bar{j} : p^{-1}(x)/G_x \to p^{-1}(Gx)/G$. Let $\{a_\iota \mid \iota \in \mathcal{I}\} \subset p^{-1}(x)$ be a set of representatives one for each G_x-orbit of $p^{-1}(x)$. Let $\gamma_x^G(l)$ be a generator of $F^G(X^+, M)$. Since its value on points which do not belong to Gx is zero, and $\gamma_x^G(l)(x) = l$, we have that

$$t_p^G\big(\gamma_x^G(l)\big) = \sum_{\iota \in \mathcal{I}} \gamma_{a_\iota}^G M^*(\hat{p}_{a_\iota})(l) \in F^G(E^+, M). \tag{1.11}$$

Remark 1.11. For any G-set S, let $\beta_S : F(S, M) \to F^G(S, M)$ be given on generators by $\beta\gamma_x(l) = \gamma_x^G(l)$. This is a surjective homomorphism. There is another transfer $\underline{t}_p^G : F^G(X^+, M) \to F^G(E^+, M)$, which is studied in [2], given by the commutativity of the diagram

$$\begin{tikzcd} F(X^+, M) \arrow[r, "t_p"] \arrow[d, "\beta_{X^+}"'] & F(E^+, M) \arrow[d, "\beta_{E^+}"] \\ F^G(X^+, M) \arrow[r, "\underline{t}_p^G"'] & F^G(E^+, M). \end{tikzcd}$$

Equivalently,

$$\underline{t}_p^G\big(\gamma_x^G(l)\big) = \sum_{i=1}^{n} \gamma_{a_i}^G M^*(\hat{p}_{a_i})(l).$$

It is clear that this transfer differs from the one given by (1.11).

Remark 1.12. Assume that $p : E \to X$ and $q : X \to Y$ are G-functions with finite fibers. Then one has that $\widehat{(q \circ p)}_\sigma = \hat{q}_{p(\sigma)} \circ \hat{p}_\sigma$. Using this, one easily verifies that the transfer is functorial in the sense that $t_{q\circ p}^G = t_p^G \circ t_q^G$.

Theorem 1.13. *Let M be a Mackey functor for G and S be a finite G-set. Then there is a canonical isomorphism $\Gamma_S : M(S) \to F^G(S^+, M)$.*

Proof. First observe that the functor $F^G(-^+, M)$ sends finite disjoint unions to direct sums. Namely, consider $S \sqcup T$ and take the inclusions $S^+ \overset{i}{\hookrightarrow} (S \sqcup T)^+ \overset{j}{\hookleftarrow} T^+$ and the retractions $S^+ \overset{r}{\twoheadleftarrow} (S \sqcup T)^+ \overset{s}{\twoheadrightarrow} T^+$, where $r|_S$, $s|_T$ are the inclusions, and $r(T) = s(S) = *$. Then

$$F^G\big((S \sqcup T)^+, M\big) \cong F^G(S^+, M) \oplus F^G(T^+, M),$$

where the isomorphism is given by $w \mapsto (w \circ i, w \circ j) = (t_i(w), t_j(w))$ with inverse $(u, v) \mapsto u \circ r + v \circ s = i_*^G(u) + j_*^G(v)$.

Let now $S/G = \{[\sigma_i]\}$, where i belongs to some finite set of indexes. Then $S = \bigsqcup_i G\sigma_i$ and there is a canonical G-bijection $\rho_i : G/G_{\sigma_i} \to G\sigma_i$ given by $\rho_i[g] = g\sigma_i$, and let $\beta_i : G\sigma_i \hookrightarrow S$ be the inclusion. The isomorphism is given by the following diagram of isomorphisms:

$$\begin{tikzcd} \bigoplus_i M(G/G_{\sigma_i}) \arrow[r, "\oplus M_*(\rho_i)", "\cong"'] \arrow[d, "\oplus\gamma_{G_{\sigma_i}}^G"'] & \bigoplus_i M(G\sigma_i) \arrow[r, "(M_*(\beta_i))", "\cong"'] & M(S) \arrow[d, dotted, "\Gamma_S"] \\ \bigoplus_i F^G(G/G_{\sigma_i}^+, M) \arrow[r, "\cong", "\oplus(\rho_i)_*^G"'] & \bigoplus_i F^G(G\sigma_i^+, M) \arrow[r, "\cong", "((\beta_i)_*^G)"'] & F^G(S^+, M). \end{tikzcd}$$

We only need to remark that the vertical arrow on the left is an isomorphism by Proposition 1.9.

The isomorphism Γ_S does not depend on the choice of representatives in S/G. Namely, let $G\sigma_i' = G\sigma_i$, then $\sigma_i' = g_i\sigma_i$, and $\rho_{\sigma_i'} \circ R_{g_i^{-1}} = \rho_i : G/G_{\sigma_i} \to G\sigma_i$. The assertion follows using this and 1.9. □

By 1.9 and the previous result, we have the following.

Proposition 1.14. *The isomorphism $\Gamma_S : M(S) \to F^G(S^+, M)$ is natural with respect to the covariant structure, namely, if $\alpha : S \to T$ is a G-function, then the following diagram commutes*:

$$\begin{tikzcd} M(S) \arrow[r, "\Gamma_S"] \arrow[d, "M_*(\alpha)"'] & F^G(S^+, M) \arrow[d, "\alpha_*^G"] \\ M(T) \arrow[r, "\Gamma_T"'] & F^G(T^+, M). \end{tikzcd}$$

Lemma 1.15. *Given a Mackey functor M and a G-function $\alpha : G/H \to G/K$, then the following diagram commutes*:

$$\begin{tikzcd} F^G(G/K^+, M) \arrow[r, "\cong"] \arrow[d, "t^G_\alpha"'] & M(G/K) \arrow[d, "M^*(\alpha)"] \\ F^G(G/H^+, M) \arrow[r, "\cong"] & M(G/H), \end{tikzcd}$$

where the isomorphisms are given in 1.9.

Proof. It is enough to check it for the cases $\alpha : G/H \to G/K$, the quotient function, and for $\alpha = R_{g^{-1}} : G/H \to G/gHg^{-1}$. The first case follows readily, since $\hat{\alpha}_{[e]_H} = \alpha$, therefore we only have to prove the second case. Put $K = gHg^{-1}$. Take $v \in F^G(G/K^+, M)$, so that $v([e]_K) \in M(G/K)$. Chasing v along the top and then down vertically in the diagram, we arrive to the element $M^*(R_{g^{-1}})(v([e]_K)) \in M(G/H)$. On the other hand, observe that if $\alpha = R_{g^{-1}}$ and $\sigma = [e]_H$, then $\hat{\alpha}_\sigma = \mathrm{id}$. Hence, chasing v down the vertical arrow on the left and then along the bottom of the diagram we obtain

$$\begin{aligned} t^G_{R_{g^{-1}}}(v)\big([e]_H\big) &= M^*(\hat{\alpha}_\sigma)\big(v\big(R_{g^{-1}}\big([e]_H\big)\big)\big) \\ &= v\big(g^{-1}[e]_K\big) \\ &= M_*(R_g)\big(v\big([e]_K\big)\big). \end{aligned}$$

The commutativity follows since $M_*(R_g)(v([e]_K)) = M^*(R_{g^{-1}})(v([e]_K))$. To see this, just observe that the following commutative diagram of bijections is obviously a pullback diagram.

$$\begin{tikzcd} G/H \arrow[r, "\mathrm{id}"] \arrow[d, "R_{g^{-1}}"'] & G/H \arrow[d, "\mathrm{id}"] \\ G/K \arrow[r, "R_g"'] & G/H. \end{tikzcd}$$

Hence $M_*(R_g) = M^*(R_{g^{-1}})$. □

Proposition 1.16. *The isomorphism $\Gamma_S : M(S) \to F^G(S^+, M)$ is natural with respect to the contravariant structure, namely, if $\alpha : S \to T$ is a G-function, then the following diagram commutes*:

$$\begin{tikzcd} M(T) \arrow[r, "\Gamma_T"] \arrow[d, "M^*(\alpha)"'] & F^G(T^+, M) \arrow[d, "t^G_\alpha"] \\ M(S) \arrow[r, "\Gamma_S"'] & F^G(S^+, M). \end{tikzcd}$$

Proof. First observe that the inverse of Γ_S is given by the transfers of the morphisms which define Γ_S and the product of the inverses of the isomorphisms $\gamma^G_{G_{\sigma_i}}$. Therefore the result follows from the functoriality of M^* and of the transfer (see 1.12), and from Proposition 1.15. □

2. Function groups of simplicial *G*-sets

We denote by Δ the category whose objects are the sets $\bar{n} = \{0, 1, 2, \ldots, n\}$ and whose morphisms $f \in \Delta(\bar{m}, \bar{n})$ are monotonic functions $f : \bar{m} \to \bar{n}$. Recall that a *simplicial set* is a contravariant functor $K : \Delta \to \mathcal{S}\mathrm{et}$; we denote the set $K(\bar{n})$ simply by K_n. The function induced by f is denoted by $f^K : K_n \to K_m$. Let $\Delta[q]$ be the simplicial set $\Delta(-, \bar{q})$. We write $|K|$ for the *geometrical realization* given by

$$|K| = \bigsqcup_n (K_n \times \Delta^n)/\sim,$$

where $\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geqslant 0, i = 0, 1, 2, \dots, n, t_0 + t_1 + \cdots + t_n = 1\}$ is the standard n-simplex, and the equivalence relation is given by $(f^K(\sigma), t) \sim (\sigma, f_\#(t))$, $\sigma \in K_n$, $t \in \Delta^m$. Here $f_\#$ denotes the map affinely induced by f in the standard simplices. Denote the elements of $|K|$ by $[\sigma, t]$, $\sigma \in K_n$ and $t \in \Delta^n$.

Observe that fixing $\sigma \in K_n$, the map $\Delta^n \to |K|$ given by $t \mapsto [\sigma, t]$ is continuous.

We say that a simplicial set K is *pointed*, if it is provided with a morphism (natural transformation) $\Delta[0] \to K$. This means that each set K_n has a base point and that for each monotonic function $f : \bar{m} \to \bar{n}$, the induced function $f^K : K_n \to K_m$ is base-point preserving.

Definition 2.1. Let G be a finite group. A (*pointed*) *simplicial G-set* is a (pointed) simplicial set K such that G acts on each K_n and the action of every $g \in G$ determines a (pointed) isomorphism of K. In other words, it is a functor $K : \Delta \to G\text{-}\mathcal{S}\mathrm{et}_*$. This means that for every monotonic function $f : \bar{m} \to \bar{n}$ the functions $f^K : K_n \to K_n$ are G-functions.

Definition 2.2. Given a pointed simplicial G-set K and a Mackey functor M, we define the simplicial abelian group $F(K, M)$ by $F(K, M)_n = F(K_n, M)$, where $F(K_n, M)$ is as defined in 1.2. The homomorphism induced by $f : \bar{m} \to \bar{n}$ is $f_*^K : F(K_n, M) \to F(K_m, M)$. Thus, in particular, $F(K, M)$ is again a pointed simplicial G-set.

Note that if K is a (pointed) simplicial G-set, then the geometric realization $|K|$ is a (pointed) G-space (in fact, a G-CW-complex). Thus, in particular, if K is a pointed simplicial G-set, then $|F(K, M)|$ is a pointed G-space. On the other hand, since $|K|$ is a pointed G-space, $F(|K|, M)$ is also a pointed G-space.

The proof of the following uses results of Milnor (see [9]).

Lemma 2.3. *Let K be a simplicial G-set. Then the geometric realization $|F(K, M)|$ is an abelian topological group such that $[v, t] + [v', t] = [v + v', t]$.*

Proof. Consider the projections $p_i : F(K, M) \times F(K, M) \to F(K, M)$, $i = 1, 2$, and the induced maps $|p_i| : |F(K, M) \times F(K, M)| \to |F(K, M)|$, and define $\eta : |F(K, M) \times F(K, M)| \to |F(K, M)| \times |F(K, M)|$ (here the topology of the product is the k-topology; see next section) by

$$\begin{aligned} \eta\big[(v, v'), t\big] &= \big(|p_1|\big[(v, v'), t\big], |p_2|\big[(v, v'), t\big]\big) \\ &= \big(\big[p_1(v, v'), t\big], \big[p_2(v, v'), t\big]\big) \\ &= \big([v, t], [v', t]\big). \end{aligned}$$

By [9, 14.3], η is a homeomorphism. The group structure $+$ in $|F(K, M)|$ is then given by the diagram

$$\begin{tikzcd} \big|F(K,M)\big| \times \big|F(K,M)\big| \arrow[r, "\eta^{-1}"] \arrow[dr, "+"'] & \big|F(K,M) \times F(K,M)\big| \arrow[d, "|\mu|"] \\ & \big|F(K,M)\big|, \end{tikzcd}$$

where $\mu : F(K, M) \times F(K, M) \to F(K, M)$ is the simplicial group structure. $\square$

Let K be a simplicial set. An element $\sigma \in K_n$ is said to be *nondegenerate* if there is no $\tau \in K_{n-1}$ and no $i = 1, \dots, n-1$ such that $\sigma = s_i^K(\tau)$, where s_i^K is the ith degeneracy operator of K. Moreover, the representative (σ, t) of an element in $|K|$ is said to be *nondegenerate* if $\sigma \in K_n$ is nondegenerate and $t \in \mathring{\Delta}^n$. If K is a simplicial G-set, then these definitions extend to the simplicial set (group) $F(K, M)$. It is a result of Milnor (see [9]) that for every element $\sigma \in K_n$ there is a unique nondegenerate element $\sigma' \in K_m$ and a unique function $f : \bar{n} \to \bar{m}$ such that $f^K(\sigma') = \sigma$. Moreover, every element $[\sigma, t] \in |K|$ has a unique nondegenerate representative (σ', t').

We have the following results on nondegeneracy.

Proposition 2.4. *Let K be a simplicial G-set. If (σ, t) is a nondegenerate representative, then $G_\sigma = G_{[\sigma,t]}$.*

Proof. Clearly $G_\sigma \subset G_{[\sigma,t]}$ for all $\sigma \in K_n$ and $t \in \Delta^n$. To show that $G_{[\sigma,t]} \subset G_\sigma$, recall (see [9]) that $|K|$ is a CW-complex, whose open cells are given by

$$|K| = \coprod_{\sigma \in K'_n, n \geqslant 0} \varphi_\sigma\big(\mathring{\Delta}^n\big), \tag{2.5}$$

where $\varphi_\sigma(t) = [\sigma, t]$ and K'_n is the subset of K_n of nondegenerate elements. Since the degeneracy operators are G-functions, if $\sigma \in K_n$ is nondegenarate, then $g\sigma$ is also nondegenerate.

Assume that (σ, t) is a nondegenerate representative and that $g[\sigma, t] = [g\sigma, t] = [\sigma, t]$. Then by the above $(g\sigma, t)$ is also nondegenerate. Therefore, $\varphi_\sigma(t) = \varphi_{g\sigma}(t)$ and so, by (2.5), $g\sigma = \sigma$. □

Definition 2.5. Given a pointed simplicial G-set K and a Mackey functor M, we define the simplicial abelian group $F^G(K, M)$ by $F^G(K, M)_n = F^G(K_n, M)$, where $F^G(K_n, M)$ is as defined in 1.2. The homomorphism induced by $f : \bar{m} \to \bar{n}$ is given by the homomorphism $(f^K)^G_* : F^G(K_n, M) \to F^G(K_m, M)$ defined in 1.7.

Let K be a simplicial G-set. Then, for each $t \in \Delta^n$, the function $K_n \to |K|$ given by $\sigma \mapsto [\sigma, t]$ is a G-function. Thus we have for the isotropy groups that $G_\sigma \subset G_{[\sigma,t]}$. Call $\hat{q}_{\sigma,t} : G/G_\sigma \to G/G_{[\sigma,t]}$ the quotient function.

Proposition 2.6. *Let K be a simplicial pointed G-set. Then*

(a) *the groups $F(|K|, M)$ and $|F(K, M)|$ are naturally G-isomorphic, and*
(b) *the groups $F^G(|K|, M)$ and $|F^G(K, M)|$ are naturally isomorphic.*

Proof. We prove (b). Define $\varphi' : F^G(|K|, M) \to |F^G(K, M)|$ as follows. For each $[\sigma, t] \in |K|$ ((σ, t) a nondegenerate representative), let

$$\varphi'_{[\sigma,t]} : M(G/G_{[\sigma,t]}) \to \big|F^G(K, M)\big| \quad \text{be given by} \quad l \longmapsto \big[\gamma^G_\sigma(l), t\big].$$

Since by Proposition 2.4, $G_\sigma = G_{[\sigma,t]}$, this is well defined, and the universal property 1.6 allows us to define φ'.

On the other hand, define $\psi' : |F^G(K, M)| \to F^G(|K|, M)$ as follows. First recall from the proof of 1.6 that an element $u \in F^G(K_n, M)$ can be written as $u = \sum_{\alpha \in \Lambda} \gamma^G_{\sigma_\alpha}(u(\sigma_\alpha))$, where $K_n/G = \{[\sigma_\alpha] \mid \alpha \in \Lambda\}$ is the set of orbits of K_n. Consider the mapping $F^G(K_n, M) \times \Delta^n \to F^G(|K|, M)$ given by

$$(u, t) \longmapsto \sum_{\alpha \in \Lambda} \gamma^G_{[\sigma_\alpha, t]}\big(M_*(\hat{q}_{\sigma_\alpha, t})\big(u(\sigma_\alpha)\big)\big).$$

This mapping depends only on the class of $[u, t] \in |F^G(K, M)|$; namely, if $f : \bar{n} \to \bar{m}$ is a morphism in Δ, then it induces $f^K : K_m \to K_n$, and with it also $(f^K)^G_* : F^G(K_m, M) \to F^G(K_n, M)$, whose value on $v = \sum_{\beta \in \Lambda'} \gamma^G_{\sigma_\beta}(v(\sigma_\beta)) \in F(K_m, M)$ is given by

$$\big(f^K\big)^G_*(v) = \sum_{\beta \in \Lambda'} \gamma^G_{f^K(\sigma_\beta)}\big(M_*\big(\hat{f}^K_{\sigma_\beta}\big)\big(v(\sigma_\beta)\big)\big).$$

The elements

$$\big(\big(f^K\big)^G_*(v), t\big) = \left(\sum_{\beta \in \Lambda'} \gamma^G_{f^K(\sigma_\beta)}\big(M_*\big(\hat{f}^K_{\sigma_\beta}\big)\big(v(\sigma_\beta)\big)\big), t\right) \quad \text{and} \quad \big(v, f_\#(t)\big)$$

represent the same element in $|F^G(K, M)|$, and each maps to

$$\sum_{\beta \in \Lambda'} \gamma^G_{[f^K(\sigma_\beta), t]}\big(M_*(\hat{q}_{f^K(\sigma_\beta), t}) M_*\big(\hat{f}^K_{\sigma_\beta}\big)\big(v(\sigma_\beta)\big)\big) = \sum_{\beta \in \Lambda'} \gamma^G_{[f^K(\sigma_\beta), t]}\big(M_*(\hat{q}_{f^K(\sigma_\beta), t} \circ \hat{f}^K_{\sigma_\beta}\big)\big(v(\sigma_\beta)\big)$$

and

$$\sum_{\beta \in \Lambda'} \gamma^G_{[\sigma_\beta, f_\#(t)]}\big(M_*(\hat{q}_{\sigma_\beta, f_\#(t)})\big(v(\sigma_\beta)\big)\big) = \sum_{\beta \in \Lambda'} \gamma^G_{[f^K(\sigma_\beta), t]}\big(M_*(\hat{q}_{\sigma_\beta, f_\#(t)})\big(v(\sigma_\beta)\big)\big),$$

respectively. These last two are equal by the commutativity of the following triangle:

$$\begin{tikzcd}
G/G_{\sigma_\beta} \arrow[d, "\hat{f}^K_{\sigma_\beta}"'] \arrow[dr, "\hat{q}_{\sigma_\beta, f_\#(t)}"] & \\
& G/G_{[f^K(\sigma_\beta),t]}. \\
G/G_{f^K(\sigma_\beta)} \arrow[ur, "\hat{q}_{f^K(\sigma_\beta),t}"'] &
\end{tikzcd}$$

Therefore, for $u=\sum_{\alpha\in\Lambda}\gamma^G_{\sigma_\alpha}(u(\sigma_\alpha))$, we define

$$\psi'\big([u,t]\big)=\sum_{\alpha\in\Lambda}\gamma^G_{[\sigma_\alpha,t]}\big(M_*(\hat{q}_{\sigma_\alpha,t})\big(u(\sigma_\alpha)\big)\big).$$

Since the mapping $F^G(K_n,M)\to|F^G(K,M)|$ given by $u\mapsto[u,t]$ is clearly additive, ψ' is also determined by its value on the generators $u=\gamma^G_\sigma(l)$, namely by $\psi'([\gamma^G_\sigma(l),t])=\gamma^G_{[\sigma,t]}(M_*(\hat{q}_{\sigma,t})(l))$.

We now show that the maps φ' and ψ' are inverse to each other. Take a generator $\gamma^G_{[\sigma,t]}(l)\in F^G(|K|,M)$, where $l\neq 0$ and $[\sigma,t]$ is nondegenerate. Then $\psi'\varphi'(\gamma^G_{[\sigma,t]}(l))=\psi'[\gamma^G_\sigma(l),t]=\gamma^G_{[\sigma,t]}(l)$, the last equality follows since $[\sigma,t]$ is nondegenerate. Hence

$$\psi'\circ\varphi'=1.$$

On the other hand, take an element $[\sum_{\alpha\in\Lambda}\gamma^G_{\sigma_\alpha}(l_{\sigma_\alpha}),t]\in|F^G(K,M)|$. We may assume that the representative of this element is nondegenerate, so that $t\in\mathring{\Delta}^n$.

For each σ_α, there exists a unique nondegenerate σ'_α such that $\sigma_\alpha=s^K(\sigma'_\alpha)$, where s^K is a composite of degeneracy operators. Therefore, $[\sigma_\alpha,t]=[\sigma'_\alpha,s_\#(t)]$ where the right-hand side is given by a nondegenerate representative. Now consider

$$\begin{aligned}
\varphi'\psi'\left(\left[\sum_{\alpha\in\Lambda}\gamma^G_{\sigma_\alpha}(l_{\sigma_\alpha}),t\right]\right)&=\sum_{\alpha\in\Lambda}\varphi'\gamma^G_{[\sigma_\alpha,t]}\big(M_*(\hat{q}_{\sigma_\alpha,t})(l_{\sigma_\alpha})\big)\\
&=\sum_{\alpha\in\Lambda}\big[\gamma^G_{\sigma'_\alpha}\big(M_*(\hat{q}_{\sigma_\alpha,t})(l_{\sigma_\alpha})\big),s_\#(t)\big]\\
&=\sum_{\alpha\in\Lambda}\big[\big(s^K\big)^G_*\gamma^G_{\sigma'_\alpha}\big(M_*(\hat{q}_{\sigma_\alpha,t})(l_{\sigma_\alpha})\big),t\big]\\
&=\sum_{\alpha\in\Lambda}\big[\gamma^G_{s^K(\sigma'_\alpha)}\big(M_*\big(\hat{s}^K_{\sigma'_\alpha}\big)M_*(\hat{q}_{\sigma_\alpha,t})(l_{\sigma_\alpha})\big),t\big]\\
&=\sum_{\alpha\in\Lambda}\big[\gamma^G_{\sigma_\alpha}(l_{\sigma_\alpha}),t\big]=\left[\sum_{\alpha\in\Lambda}\gamma^G_{\sigma_\alpha}(l_{\sigma_\alpha}),t\right],
\end{aligned}$$

where the next to the last equality follows from the fact that $\hat{s}^K_{\sigma'_\alpha}\circ\hat{q}_{\sigma_\alpha,t}=\mathrm{id}_{G/G_{\sigma_\alpha}}$. Thus

$$\varphi'\circ\psi'=1.$$

The proof of (a) is similar, defining $\varphi:F(|K|,M)\to|F(K,M)|$ and $\psi:|F(K,M)|\to F(|K|,M)$ using the homomorphisms γ and the universal property 1.4. □

3. Topology for the function groups

We shall work in the category of k-spaces. We understand by a k-space a topological space X with the property that a set $C\subset X$ is closed if and only if $f^{-1}C\subset K$ is closed for any map $f:K\to X$, where K is any compact Hausdorff space (see [17]). There is a functor that associates to every topological space X a k-space $k(X)$ with the same underlying set and a finer topology defined as before. Thus the identity $k(X)\to X$ is continuous and a weak homotopy equivalence. Instead of the usual topological product, we shall take its image under the functor k; we shall use the same notation $\times$ for it. This category has the following two useful properties [17]:

1. If X is a k-space and $p : X \to X'$ is an identification, then X' is a k-space; and
2. if $p : X \to X'$ and $q : Y \to Y'$ are identifications between k-spaces, then $p \times q : X \times Y \to X' \times Y'$ is an identification.

Observe that this category includes Steenrod's category of compactly generated Hausdorff spaces, as well as the category of weak Hausdorff k-spaces [10]. Thus, in what follows, space will always mean k-space.

Let now X be a pointed G-space and M a Mackey functor. In this section, we shall define the topology of the function groups $F(X,M)$ and $F^G(X,M)$ and prove that induced homomorphisms and transfers are continuous. We start by considering the special case, where the G-space is the geometric realization $|K|$ of a simplicial G-set K.

In order to define a topology on $F(|K|,M)$, consider

$$F_n\big(|K|,M\big) = \big\{u \in F\big(|K|,M\big) \mid u(x) \neq 0 \text{ for at most } n \text{ values of } x \in |K|\big\}.$$

Take

$$P\big(|K|,M\big) = \big\{(l,x) \mid l \in M(G/G_x)\big\} \subset \widehat{M} \times |K|$$

with the relative topology ($\widehat{M}$ is discrete), and the nth power $P(|K|,M)^n$. There is a surjection $\mu_n : P(|K|,M)^n \to F_n(|K|,M)$ given by

$$\mu_n\big((l_1,x_1),\dots,(l_n,x_n)\big) = \gamma_{x_1}(l_1) + \cdots + \gamma_{x_n}(l_n) = l_1x_1 + \cdots + l_nx_n.$$

Give $F_n(|K|,M)$ the identification topology. By Property 1 of the category of k-spaces this is a k-space.

Definition 3.1. Let K be a simplicial pointed G-set. As a topological space,

$$F\big(|K|,M\big) = \bigcup_n F_n\big(|K|,M\big)$$

will be provided with the weak (union) topology, so it is clearly a k-space. Moreover, by Property 2 of the category of k-spaces, the continuous map

$$P\big(|K|,M\big)^m \times P\big(|K|,M\big)^n \xrightarrow{s} P\big(|K|,M\big)^{m+n} \xrightarrow{\mu_{m+n}} F_{m+n}\big(|K|,M\big) \subset F\big(|K|,M\big)$$

given by

$$s\big(\big((l_1,x_1),\dots,(l_m,x_m)\big),\big((l'_1,x'_1),\dots,(l'_n,x'_n)\big)\big) = \big((l_1,x_1),\dots,(l_m,x_m),(l'_1,x'_1),\dots,(l'_n,x'_n)\big)$$

induces in the identification space a continuous map

$$F_m\big(|K|,M\big) \times F_n\big(|K|,M\big) \to F\big(|K|,M\big),$$

which defines in the union the sum. Hence

$$+ : F\big(|K|,M\big) \times F\big(|K|,M\big) \longrightarrow F\big(|K|,M\big)$$

is a continuous map and so $F(|K|,M)$ is a topological abelian group.

In short, for the geometric realization $|K|$ of a simplicial G-set K, we can give $F(|K|,M)$ the identification topology of the map

$$\mu : \bigsqcup_n \big(P\big(|K|,M\big)\big)^n \twoheadrightarrow F\big(|K|,M\big).$$

Similarly, we may define on $F^G(|K|,M)$ the identification topology of the map

$$\mu' : \bigsqcup_n \big(P\big(|K|,M\big)\big)^n \twoheadrightarrow F^G\big(|K|,M\big),$$

where $\mu'(l_1,x_1,\dots,l_n,x_n) = \sum_{i=1}^n \gamma_{x_i}^G(l_i)$, and $\gamma_x^G : M(G/G_x) \to F^G(|K|,M)$ is as in Definition 1.5. Thus $F^G(|K|,M)$ is also a k-space, and we can also show in the same way as above that it is a topological abelian group.

We have the following.

Proposition 3.2. *Let K be a simplicial G-set and let $\beta_{|K|}: F(|K|, M) \to F^G(|K|, M)$ be given on generators by $\beta_{|K|}(lx) = \gamma_x^G(l)$. Then $F^G(|K|, M)$ has the identification topology given by the surjective map $\beta_{|K|}$.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd} \bigsqcup_n \big(P\big(|K|, M\big)\big)^n \arrow[d, "\mu"'] \arrow[dr, "\mu'"] & \\ F\big(|K|, M\big) \arrow[r, "\beta_{|K|}"'] & F^G\big(|K|, M\big). \end{tikzcd} \qquad \square$$

The following result is the topological counterpart of 2.6.

Proposition 3.3. *Let K be a simplicial pointed G-set. Then*

(a) *the group isomorphisms*

$$\varphi : F\big(|K|, M\big) \to \big|F(K, M)\big| \quad \text{and} \quad \psi : \big|F(K, M)\big| \to F\big(|K|, M\big)$$

are continuous, and

(b) *the group isomorphisms*

$$\varphi' : F^G\big(|K|, M\big) \to \big|F^G(K, M)\big| \quad \text{and} \quad \psi' : \big|F^G(K, M)\big| \to F^G\big(|K|, M\big)$$

are continuous.

Proof. To prove that φ is continuous, it suffices to consider the diagram

$$\begin{tikzcd} \Big(P\Big(\bigsqcup_n K_n \times \Delta^n, M\Big)\Big)^k \arrow[d] \arrow[r, equal] & \Big(\bigsqcup_n P\big(K_n \times \Delta^n, M\big)\Big)^k \arrow[r] & \Big(\bigsqcup_n F(K_n, M) \times \Delta^n\Big)^k \arrow[d] \\ \big(P\big(|K|, M\big)\big)^k \arrow[r] & F_k\big(|K|, M\big) \arrow[r, "\varphi|"'] & \big|F(K, M)\big|, \end{tikzcd}$$

where the top arrow maps

$$\big(\big(l_1, (\sigma_1, t_1)\big), \ldots, \big(l_k, (\sigma_k, t_k)\big)\big) \quad \text{to} \quad \big(\big(\gamma_{\sigma_1}(l_1), t_1\big), \ldots, \big(\gamma_{\sigma_k}(l_k), t_k\big)\big),$$

$l_i \in M(G/G_{\sigma_i})$, which is clearly continuous, while the bottom composite maps

$$\big(\big(l_1, [\sigma_1, t_1]\big), \ldots, \big(l_k, [\sigma_k, t_k]\big)\big) \quad \text{to} \quad \big[\gamma_{\sigma_1}(l_1), t_1\big] + \cdots + \big[\gamma_{\sigma_k}(l_k), t_k\big],$$

the vertical arrow on the left maps

$$\begin{gathered} \big(\big(l_1, (\sigma_1, t_1)\big), \ldots, \big(l_k, (\sigma_k, t_k)\big)\big) \quad \text{to} \\ \big(\big(M_*(\hat{q}_{\sigma_1, t_1})(l_1), [\sigma_1, t_1]\big), \ldots, \big(M_*(\hat{q}_{\sigma_k, t_k})(l_k), [\sigma_k, t_k]\big)\big), \end{gathered}$$

and the vertical arrow on the right maps

$$\big((l_1\sigma_1, t_1), \ldots, (l_k\sigma_k, t_k)\big) \quad \text{to} \quad \big[\gamma_{\sigma_1}(l_1), t_1\big] + \cdots + \big[\gamma_{\sigma_k}(l_k), t_k\big].$$

In order to verify the commutativity of the diagram it is enough to check that

$$\varphi\big(M(\hat{q}_{\sigma_1, t_1})(l_1)[\sigma_1, t_1] + \cdots + M(\hat{q}_{\sigma_k, t_k})(l_k)[\sigma_k, t_k]\big) = \big[\gamma_{\sigma_1}(l_1), t_1\big] + \cdots + \big[\gamma_{\sigma_k}(l_k), t_k\big].$$

To do this, write each $[\sigma_i, t_i]$ as the class of a nondegenerate representative $[\sigma_i', t_i']$ and for simplicity, $l_i' = M_*(\hat{q}_{\sigma_i, t_i})(l_i)$. Therefore, for each i there are $d : \bar{r} \to \bar{n}$, $s : \bar{r} \to \bar{m}$, $r \leqslant n$, $m \leqslant r$, such that $t_i = d_\#(t_i'')$, $d^K(\sigma_i) = s^K(\sigma_i')$, and $t_i' = s_\#(t_i'')$, $t_i'' \in \mathring{\Delta}^r$ (d^K is a composite of some face operators, and s^K is a composite of some degeneracy operators). Thus $\varphi(l'_1[\sigma_1,t_1]+\cdots+l'_k[\sigma_k,t_k])=\varphi(l'_1[\sigma'_1,t'_1]+\cdots+l'_k[\sigma'_k,t'_k])=[\gamma_{\sigma'_1}(l'_1),t'_1]+\cdots+[\gamma_{\sigma'_k}(l'_k),t'_k]$. We will show that for each i, that $[\gamma_{\sigma'_i}(l'_i),t'_i]=[\gamma_{\sigma_i}(l_i),t_i]$. We have

$$\begin{aligned}\big[\gamma_{\sigma'_i}(l'_i),t'_i\big]&=\big[\gamma_{\sigma'_i}(l'_i),s_\#(t''_i)\big]\\&=\big[\big(s^F\big)_*\big(\gamma_{\sigma'_i}(l'_i)\big),t''_i\big]\\&=\big[\gamma_{d^K(\sigma_i)}\big(M_*\big(\hat{s}^K_{\sigma'_i}\big)M_*(\hat{q}_{\sigma_i,t_i})(l_i)\big),t''_i\big]\\&=\big[\gamma_{d^K(\sigma_i)}\big(M_*\big(\hat{s}^K_{\sigma'_i}\circ\hat{q}_{\sigma_i,t_i}\big)(l)\big),t''_i\big].\end{aligned}$$

On the other hand

$$\begin{aligned}\big[\gamma_{\sigma_i}(l_i),t_i\big]&=\big[\gamma_{\sigma_i}(l_i),d_\#(t''_i)\big]\\&=\big[\big(d^F\big)_*\big(\gamma_{\sigma_i}(l_i)\big),t''_i\big]\\&=\big[\gamma_{d^K(\sigma_i)}\big(M_*\big(\hat{d}^K_{\sigma_i}\big)(l_i)\big),t''_i\big].\end{aligned}$$

But $\hat{d}^K_{\sigma_i}=\hat{s}^K_{\sigma'_i}\circ q_{\sigma_i,t_i}:G/G_{\sigma_i}\to G/G_{s^K(\sigma'_i)}=G/G_{\sigma'_i}$, since both are quotient functions and $G_{s^K(\sigma'_i)}=G_{\sigma'_i}$ because s^K is injective.

In order to check the continuity of ψ, it is enough to consider the diagram

$$\begin{array}{ccc} P(K_n,M)^m\times\Delta^n & \longrightarrow & P(|K|,M)^m\\ \downarrow & & \\ \bigsqcup_n\big(F^G(K_n,M)\times\Delta^n\big) & & \downarrow\\ \downarrow & & \\ \big|F^G(K,M)\big| & \xrightarrow[\psi]{} & F^G\big(|K|,M\big),\end{array}$$

where the top arrow is given by

$$\big((l_1,\sigma_1,\dots,l_m,\sigma_m),t\big)\longmapsto\big(l_1,[\sigma_1,t],\dots,l_m,[\sigma_m,t]\big),$$

which is continuous.

To see that φ' and ψ' are continuous it is enough to consider the commutative diagrams

$$\begin{array}{ccc} F\big(|K|,M\big) & \xrightarrow{\varphi} & \big|F(K,M)\big|\\ {\scriptstyle\beta_{|K|}}\downarrow & & \downarrow{\scriptstyle|\beta_K|}\\ F^G\big(|K|,M\big) & \xrightarrow[\varphi']{} & \big|F^G(K,M)\big|\end{array}\quad\text{and}\quad\begin{array}{ccc} \big|F(K,M)\big| & \xrightarrow{\psi} & F\big(|K|,M\big)\\ {\scriptstyle|\beta_K|}\downarrow & & \downarrow{\scriptstyle\beta_{|K|}}\\ \big|F^G(K,M)\big| & \xrightarrow[\psi']{} & F^G\big(|K|,M\big),\end{array}$$

where the vertical arrows are identifications. □

In order to define the topology of $F(X,M)$ and $F^G(X,M)$ for an arbitrary G-space X, we shall need the following.

Definition 3.4. Let G be a finite group and X a pointed G-space and let $\mathcal{S}(X)$ denote the *singular simplicial set* of X given for each q by

$$\mathcal{S}_q(X)=\big\{\sigma\colon\,\Delta^q\to X\mid\sigma\text{ is a map}\big\}.$$

Then, in fact, $\mathcal{S}(X)$ is a simplicial pointed G-set with the usual simplicial structure. There is a natural G-map $\rho_X:|\mathcal{S}(X)|\to X$ given by $\rho_X[\sigma,t]=\sigma(t)$. We may consider the group $F(|\mathcal{S}(X)|,M)$ and the group $F^G(|\mathcal{S}(X)|,M)$. The homomorphisms

$$\rho_{X*}:F\big(\big|\mathcal{S}(X)\big|,M\big)\to F(X,M)\quad\text{and}\quad(\rho_X)^G_*:F^G\big(\big|\mathcal{S}(X)\big|,M\big)\to F^G(X,M)$$

are clearly surjective. Then give both $F(X,M)$ and $F^G(X,M)$ the respective identification topology.

We have to verify that this topology reduces to the one defined previously, in the case of $X = |K|$. We need the following.

Lemma 3.5. *Let K be a simplicial G-set. Then the canonical map $\rho_{|K|} : |\mathcal{S}(|K|)| \to |K|$ is an identification.*

Proof. Consider the map $i : |K| \to |\mathcal{S}(|K|)|$ given by $i[\alpha, t] = [\sigma_\alpha, t]$, where $\sigma_\alpha : \Delta^n \to |K|$ is defined by $\sigma_\alpha(s) = [\alpha, s]$. One can easily check that this map is well defined and continuous. The composite $\rho_{|K|} \circ i : |K| \to |K|$ is the identity, since $\rho_{|K|} i[\alpha, t] = \rho_{|K|}[\sigma_\alpha, t] = \sigma_\alpha(t) = [\alpha, t]$. Therefore, $\rho_{|K|}$ is a retraction and so it is also an identification. $\square$

Proposition 3.6. *Let K be a simplicial pointed G-set and assume that $F(|K|, M)$ and $F^G(|K|, M)$, as well as $F(|\mathcal{S}(|K|)|, M)$ and $F^G(|\mathcal{S}(|K|)|, M)$, have the topology given in* 3.1. *Then the maps*

$$\rho_{|K|*} : F\big(\big|\mathcal{S}\big(|K|\big)\big|, M\big) \twoheadrightarrow F\big(|K|, M\big),$$
$$\rho^G_{|K|*} : F^G\big(\big|\mathcal{S}\big(|K|\big)\big|, M\big) \twoheadrightarrow F^G\big(|K|, M\big)$$

are identifications.

Proof. We just prove the first case, since the second is similar. Consider the map $\rho_{|K|} : |\mathcal{S}(|K|)| \to |K|$, which by 3.5 is an identification. We have a commutative diagram

$$\begin{tikzcd}
\bigsqcup_k P\big(\big|\mathcal{S}\big(|K|\big)\big|, M\big)^k \arrow[r, two heads] \arrow[d] & \bigsqcup_k P\big(|K|, M\big)^k \arrow[d] \\
F\big(\big|\mathcal{S}\big(|K|\big)\big|, M\big) \arrow[r, two heads, "\rho_{|K|*}"'] & F\big(|K|, M\big),
\end{tikzcd}$$

where the top arrow is defined by the identification $\rho_{|K|}$ and, therefore, it is also an identification (since we are working in the category of k-spaces). Hence, the bottom arrow is an identification. $\square$

Therefore, we have two different ways to describe the topology on $F(|K|, M)$; namely, as in Definition 3.1 or as in Definition 3.4. Thus the topology of $F(|K|, M)$ for the geometric realization $|K|$ of a simplicial G-set K is well defined.

Once again, as in 3.2, for the general case we have the following.

Proposition 3.7. *Let X be any G-space and let $\beta_X : F(X, M) \to F^G(X, M)$ be given on generators by $\beta_X(lx) = \gamma^G_x(l)$. Then $F^G(X, M)$ has the identification topology given by the surjective map β_X.*

Proof. Just consider the following commutative diagram

$$\begin{tikzcd}
F\big(\big|\mathcal{S}(X)\big|, M\big) \arrow[r, two heads, "\beta_{|\mathcal{S}(X)|}"] \arrow[d, "\rho_{X*}"'] & F^G\big(\big|\mathcal{S}(X)\big|, M\big) \arrow[d, "\rho^G_{X*}"] \\
F(X, M) \arrow[r, two heads, "\beta_X"'] & F^G(X, M),
\end{tikzcd}$$

where the top arrow is an identification by 3.2, the right arrow is also an identification by definition. Therefore the bottom arrow is also an identification. $\square$

Remark 3.8. The groups $F(X,M)$ and $F^G(X,M)$ are topological groups. This follows from the commutativity of the next diagram.

$$\begin{tikzcd}
F\big(|\mathcal{S}(X)|,M\big)\times F\big(|\mathcal{S}(X)|,M\big) \arrow[r, "+"] \arrow[d, "\rho_{X*}\times\rho_{X*}"'] & F\big(|\mathcal{S}(X)|,M\big) \arrow[d, "\rho_{X*}"] \\
F(X,M)\times F(X,M) \arrow[r, "+"] \arrow[d, "\beta_X\times\beta_X"'] & F(X,M) \arrow[d, "\beta_X"] \\
F^G(X,M)\times F^G(X,M) \arrow[r, "+"] & F^G(X,M).
\end{tikzcd}$$

In order to show now that the topological groups $F(X,M)$ and $F^G(X,M)$ are indeed functors of X we have the following.

Proposition 3.9. *Let $f:X\to Y$ be a continuous G-map of pointed G-spaces. Then*

$$f_*:F(X,M)\to F(Y,M) \quad \textit{and} \quad f_*^G:F^G(X,M)\to F^G(Y,M)$$

are continuous homomorphisms.

Proof. Let $\mathcal{S}(f):\mathcal{S}(X)\to\mathcal{S}(Y)$ be the map of simplicial G-sets induced by f. We have the following commutative diagram.

$$\begin{tikzcd}
\big|F\big(\mathcal{S}(X),M\big)\big| \arrow[r, "|\mathcal{S}(f)_*|"] \arrow[d, "\psi"', "\cong"] & \big|F\big(\mathcal{S}(Y),M\big)\big| \arrow[d, "\cong"', "\psi"] \\
F\big(|\mathcal{S}(X)|,M\big) \arrow[r, "|\mathcal{S}(f)|_*"] \arrow[d, "\rho_{X*}"'] & F\big(|\mathcal{S}(Y)|,M\big) \arrow[d, "\rho_{Y*}"] \\
F(X,M) \arrow[r, "f_*"] \arrow[d, "\beta_X"'] & F(Y,M) \arrow[d, "\beta_Y"] \\
F^G(X,M) \arrow[r, "f_*^G"'] & F^G(Y,M).
\end{tikzcd}$$

Since the top map is continuous and the maps ψ are homeomorphisms by 3.3(a), the map $|\mathcal{S}(f)|_*$ is also continuous. Moreover, by 3.4, the maps ρ_{X*} and ρ_{Y*} are identifications. Hence f_* is continuous. By 3.7, the maps β_X and β_Y are identifications, therefore, f_*^G is also continuous. □

Recall the definition of the transfers

$$t_p:F(X^+,M)\longrightarrow F(E^+,M) \quad \text{and} \quad t_p^G:F^G(X^+,M)\longrightarrow F^G(E^+,M)$$

given for finite-to-one maps $p:E\to X$ in 1.10. We now study its topological counterpart.

In what follows, we shall prove that if $p:E\to X$ is an *n-fold G-equivariant covering map*, namely an ordinary covering map, such that E and X are G-spaces and p is G-equivariant, then t_p is continuous on $F(X,M)$ and t_p^G is continuous on $F^G(X,M)$. To that end we shall need the simplicial map

$$\mathcal{S}(p):\mathcal{S}(E)\to\mathcal{S}(X).$$

For every n, the G-function $\mathcal{S}_n(p):\mathcal{S}_n(E)\to\mathcal{S}_n(X)$ is n-to-one, since every k-simplex $\sigma:\Delta^k\to X$ has exactly n liftings $\tilde\sigma_i:\Delta^k\to E$, $i=1,\dots,n$. Hence $\mathcal{S}_n(p)$ has transfers (see 1.10)

$$t_{\mathcal{S}_n(p)}:F\big(\mathcal{S}_n(X)^+,M\big)\longrightarrow F\big(\mathcal{S}_n(E)^+,M\big),$$
$$t_{\mathcal{S}_n(p)}^G:F^G\big(\mathcal{S}_n(X)^+,M\big)\longrightarrow F^G\big(\mathcal{S}_n(E)^+,M\big).$$

Proposition 3.10. *The homomorphisms $t_{\mathcal{S}_n(p)}$ and $t^G_{\mathcal{S}_n(p)}$ define maps of simplicial sets*

$$t_{\mathcal{S}(p)} : F\big(\mathcal{S}(X)^+, M\big) \longrightarrow F\big(\mathcal{S}(E)^+, M\big),$$
$$t^G_{\mathcal{S}(p)} : F^G\big(\mathcal{S}(X)^+, M\big) \longrightarrow F^G\big(\mathcal{S}(E)^+, M\big).$$

Proof. We prove the second part. Let $f : \bar{m} \to \bar{n}$ be a morphism in the category Δ. Consider the diagram

$$\begin{tikzcd} F^G\big(\mathcal{S}_n(X), M\big) \arrow[r, "(f^{\mathcal{S}(X)})^G_*"] \arrow[d, "t^G_{\mathcal{S}_n(p)}"'] & F^G\big(\mathcal{S}_m(X), M\big) \arrow[d, "t^G_{\mathcal{S}_m(p)}"] \\ F^G\big(\mathcal{S}_n(E), M\big) \arrow[r, "(f^{\mathcal{S}(E)})^G_*"'] & F^G\big(\mathcal{S}_m(E), M\big). \end{tikzcd}$$

To see that it commutes, take a generator $\gamma^G_\sigma(l) \in F^G(\mathcal{S}_n(X), M)$. Using the formula (1.11), we have

$$\big(f^{\mathcal{S}(E)}\big)^G_* t^G_{\mathcal{S}_n(p)}\big(\gamma^G_\sigma(l)\big) = \sum_{\iota \in \mathcal{I}} \gamma^G_{f^{\mathcal{S}(E)}(\tilde{\sigma}_\iota)} M_*\big(\widehat{f^{\mathcal{S}(E)}}_{\tilde{\sigma}_\iota}\big) M^*\big(\widehat{\mathcal{S}_n(p)}_{\tilde{\sigma}_\iota}\big)(l),$$

where $\mathcal{S}_n(p)^{-1}(\sigma)/G_\sigma = \{[\tilde{\sigma}_\iota] \mid \iota \in \mathcal{I}\}$. On the other hand,

$$t^G_{\mathcal{S}_m(p)}\big(f^{\mathcal{S}(X)}\big)^G_*\big(\gamma^G_\sigma(l)\big) = \sum_{\iota' \in \mathcal{I}'} \gamma^G_{f^{\mathcal{S}(E)}(\tilde{\sigma}_{\iota'})} M^*\big(\widehat{\mathcal{S}_m(p)}_{f^{\mathcal{S}(E)}(\tilde{\sigma}_{\iota'})}\big) M_*\big(\widehat{f^{\mathcal{S}(X)}}_\sigma\big)(l),$$

where $\mathcal{S}_m(p)^{-1}(f^{\mathcal{S}(X)}(\sigma))/G_{f^{\mathcal{S}(X)}(\sigma)} = \{[\tilde{\sigma}_{\iota'}] \mid \iota' \in \mathcal{I}'\}$.

We show that both composites coincide. On the one hand,

$$M_*\big(\widehat{f^{\mathcal{S}(E)}}_{\tilde{\sigma}_\iota}\big) M^*\big(\widehat{\mathcal{S}_n(p)}_{\tilde{\sigma}_\iota}\big) = M^*\big(\widehat{\mathcal{S}_m(p)}_{f^{\mathcal{S}(E)}(\tilde{\sigma}_{\iota'})}\big) M_*\big(\widehat{f^{\mathcal{S}(X)}}_\sigma\big)$$

by the pullback property of the Mackey functor. On the other hand, there is a bijection between

$$\big(\mathcal{S}_n(p)\big)^{-1}(\sigma)/G_\sigma \quad \text{and} \quad \mathcal{S}_m(p)^{-1}\big(f^{\mathcal{S}(X)}(\sigma)\big)/G_{f^{\mathcal{S}(X)}(\sigma)}$$

induced by $\tilde{\sigma}_\iota \mapsto \tilde{\sigma}_\iota \circ f_\# = f^{\mathcal{S}_n(E)}(\tilde{\sigma}_\iota)$. This follows from the fact that the orbits $G_\sigma\tilde{\sigma} \cong G_\sigma/G_{\tilde{\sigma}}$ and $G_{f^{\mathcal{S}_n(X)}(\sigma)}(\tilde{\sigma}_\iota \circ f_\#) \cong G_{f^{\mathcal{S}_n(X)}(\sigma)}/G_{\tilde{\sigma}_\iota \circ f_\#}$ are isomorphic, since $G_\sigma \cap G_{\tilde{\sigma}_\iota \circ f_\#} = G_{\tilde{\sigma}_\iota}$, which follows easily, since $p : E \to X$ is a G-equivariant covering map.

The proof of the first part is similar but much simpler. □

Remark 3.11. The transfers $\underline{t}^G_{\mathcal{S}_n(p)} : F^G(\mathcal{S}_n(X), M) \to F^G(\mathcal{S}_n(E), M)$ determine also a map of simplicial sets $\underline{t}^G_{\mathcal{S}(p)} : F^G(\mathcal{S}(X), M) \to F^G(\mathcal{S}(E), M)$, as follows easily from the first part of the previous result.

Proposition 3.12. *Let $p : E \to X$ be an n-fold G-equivariant covering map. Then*

(a) *$t_p : F(X^+, M) \to F(E^+, M)$ is continuous, and*
(b) *$t^G_p : F^G(X^+, M) \to F^G(E^+, M)$ is continuous.*

Proof. We prove (b). It follows from the fact that the following diagram commutes:

$$\begin{tikzcd} \big|F^G\big(\mathcal{S}(X^+), M\big)\big| \arrow[r, "|t^G_{\mathcal{S}(p)}|"] \arrow[d, "\psi"', "\cong"] & \big|F^G\big(\mathcal{S}(E^+), M\big)\big| \arrow[d, "\cong"', "\psi"] \\ F^G\big(\big|\mathcal{S}(X^+)\big|, M\big) \arrow[d, "(\rho_{X^+})^G_*"'] & F^G\big(\big|\mathcal{S}(E^+)\big|, M\big) \arrow[d, "(\rho_{E^+})^G_*"] \\ F^G(X^+, M) \arrow[r, "t^G_p"'] & F^G(E^+, M). \end{tikzcd}$$

Since the vertical arrows are identifications and the top arrow is continuous by Proposition 3.10, so the bottom arrow is continuous too.

The proof of (a) is similar. □

Remark 3.13. Since $\underline{t}_p^G \circ \beta_{X^+} = \beta_{E^+} \circ t_p : F(X^+, M) \to F^G(E^+, M)$ and β_{X^+} and β_{E^+} are identifications, the transfer $\underline{t}_p^G : F^G(X^+, M) \to F^G(E^+, M)$ is also continuous.

The functors $F(X, M)$ and $F^G(X, M)$ are homotopy invariant. Namely, we have the following.

Proposition 3.14. *If $f_0, f_1 : X \to Y$ are G-homotopic pointed maps, then*

(a) $f_{0*}, f_{1*} : F(X, M) \to F(Y, M)$ *are homotopic homomorphisms, and*
(b) $f_{0*}^G, f_{1*}^G : F^G(X, M) \to F^G(Y, M)$ *are homotopic homomorphisms.*

Proof. We prove (b); (a) is similar. Let $H : X \times I \to Y$ be a pointed G-homotopy from f_0 to f_1. Define a homotopy $\overline{H}$ by the commutative diagram

$$\begin{tikzcd} F^G(X, M) \times I \arrow[r, dashed, "\overline{H}"] \arrow[d, "\bar{\alpha}"'] & F^G(Y, M) \\ F^G(X \times I, M), \arrow[ur, "H_*^G"'] & \end{tikzcd}$$

where $\bar{\alpha}(\gamma_x^G(l), t) = \gamma_{(x,t)}^G(l)$. The function $\bar{\alpha}$ is continuous since it is induced by the continuous map $\alpha : |\mathcal{S}(X)| \times I \to |\mathcal{S}(X \times I)|$ given by $\alpha([\sigma, s], t) = [\sigma_t, s]$, where $\sigma \in \mathcal{S}_n(X)$, $s \in \Delta^n$, $t \in I$, and $\sigma_t(s) = (\sigma(s), t)$. Hereby we are using the description of the topology of $F^G(X, M)$ given in 3.4. Then $\overline{H}$ is a pointed homotopy from f_{0*}^G to f_{1*}^G. □

Definition 3.15. Let G be a finite group, X a G-space and M a Mackey functor for G. Define

$$\mathbb{H}_q^G(X; M) = \pi_q\big(F^G(X^+, M)\big).$$

These are homotopy invariant groups.

Propositions 3.9 and 3.12 show that $X \mapsto F^G(X^+, M)$ is a *bifunctor* on G-$\mathcal{T}$op in the sense that continuous G-maps $f : X \to Y$ define continuous homomorphisms $(f^+)_*^G : F^G(X^+, M) \to F(Y^+, M)$, and n-fold G-equivariant covering maps $p : E \to B$ define continuous homomorphisms $t_p^G : F^G(X^+, M) \to F^G(E^+, M)$. By using Propositions 1.14 and 1.16, we have the first main theorem of this paper.

Theorem 3.16. *The bifunctor $F^G(-^+, M)$ on G-$\mathcal{T}$op restricted to the category G-$\mathcal{S}$et$_{\text{fin}}$ is naturally isomorphic to the Mackey functor M. Therefore, the group functors $\mathbb{H}_q^G(X; M)$ restricted to the category G-$\mathcal{S}$et$_{\text{fin}}$ are naturally isomorphic to the Mackey functor M, when $q = 0$, and 0 otherwise.*

4. The second main theorem

The second main theorem of this paper is the following result.

Theorem 4.1. *Let M be a Mackey functor and X a pointed G-space of the same homotopy type of a G-CW-complex. Then the homotopy groups*

$$\widetilde{\mathbb{H}}_q^G(X; M) = \pi_q\big(F^G(X, M)\big)$$

are naturally isomorphic to the (reduced) Bredon–Illman G-equivariant homology groups $\tilde{H}_q^G(X; M_*)$ with coefficients in the covariant part of M.

In what follows we shall prove this theorem.

Let $\mathcal{A}$ be a simplicial abelian group. Recall that the q-homotopy group of $\mathcal{A}$ is defined by

$$\pi_q(\mathcal{A}) = H_q\big(N(\mathcal{A}), \tilde{\partial}\big),$$

where $N(\mathcal{A})_q = \mathcal{A}_q \cap \ker d_0 \cap \cdots \cap \ker d_{q-1}$ and $\tilde{\partial}_q = (-1)^q d_q$; here d_i is the ith face operator of $\mathcal{A}$. On the other hand, $\mathcal{A}$ can be seen as a chain complex, with $\partial : \mathcal{A}_q \to \mathcal{A}_{q-1}$ given by $\sum_{i=0}^{q}(-1)^i d_i$.

We have the following result (cf. [9, 22.1]).

Proposition 4.2. *The canonical inclusion of chain complexes $N(\mathcal{A}) \hookrightarrow \mathcal{A}$ induces an isomorphism in homology.*

Proof. The chain complex $\mathcal{A}$ is filtered by chain complexes $\mathcal{A}^p$, where

$$\mathcal{A}_q^p = \big\{u \in \mathcal{A}_q \mid d_i(u) = 0,\ 0 \leqslant i < \min\{q, p\}\big\}.$$

The canonical inclusion $i^p : \mathcal{A}^{p+1} \hookrightarrow \mathcal{A}^p$ is a chain homotopy equivalence with inverse $r^p : \mathcal{A}^p \to \mathcal{A}^{p+1}$ given by $r^p(u) = u - s_p d_p(u)$, where s_p is the pth degeneracy operator of $\mathcal{A}$. Obviously, $r^p \circ i^p = 1_{\mathcal{A}^{p+1}}$; conversely, $i^p \circ r^p$ is chain homotopic to $1_{\mathcal{A}^p}$ via the chain homotopy $h^p : \mathcal{A}_q^p \to \mathcal{A}_{q+1}^p$ given by

$$h^p(u) = \begin{cases} 0 & \text{if } q < p, \\ (-1)^p s_p(u) & \text{if } q \geqslant p. \end{cases} \qquad \square$$

Definition 4.3. For a pointed G-space X, let $\mathcal{T}_q^G(X)$ be the set G-maps $T : \Delta^q \times G/H \to X$, where G acts only on G/H, and let $T_0 : \Delta^q \times G/G \to X$ be the constant map with value the base point of X. Now let M be a Mackey functor. Define

$$\widehat{F}\big(\mathcal{T}_q^G(X), M\big) = \big\{v\colon \mathcal{T}_q^G(X) \to \widehat{M} \mid \text{ if } T : \Delta^q \times G/H \to X \\ \text{then } v(T) \in M(G/H),\ v(T_0) = 0,\ v(T) = 0 \text{ for almost all } T \in \mathcal{T}_q^G(X)\big\}.$$

One easily sees that the groups $\widehat{F}(\mathcal{T}_q^G(X), M)$ are exactly Illman's groups $\widehat{C}_q^G(X; M_*)$ [6, Def. 3.3], where M_* denotes the covariant coefficient system associated to the Mackey functor M. As Illman does, we say that the generator lT is *related* to $l'T'$ if there exists a G-function $\alpha : G/H \to G/H'$ such that the following diagram commutes

$$\begin{tikzcd} \Delta^q \times G/H \arrow[rr, "\mathrm{id}\times\alpha"] \arrow[dr, "T"'] & & \Delta^q \times G/H' \arrow[dl, "T'"] \\ & X & \end{tikzcd}$$

and $l' = M_*(\alpha)(l) \in M(G/H')$. Divide the group $\widehat{F}(\mathcal{T}_q^G(X), M)$ by the subgroup generated by the differences $lT - l'T'$ where either lT is related to $l'T'$ or $l'T'$ is related to lT, as well as by all elements lT such that $T : \Delta^q \times G/H \to X$ is constant with value the base point $*$, to obtain the group $F'(\mathcal{T}_q^G(X), M)$.

Lemma 4.4. *The simplicial group $F^G(\mathcal{S}(X), M)$ and the graded group $F'(\mathcal{T}^G(X), M)$ are chain complexes.*

Proof. Since $F^G(\mathcal{S}(X), M)$ is a simplicial abelian group, then by [9] it can be regarded as a chain complex with the differential

$$\partial_q^G : F^G\big(\mathcal{S}_q(X), M\big) \longrightarrow F^G\big(\mathcal{S}_{q-1}(X), M\big) \quad \text{given by} \quad \partial_q^G = \sum_{i=0}^{q}(-1)^i \big(d_i^{\mathcal{S}}\big)_*^G,$$

where $d_i^{\mathcal{S}} : \mathcal{S}_q(X) \to \mathcal{S}_{q-1}(X)$ is defined by $d_i^{\mathcal{S}}(\sigma) = \sigma \circ d_{i\#}$.

Now let $\delta_i : F'(\mathcal{T}_q^G(X), M) \to F'(\mathcal{T}_{q-1}^G(X), M)$ be the homomorphism given by $\delta_i[lT] = [lT^i]$, where $T : \Delta^q \times G/H \to X$, and $T^i : \Delta^{q-1} \times G/H \to X$ is given by $T^i = T \circ (d_{i\#} \times \mathrm{id}_{G/H})$. Then we have the differential

$$\partial_q : F'\big(\mathcal{T}_q^G(X), M\big) \longrightarrow F'\big(\mathcal{T}_{q-1}^G(X), M\big) \quad \text{given by} \quad \partial_q = \sum_{i=0}^{q}(-1)^i \delta_i. \qquad \square$$

In fact, the chain complex $F'(\mathcal{T}^G(X), M)$ is identical to Illman's chain complex $S^G(X, *; M_*)$ (cf. [6, p. 15]). Then we have the following.

Theorem 4.5. *The chain complexes $F'(\mathcal{T}^G(X), M)$ and $F^G(\mathcal{S}(X), M)$ are isomorphic.*

The proof of this theorem requires some preparation.

Let X be a pointed G-space. We shall show that the groups $F'(\mathcal{T}_q^G(X), M)$ have the universal property of Proposition 1.6(a) with respect to the G-set $\mathcal{S}_q(X)$. Namely, take for each $\sigma \in \mathcal{S}_q(X)$ the homomorphisms $\nu_\sigma : M(G/G_\sigma) \to F'(\mathcal{T}_q^G(X), M)$ given by $\nu_\sigma(l) = [lT_\sigma]$, where $T_\sigma : \Delta^q \times G/G_\sigma \to X$ is defined by $T_\sigma(t, [g]) = g\sigma(t)$. Then, for the base point $\sigma_0 \in \mathcal{S}_q(X)$, $\nu_{\sigma_0} = 0$ and $\nu_{g\sigma} = \nu_\sigma \circ M_*(R_g)$. Thus the universal property of $F'(\mathcal{T}_q^G(X), M)$ is given by the following.

Proposition 4.6. *If A is an abelian group and there is a family of homomorphisms $f_\sigma : M(G/G_\sigma) \to A$, $\sigma \in \mathcal{S}_q(X)$, satisfying $f_{\sigma_0} = 0$ and $f_{g\sigma} = f_\sigma \circ M_*(R_g)$, then there is a unique $f : F'(\mathcal{T}^G(X), M) \to A$ such that $f \circ \nu_\sigma = f_\sigma$. Thus we have*

$$\begin{tikzcd} M(G/G_\sigma) \arrow[r, "\nu_\sigma"] \arrow[dr, "f_\sigma"'] & F'\big(\mathcal{T}_q^G(X), M\big) \arrow[d, dotted, "f"] \\ & A. \end{tikzcd}$$

Proof. Given a G-map $T : \Delta^q \times G/H \to X$, define $\sigma_T : \Delta^q \to X$ by $\sigma_T(t) = T(t, [e]_H)$, where $e \in G$ is the neutral element.

Define $f : F'(\mathcal{T}^G(X), M) \to A$ by $f[lT] = f_{\sigma_T} M_*(p)(l)$, where $p : G/H \to G/G_{\sigma_T}$ is the quotient function. To show that this is well defined, suppose lT is related to $l'T'$. We consider two cases.

We assume first that $H \subset H'$ and that $\alpha : G/H \to G/H'$ is the quotient function. Then $T' \circ (\mathrm{id} \times \alpha) = T$, thus $T(t, [g]_H) = T'(t, [g]_{H'})$, and $l' = M_*(\alpha)(l)$. Notice that $\sigma_{T'} = \sigma_T$, so that we have $H \subset H' \subset G_{\sigma_T}$, and so $p' \circ \alpha = p$. Therefore, $f[l'T] = f_{\sigma_{T'}} M_*(p') M_*(\alpha)(l) = f_{\sigma_T} M_*(p)(l) = f[lT]$.

Now assume that $H' = gHg^{-1}$ and $\alpha = R_{g^{-1}} : G/H \to G/H'$. Then $\sigma_{T'} = g\sigma_T$, $p' \circ R_{g^{-1}} = R_{g^{-1}} \circ p$, and $l' = \tilde{M}_*(R_{g^{-1}})(l)$. Hence $f[l'T'] = f_{\sigma_{T'}} M_*(p') M_*(R_{g^{-1}})(l) = f_{g\sigma_T} M_*(R_{g^{-1}}) M_*(p)(l) = f_{\sigma_T} M_*(p)(l) = f[lT]$.

Clearly, f is the unique homomorphism such that $f \circ \nu_\sigma = f_\sigma$ for all $\sigma \in \mathcal{S}_q(X)$. $\square$

Proof of Theorem 4.5. The isomorphism $\beta : F'(\mathcal{T}_q^G(X), M) \to F^G(\mathcal{S}_q(X), M)$ provided by the universal property proved above, is given by

$$[lT] \longmapsto \gamma_{\sigma_T}^G\big(M_*(p)(l)\big),$$

where $T : \Delta^q \times G/H \to X$, $p : G/H \to G/G_{\sigma_T}$ is the quotient function, and $l \in M(G/H)$. In order to show that β is a chain map, consider the following diagram:

$$\begin{tikzcd} F'\big(\mathcal{T}_q^G(X), M\big) \arrow[r, "\beta", "\cong"'] \arrow[d, "\delta_i"'] & F^G\big(\mathcal{S}_q(X), M\big) \arrow[d, "(d_i^{\mathcal{S}})_*^G"] \\ F'\big(\mathcal{T}_{q-1}^G(X), M\big) \arrow[r, "\beta", "\cong"'] & F^G\big(\mathcal{S}_{q-1}(X), M\big). \end{tikzcd} \tag{4.7}$$

By Definition 1.7, we have that $(d_i^{\mathcal{S}})_*^G \circ \gamma_{\sigma_T}^G = \gamma_{d_i^{\mathcal{S}}(\sigma_T)}^G \circ M_*(\widehat{(d_i^{\mathcal{S}})}_{\sigma_T})$. Now consider a map $T : \Delta^q \times G/H \to X$; since $\sigma_T : \Delta^q \to X$ is given by $\sigma_T(t) = T(t, [e]_H)$, then $\sigma_T^i = d_i^{\mathcal{S}}(\sigma_T)$. Furthermore, the following diagram is commutative:

$$\begin{tikzcd} G/H \arrow[dr, "q"] \arrow[dd, "p"'] & \\ & G/G_{\sigma_T^i} = G/G_{d_i^{\mathcal{S}}(\sigma_T)}. \\ G/G_{\sigma_T} \arrow[ur, "\widehat{(d_i^{\mathcal{S}})}_{\sigma_T}"'] & \end{tikzcd}$$

Using this, we can write the value of $(d_i^{\mathcal{S}})_*^G \circ \beta$ on a generator $[lT]$ in $F'(\mathcal{T}_q^G(X), M)$ as follows:

$$\begin{aligned}
\big(d_i^{\mathcal{S}}\big)_*^G \beta[lT] &= \big(d_i^{\mathcal{S}}\big)_*^G \gamma_{\sigma_T}^G M_*(p)(l) \\
&= \gamma_{d_i^{\mathcal{S}}(\sigma_T)}^G M_*\big(\widehat{\big(d_i^{\mathcal{S}}\big)}_{\sigma_T}\big) M_*(p)(l) \\
&= \gamma_{\sigma_{Ti}}^G M_*(q)(l).
\end{aligned}$$

Therefore, diagram (4.7) commutes. Since ∂^G as well as ∂ are linear combinations of the operators $(d_i^{\mathcal{S}})_*^G$ and δ_i, respectively, then β is a chain map. □

Proposition 4.7. *Let X be a pointed G-space of the same homotopy type of a G-CW-complex. Then $\rho: |\mathcal{S}(X)| \to X$ given by $\rho[\sigma, t] = \sigma(t)$ is a G-homotopy equivalence.*

Proof. Let $H \subset G$ be any subgroup. Note first, as already mentioned before, that the identity induces a homeomorphism $|K^H| \approx |K|^H$ for any simplicial G-set K. On the other hand, one also has a canonical isomorphism of simplicial sets $\mathcal{S}(X^H) \cong \mathcal{S}(X)^H$. We have that the map $\rho: |\mathcal{S}(X)| \to X$, being natural, is a G-map. By the naturality of the map ρ, it restricts to $\rho_H: |\mathcal{S}(X^H)| \to X^H$, which by a theorem of Milnor is a homotopy equivalence. Therefore, the map $\rho^H: |\mathcal{S}(X)|^H \approx |\mathcal{S}(X^H)| \to X^H$ is a homotopy equivalence for every $H \subset G$. Then, by a result of Bredon [3, II(5.5)], ρ is a G-homotopy equivalence. □

Note that a nice consequence of the previous result is the following.

Proposition 4.8. *Let X be a pointed G-space of the same homotopy type of a G-CW-complex. Then $F^G(X, M)$ has the same homotopy type of a CW-complex.*

Proof. This follows from 3.3, 4.7, and the homotopy invariance 3.14. □

Proof of Theorem 4.1. We shall give an isomorphism

$$\widetilde{H}_q^G(X; M) \cong H_q\big(F^G\big(\mathcal{S}_q(X), M\big)\big) \longrightarrow \pi_q\big(F^G(X, M)\big) = \widetilde{\mathbb{H}}_q^G(X; M).$$

Here the left-hand side is the Bredon–Illman (reduced) homology of X, and the first isomorphism follows from the natural isomorphism of Theorem 4.5.

To construct the arrow, we shall give several isomorphisms as depicted in the following diagram.

$$\begin{tikzcd}
H_q\big(F^G\big(\mathcal{S}(X), M\big)\big) \arrow[d, dotted] & \pi_q\big(F^G\big(\mathcal{S}(X), M\big)\big) \arrow[l, "i_*"', "\cong"] \arrow[r, "\Psi", "\cong"'] & \pi_q\big(\mathcal{S}\big(\big|F^G\big(\mathcal{S}(X), M\big)\big|\big)\big) \arrow[d, "\Phi", "\cong"'] \\
\pi_q\big(F^G(X, M)\big) & \pi_q\big(F^G\big(\big|\mathcal{S}(X)\big|, M\big)\big) \arrow[l, "\cong"', "(\rho_*^G)_*"] & \pi_q\big(\big|F^G\big(\mathcal{S}(X), M\big)\big|\big). \arrow[l, "\cong"', "\psi_*"]
\end{tikzcd}$$

By Proposition 4.2, i_* is an isomorphism. In particular, this shows that every cycle in $\widetilde{H}^G(X; M)$ is represented by a chain u, all of whose faces are zero. We call this a *special chain*.

The homomorphism Ψ, which is given by $\Psi(u)[t] = [u, t]$, where u is a special q-chain and $t \in \Delta^q$, is an isomorphism, as follows from [9, 16.6].

In order to define Φ, we must express $\Psi(u)$ as a map $\gamma: (\Delta[q], \dot{\Delta}[q]) \to (\mathcal{S}|F^H(\mathcal{S}(X), M)|, *)$. By the Yoneda lemma, γ is the unique map such that $\gamma(\delta_q) = \Psi(u)$, where $\delta_q = \mathrm{id}: \bar{q} \to \bar{q}$. The homomorphism Φ, defined by $\Phi[\gamma][f, s] = \gamma(f)(s)$, for $f \in \Delta[q]_n$ and $s \in \Delta^n$, is given by the adjunction between the realization functor and the singular complex functor (see [9, 16.1]).

That ψ_* is an isomorphism follows from Proposition 3.3. Finally, the homomorphism $(\rho_*^G)_*$ is an isomorphism by 4.7 and 3.14. □

Chasing along the diagram and using the homeomorphism $|\Delta[q]| \to \Delta^q$ given by $[f, t] \mapsto f_\#(t)$, one obtains the following.

Corollary 4.9. *The isomorphism* $H_q(F^G(\mathcal{S}(X),M)) \to \pi_q(F^G(X,M))$ *sends a homology class* $[u]$ *represented by a special chain* $u = \sum_\alpha \gamma^G_{\sigma_\alpha}(u(\sigma_\alpha))$ *to the homotopy class* $[\bar u]$ *given by* $\bar u(t) = \sum_\alpha \gamma^G_{\sigma_\alpha(t)}(M_*(p_\alpha)(u(\sigma_\alpha)))$*, where* $p_\alpha : G/G_{\sigma_\alpha} \to G/G_{\sigma_\alpha(t)}$ *is the quotient function and* $\mathcal{S}(X)/G = \{[\sigma_\alpha]\}$.

In [11, IX.5.2] it is proved that whenever the coefficients of a $\mathbb{Z}$-graded G-equivariant homology theory can be extended to a Mackey functor, then the grading can be extended to an RO(G)-grading (see also [7]). Therefore we have the following:

Theorem 4.10. *The grading of the theory* $\mathbb{H}^G_*(-;M)$ *can be extended to an* RO(G)*-grading.*

Example 4.11. We shall give here the example of the previous constructions for the Mackey functor M_L, where L is a G-module. We define $M_L(G/H) = L^H$. Let $H \subset K$ and $q : G/H \to G/K$ be the quotient map. Then $(M_L)_*(q) : L^H \to L^K$ is given by $(M_L)_*(q)(l) = \sum_{i=1}^r k_i l$, where $K/H = \{[k_i]\}_{i=1}^r$. Moreover, $(M_L)^*(q) : L^K \to L^H$ is the inclusion.

On the other hand, let $R_{g^{-1}} : G/H \to G/gHg^{-1}$ be right translation by g^{-1}. Then $(M_L)_*(R_{g^{-1}}) : L^H \to L^{gHg^{-1}}$ is given by $(M_L)_*(R_{g^{-1}})(l) = gl$. Moreover, $(M_L)^*(R_{g^{-1}}) : L^{gHg^{-1}} \to L^H$ is given by $(M_L)^*(R_{g^{-1}})(l) = g^{-1}l$.

Let X be a pointed G-space. The group $F^G(X, M_L)$ consists of pointed almost-zero G-equivariant functions $u : X \to \widehat{M}_L = L$. Hence, algebraically, this group is equal to the group $F^G(X, L)$ defined in [1]. However, as a topological group $F^G(X, M_L)$ need not be equal to $F^G(X, L)$, since their topologies are given in very different ways. They only have the same homotopy groups, since they both yield they same equivariant homology theory. Therefore, they are homotopy equivalent, because their Postnikov invariants are zero.

5. Mackey functors of *R*-modules and homological Mackey functors

We start this section by recalling what is a morphism between Mackey functors. Let M and N be Mackey functors for the finite group G. A *morphism* from M to N is a transformation $\varphi : M \to N$, that is natural with respect to both the covariant and the contravariant structures. Namely, if $f : S \to T$ is a G-function between G-sets, then

$$\begin{aligned} &\varphi_T \circ M_*(f) = N_*(f) \circ \varphi_S : M(S) \longrightarrow N(T), \\ &N^*(f) \circ \varphi_T = \varphi_S \circ M^*(f) : M(T) \longrightarrow N(S). \end{aligned}$$

Let X be a pointed G-space and $\varphi : M \to N$ a morphism of Mackey functors. We have the following.

Proposition 5.1. *Let X be a pointed G-space. If* $\varphi : M \to N$ *is a morphism of Mackey functors, then* $\varphi^G_* : F^G(X,M) \to F^G(X,N)$ *is a continuous homomorphism of topological groups. This converts* $M \mapsto F^G(X,M)$ *into a covariant functor from the category of Mackey functors for G to the category of topological abelian groups.*

Proof. The naturality of φ guarantees that the induced function

$$\tilde\varphi_n : F^G(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(X), N),$$

given by $\tilde\varphi_n(\gamma^G_\sigma(l)) = \gamma^G_\sigma(\varphi_{G/G_\sigma}(l))$, determines a morphism $\tilde\varphi$ of simplicial groups. Therefore $\tilde\varphi$ determines a continuous homomorphism

$$|\tilde\varphi| : |F^G(\mathcal{S}(X), M)| \longrightarrow |F^G(\mathcal{S}(X), N)|.$$

Using the isomorphisms of Proposition 3.3, one easily verifies that $|\tilde\varphi|$ corresponds to the homomorphism

$$\varphi^G_* : F^G(|\mathcal{S}(X)|, M) \longrightarrow F^G(|\mathcal{S}(X)|, N);$$

hence this is continuous. Since $F^G(X,M)$ and $F^G(X,N)$ have the identification topology induced by the maps $(\rho_X)^G_*$, the result follows. □

Let R be a commutative ring with one. We shall consider Mackey functors with values in the category R-$\mathcal{M}$od of left R-modules. Thus, each $r \in R$ behaves like morphism $r : M \to M$ between the underlying Mackey functors of groups. By Proposition 5.1, we have the next.

Corollary 5.2. *Let X be a pointed G space and M be a Mackey functor with values in R-$\mathcal{M}$od. Then $F^G(X, M)$ is a topological R-module. Thus $X \mapsto F^G(X, M)$ is a functor from the category of pointed G-spaces to the category of topological R-modules.*

Remark 5.3. It is not difficult to verify that all results in the previous sections hold as well for Mackey functors with values in R-modules.

In what follows, we shall always assume that M is a Mackey functor for a finite group G with values in R-$\mathcal{M}$od.

Let now $p: E \to X$ be an n-fold G-equivariant covering map. We shall compute the composite $p_*^G \circ t_p^G$: $F^G(X^+, M) \to F^G(X^+, M)$, which will have a nice formula when M is a homological Mackey functor. To start, recall that the transfer t_p^G is the restriction to $F^G(X^+, M)$ of the transfer $t_p: F(X^+, M) \to F(E^+, M)$, which for monomials is given by

$$t_p(lx) = \sum_{a \in p^{-1}(x)} M^*(\hat{p}_a)(l)a \quad \text{where } l \in M(G/G_x). \tag{5.4}$$

Let $\gamma_x^G(l)$ be a generator of $F^G(X^+, M)$. Then

$$\begin{aligned} p_*^G t_p^G\big(\gamma_x^G(l)\big) &= p_*^G t_p^G\left(\sum_{i=1}^m M_*(R_{g_i^{-1}})(l) g_i x\right) \\ &= p_*^G\left(\sum_{i=1}^m t_p^G\big(M_*(R_{g_i^{-1}})(l) g_i x\big)\right). \end{aligned}$$

Using (5.4), the last term is equal to

$$p_*^G\left(\sum_{i=1}^m \sum_{a \in p^{-1}(g_i x)} M^*(\hat{p}_a) M_*(R_{g_i^{-1}})(l) g_i x\right).$$

By the definition of p_*^G, we get the following.

Proposition 5.4. *Let M be a Mackey functor and let $p: E \to X$ be an n-fold G-equivariant covering map. Then*

$$p_*^G t_p^G\big(\gamma_x^G(l)\big) = \sum_{\kappa \in K} \gamma_{p(a_\kappa)}^G\big(M_*(\hat{p}_{a_\kappa}) M^*(\hat{p}_{a_\kappa}) M_*\big(R_{g_{i_\kappa}^{-1}}(l)\big)\big),$$

where $p^{-1}(Gx)/G = \{[a_\kappa] \mid \kappa \in K\}$ and $p(a_\kappa) = g_{i_\kappa} x$.

Definition 5.5. A Mackey functor M is said to be *homological* if whenever $K \subset H \subset G$ and $q: G/H \to G/K$ is the quotient function, one has $M_*(q)M^*(q) = [H{:}K]$, that is, multiplication by the index of K in H.

Remark 5.6. Thévenaz and Webb [16] call such a Mackey functor a *cohomological* Mackey functor, since they consider as the transfer the covariant part, contrary to what we do.

Coming back to the formula of Proposition 5.4, if M is a homological Mackey functor, then the composite $M_*(\hat{p}_{a_\kappa}) M^*(\hat{p}_{a_\kappa})$ is multiplication by the index $[G_{p(a_\kappa)}{:}G_{a_\kappa}]$. Since $\gamma_{p(a_\kappa)}^G(M_*(R_{g_{i_\kappa}^{-1}}(l))) = \gamma_x^G(l)$, by 3.16 we have the next.

Theorem 5.7. *Let M be a Mackey functor. Then the formula*

$$p_*^G t_p^G\big(\gamma_x^G(l)\big) = \sum_{\kappa \in K} \big[G_{p(a_\kappa)}{:}G_{a_\kappa}\big] \gamma_x^G(l),$$

where $p^{-1}(Gx)/G = \{[a_\kappa] \mid \kappa \in K\}$ and $p(a_\kappa) = g_{i_\kappa} x$, holds for every n-fold G-equivariant covering map $p: E \to X$ ($n \geqslant 0$) if and only if M is homological.

Assume now that our base ring R is a field k. Thévenaz and Webb [16] prove that a Mackey functor M for G with values in k-vector spaces, such that the field k has characteristic zero or prime to $|G|$, is homological if and only if M is the fixed point Mackey functor M_L of some kG-module L. That is $M(G/H) = L^H$. Using this and 5.7, we have the following.

Theorem 5.8. *Let M be a Mackey functor for G with values in k-vector spaces, such that the field k has characteristic zero or prime to $|G|$. Then the formula*

$$p_*^G t_p^G\big(\gamma_x^G(l)\big) = \sum_{\kappa\in K}[G_{p(a_\kappa)}:G_{a_\kappa}]\gamma_x^G(l),$$

where $p^{-1}(Gx)/G = \{[a_\kappa] \mid \kappa \in K\}$ and $p(a_\kappa) = g_{i_\kappa}x$, holds for any n-fold G-equivariant covering map $p: E \to X$ if and only if M is the fixed point Mackey functor M_L of some kG-module L.

In this case, the theory $\mathbb{H}^G(-; M_L) = \mathbb{H}^G(-; \bar{L})$ was studied in [1].

To finish, we have the following immediate consequence of Theorem 5.7.

Corollary 5.9. *Let $p: E \to X$ be n-fold G-equivariant covering map such that G acts freely on X, and let M be a homological Mackey functor. Then the composites*

$$p_*^G \circ t_p^G : F^G(X^+, M) \longrightarrow F^G(X^+, M),$$
$$p_*^G \circ \tau_p^G : \mathbb{H}_*^G(X, M) \longrightarrow \mathbb{H}_*^G(X, M)$$

are multiplication by n.

Note added in proof

The referee brought to our attention the recent paper [13] of Z. Nie, where a result similar to Theorem 4.1 was obtained using different methods.

Acknowledgement

We wish to thank the referee for his careful reading and his useful suggestions.

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