The transfer for ramified covering G-maps

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Abstract Let G be a finite group. The main objective of this paper is to study ramified covering G-maps and to construct a transfer for them in Bredon-Illman equivariant homology with coefficients in a homological Mackey functor M. We show that this transfer has the usual properties of a transfer.

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Contents

0 INTRODUCTION

Let G be a finite group. The main objective of this paper is to study ramified covering G-maps and to construct a transfer for them in Bredon-Illman equivariant homology with coefficients in a homological Mackey functor M. We show that this transfer has many of the properties of other known transfers. Notice that in order to have the property that the composite of the transfer with the

*Corresponding author, Phone: ++5255-56224489, Fax ++5255-56160348. This author was partially supported by PAPIIT grant IN105106-3 and by CONACYT grant 58049. Webpage: http://www.matem.unam.mx/cprieto projection is multiplication by the multiplicity of the ramified covering map, M must be a homological Mackey functor. The transfer for any ramified covering G-map will be given by a homomorphism between certain topological abelian groups. It cannot be given by a stable transfer map (see Remark 5.3).

To construct the transfer, we shall use the homotopical definition of Bredon-Illman homology $H_*^G(-;L)$ given in [4], when L is a G-module, and of $H_*^G(X;M)$ given in [5] when M is a homological Mackey functor. Namely, to each pointed G-space X, a G-module L and a homological Mackey functor M for G, we associate topological abelian groups $F^G(X,L)$ and $\mathbb{F}^G(X,M)$ such that $\pi_q(F^G(X,L)) \cong \widetilde{H}^G(X;L)$ and $\pi_q(\mathbb{F}^G(X,M)) \cong \widetilde{H}^G(X;M)$. The topology in $F^G(X,L)$ is a generalization of the usual topology of the infinite symmetric product $\mathrm{SP}^\infty X$. The topology in $\mathbb{F}^G(X,M)$ is defined using the singular simplicial set $\mathcal{S}(X)$ associated to X. With these topologies, the homomorphisms induced by any pointed G-map $f: X \longrightarrow Y$ turn out to be continuous. For a ramified covering G-map $p: E \longrightarrow X$ we define transfer homomorphisms $t_p^G : F^G(X,L) \longrightarrow F^G(E,L)$ and $t_p^G : \mathbb{F}^G(X,M) \longrightarrow \mathbb{F}^G(E,M)$. The first one is always continuous. When we take coefficients in a homological Mackey functor, we prove that the transfer is continuous provided that the spaces involved are strong ρ-spaces. The class of strong ρ-spaces contains all simplicial G-complexes (Proposition 4.9), as well as the class of G-ENRs (Proposition 4.11) and the class of G-CW-complexes, that are either locally compact, countable and finite-dimensional (Proposition 4.12) or regular (Proposition 4.13).

This approach to the transfer was already used by the authors in [1] in the nonequivariant case for singular homology.

The paper is organized as follows. In Section 1, we define a transfer t_p^G : $F^G(C,M) \longrightarrow F^G(A,M)$ for certain finite-to-one G-functions $p: A \longrightarrow C$ between G-sets, which we call n-fold G-functions with multiplicity (1.1). The reader should think of them as discrete ramified covering G-maps. We show that this transfer has all the usual properties, namely the pullback property (2.15), normalization (2.17), additivity (2.20), and that its composite with p_*^G is multiplication by n (2.24). In Section 3, we define the concept of a ramified covering G-map $p: E \longrightarrow X$. This is an n-fold G-function with multiplicity and some topological properties. This generalizes to the equivariant case the definition in [11]. For any G-module L, using the topology on $F^G(X,L)$ described in [4], we prove that the transfer constructed in the previous section is continuous for any p (3.6). In Section 4, using the continuity of the transfer in the case of coefficients in a G-module L, we prove the continuity of the transfer with coefficients in a homological Mackey functor M, provided that E and X are strong ρ-spaces (4.7).

Finally, in Section 5, we pass to Bredon-Illman homology (applying the homotopy-group functors) and give the transfer and its properties in homology.

1 The transfer in the category of G-sets

In this section we shall define the transfer for a certain family of G-functions.

Definition 1.1 By an *n-fold G-function with multiplicity* we understand a G-function $p : A \longrightarrow C$ between G-sets with finite fibers, together with a G-invariant function $\mu : A \longrightarrow \mathbb{N}$, called *multiplicity function*, such that for each $x \in C$,

$$\sum_{a \in p^{-1}(x)} \mu(a) = n\,.$$

We say that the n-fold G-function with multiplicity $p : A \longrightarrow C$ is *pointed* if the sets A and C have base points, which are fixed under the G-action, and p is a pointed function.

Definition 1.2 Given a G-function $p : A \longrightarrow C$ with multiplicity $\mu : A \longrightarrow \mathbb{N}$, one may define the the G-function

$$\varphi_p : C \longrightarrow \mathrm{SP}^n A$$

by

$$\varphi_p(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu(a_r)} \rangle\,.$$

This function will play an important role in Section 3.

Example 1.3 Let C be a G-set and consider the G-function $\pi : C^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n C$ given by $\pi\langle x_1, \ldots, x_n; i\rangle = \langle x_1, \ldots, x_n\rangle$, where G acts diagonally on C^n and on $\mathrm{SP}^n C$, and trivially on the set $\overline{n} = \{1, 2, \ldots, n\}$. Define $\mu : C^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N}$ by

$$\mu\langle x; i\rangle = \# x^{-1}(x(i))\,,$$

where one regards x as a function $\overline{n} \longrightarrow C$. Then p is an n-fold G-function with multiplicity, since the sets $x^{-1}x(i)$ form a partition of the set $\overline{n}$. Furthermore, μ is clearly G-invariant. The function $\varphi_\pi : \mathrm{SP}^n C \longrightarrow \mathrm{SP}^n(C^n \times_{\Sigma_n} \overline{n})$ is given in this case by

$$\varphi_\pi\langle x_1, \ldots, x_n\rangle = \langle\langle x_1, \ldots, x_n; 1\rangle, \ldots, \langle x_1, \ldots, x_n; n\rangle\rangle\,.$$

Remark 1.4 We can always assume that an n-fold G-function with multiplicity is pointed by adding isolated points $*$ to A and to C which remain fixed under the G-action and by defining $\mu(*) = n$. Therefore we shall always consider pointed n-fold G-functions with multiplicity without saying it explicitly.

We now recall the definition of the groups $F(D, M)$ and $F^G(D, M)$ for a pointed G-set D and a Mackey functor M for the group G. First recall that a *Mackey functor* consists of two functors, one covariant and one contravariant, both with the same object function $M : G\text{-}\mathcal{S}\mathrm{et}_{\mathrm{fin}} \longrightarrow \mathcal{A}\mathrm{b}$. If $\alpha : S \longrightarrow T$ is a G-function between finite G-sets, we denote the covariant part in morphisms by $M_*(\alpha) : M(S) \longrightarrow M(T)$ and the contravariant part by $M^*(\alpha) : M(T) \longrightarrow M(S)$. The functor has to be additive in the sense that the two embeddings $S \hookrightarrow S \sqcup T \hookleftarrow T$ into the disjoint union of G-sets define an isomorphism $M(S \sqcup T) \cong M(S) \oplus M(T)$ and if one has a pullback diagram of G-sets

(1.5)
$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\beta}} & S \\ {\scriptstyle \widetilde{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ T & \xrightarrow[\beta]{} & V \end{array},$$

then

(1.6)
$$M_*(\widetilde{\beta}) \circ M^*(\widetilde{\alpha}) = M^*(\alpha) \circ M_*(\beta)\,.$$

By the additivity property, the Mackey functor M is determined by its restriction $M : \mathcal{O}(G) \longrightarrow \mathcal{A}\mathrm{b}$, where $\mathcal{O}(G)$ is the full subcategory of G-orbits G/H, $H \subset G$. A particular role will be played by the G-function $R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$, given by *right translation* by $g^{-1} \in G$, namely

$$R_{g^{-1}}(g'H) = g'Hg^{-1} = g'g^{-1}(gHg^{-1})\,.$$

We shall often denote the coset gH by $[g]_H$ or simply by $[g]$, if there is no danger of confusion. Observe that if C is a G-set and $x \in C$, then the canonical bijection $G/G_x \longrightarrow G/G_{gx}$ is precisely $R_{g^{-1}}$, where as usual G_x denotes the *isotropy subgroup* of x, namely the maximal subgroup of G that leaves x fixed.

Consider the set $\widehat{M} = \cup_{H \subset G} M(G/H)$. Then $F(D, M)$ consists of functions $u : D \longrightarrow \widehat{M}$ such that $u(y) \in M(G/G_y)$, $u(*) = 0$, and $u(y) = 0$ for all but a finite number of elements $y \in D$. The canonical generators of this group are functions denoted by ly given by

$$(ly)(y') = \begin{cases} l & \text{if } y' = y\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $l \in M(G/G_y)$ and $y \in D - \{*\}$. The group $F(D,M)$ has a natural (left) action of G given by $(gu)(y) = M_*(R_{g^{-1}})(u(g^{-1}y))$. Define $F^G(D,M)$ as the subgroup of the fixed points of $F(C,M)$ under this G-action. The canonical generators of $F^G(D,M)$ are functions denoted by $\gamma_y^G(l)$ given by

$$\gamma_y^G(l) = \sum_{j=1}^{m} M_*(R_{g_j^{-1}})(l)(g_j y)\,,$$

where $l \in M(G/G_y)$, $y \in D - \{*\}$, and $G/G_y = \{[g_j] \mid j = 1, \ldots, m\}$. Given a pointed G-function $f : C \longrightarrow D$, the homomorphism $f_*^G : F^G(C,M) \longrightarrow F^G(D,M)$ is given on the generators by

$$f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G M_*(\widehat{f}_x)(l)\,,$$

where $\widehat{f}_x : G/G_x \longrightarrow G/G_{f(x)}$ is the canonical quotient function (see [3, 5] for details).

Definition 1.7 Let $p : A \longrightarrow C$ be a n-fold G-function with multiplicity μ, and let M be a Mackey functor. Define a homomorphism

$$t_p : F(C,M) \longrightarrow F(A,M)\,,$$

by

$$t_p(u)(a) = \mu(a) M^*(\widehat{p}_a) u(p(a))\,,$$

where $u \in F(C,M)$ and $a \in A$. If we assume that $u \in F^G(C,M)$, i.e., that $u(gx) = M_*(R_{g^{-1}})(u(x))$, then

$$\begin{aligned} t_p(u)(ga) &= \mu(ga) M^*(\widehat{p}_{ga})(u(p(ga))) \\ &= \mu(a) M^*(\widehat{p}_{ga}) M_*(R_{g^{-1}})(u(p(a))) \\ &= \mu(a) M_*(R_{g^{-1}}) M^*(\widehat{p}_a)(u(p(a))) \\ &= M_*(R_{g^{-1}})(t_p(u)(a))\,, \end{aligned}$$

where the next to the last equality follows from the pullback property of the Mackey functor. Thus $t_p(u) \in F^G(A,M)$. Therefore, the homomorphism t_p restricts to a *transfer* homomorphism

$$t_p^G : F^G(C,M) \longrightarrow F^G(A,M)\,.$$

Remark 1.8 Let $p : A \longrightarrow C$ be a n-fold G-function with multiplicity μ. The isotropy group G_x acts on $p^{-1}(x)$ and the inclusion $p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$. Let $\{a_\iota\} \subset p^{-1}(x)$ be a set of representatives one for each G_x-orbit. Let $\gamma_x^G(l)$ be a generator of $F^G(C,M)$. Since the value of this function is zero on points which do not belong to the orbit Gx, and $\gamma_x^G(l)(x) = l$. One can give the transfer t_p^G on the generators $\gamma_x^G(l)$ by the formula

$$t_p^G(\gamma_x^G(l)) = \sum_{[a_\iota] \in p^{-1}(x)/G_x} \mu(a_\iota)\gamma_{a_\iota}^G(M^*(\widehat{p}_{a_\iota})(l)) \,. \tag{1.9}$$

2 Properties of the transfer

In this section we shall give some properties of the transfer that do not depend on the topology. We start with a definition.

Definition 2.1 Let $p : A \longrightarrow C$ and $p' : A' \longrightarrow C'$ be n-fold G-functions with multiplicity functions μ and μ', respectively. A *morphism* from p to p' is a pair of G-functions $(\widetilde{f}, f)$ such that

(a) the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{f}} & A' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ C & \xrightarrow[f]{} & C' \,, \end{array}$$

(b) for each $x \in C$, the restriction $\widetilde{f}|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$ is surjective,

(c) for each $x \in C$ and $a' \in p'^{-1}(f(x))$, one has the equality

$$\mu'(a') = \sum_{p(a)=x\,,\ \widetilde{f}(a)=a'} \mu(a)\,, \quad \text{and} \tag{2.2}$$

(d) for each $a \in A$ one has the formula

$$G_a = G_{p(a)} \cap G_{\widetilde{f}(a)} \,. \tag{2.3}$$

We have the next useful characterization of a morphism of ramified covering G-maps.

Proposition 2.4 *Let $p : A \longrightarrow C$ and $p' : A' \longrightarrow C'$ be n-fold G-functions with multiplicity, and let $f : C \longrightarrow C'$ and $\widetilde{f} : A \longrightarrow A'$ be G-functions such that $p' \circ \widetilde{f} = f \circ p$ and for $a \in A$, $G_a = G_{p(a)} \cap G_{\widetilde{f}(a)}$. Then $(\widetilde{f}, f)$ is a morphism from p to p' if and only if*

$$\varphi_{p'} \circ f = \mathrm{SP}^n \widetilde{f} \circ \varphi_p : C \longrightarrow \mathrm{SP}^n A' \,,$$

where φ_p and $\varphi_{p'}$ are as defined in 1.2.

Proof: Assume that $\varphi_{p'} \circ f = \mathrm{SP}^n \widetilde{f} \circ \varphi_p$. This clearly implies that $\widetilde{f}$ is surjective on fibers. Take $x \in C$ and let $\{a'_1, \ldots, a'_{k'}\} = p'^{-1}(f(x))$. Moreover, let $\{a_{1\,1}, \ldots, a_{1\,r_1}\} = \widetilde{f}^{-1}(a'_1)$,...,$\{a_{k'\,1}, \ldots, a_{k'\,r_{k'}}\} = \widetilde{f}^{-1}(a'_{k'})$. Hence $p^{-1}(x) = \{a_{1\,1}, \ldots, a_{1\,r_1}, \ldots, a_{k'\,1}, \ldots, a_{k'\,r_{k'}}\}$. Writing down $\mathrm{SP}^n \widetilde{f}(\varphi_p(x))$ using this description of $p^{-1}(x)$, which is equal to $\varphi_{p'}(f(x))$, one easily obtains the equality (2.2).

Assuming now (b) and (c) in the definition and using the same labels for the elements of the fibers as in the first part, one obtains the desired equality. ■

Examples 2.5 There are two interesting examples of morphisms between G-functions with multiplicity:

(a) Let $p : A \longrightarrow C$ be a n-fold G-function with multiplicity μ, and let $f : D \longrightarrow C$ be a G-function. Consider the pullback diagram

(2.6)
$$\begin{array}{ccc} f^*A & \xrightarrow{\widetilde{f}} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ D & \xrightarrow[f]{} & C\,, \end{array}$$

where $f^*A = D \times_C A = \{(y, a) \mid f(y) = p(a)\}$. Clearly, q is also an n-fold G-function with multiplicity μ' given by $\mu'(y, a) = \mu(a)$, since $\mu'(g(y, a)) = \mu'(gy, ga) = \mu(ga) = \mu(a) = \mu'(y, a)$. Consider the restriction of $\widetilde{f}$ from the fiber $q^{-1}(y)$ to the fiber $p^{-1}(f(y))$. This function induces a surjective function

$$q^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}\,.$$

Clearly, conditions (a), (b), and (c) in the previous definition hold. Furthermore $G_{(y,a)} = G_y \cap G_a$, thus condition (d) also holds. Hence $(\widetilde{f}, f)$ is a morphism from q to p.

(b) Let C and D be G-sets and let $f : C \longrightarrow D$ be G-equivariant. We say that f is n*-permutable* if the equality

(2.7)
$$G_{\langle x_1,\ldots,x_n;i\rangle} = G_{\langle x_1,\ldots,x_n\rangle} \cap G_{\langle f(x_1),\ldots,f(x_n);i\rangle}$$

holds in terms of isotropy groups, where $\langle x_1, \ldots, x_n; i\rangle \in C^n \times_{\Sigma_n} \overline{n}$, $\langle x_1, \ldots, x_n\rangle \in \mathrm{SP}^n C$, and $\langle f(x_1), \ldots, f(x_n); i\rangle \in D^n \times_{\Sigma_n} \overline{n}$. If $\pi : C^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n C$ and $\pi' : D^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n D$ are as in 1.3, then the pair of G-functions $(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}}, \mathrm{SP}^n f)$ is a morphism from π to π'. To see this, we use Proposition 2.4 above. Namely, we have to show that

$$\varphi_{\pi'} \circ \mathrm{SP}^n f = \mathrm{SP}^n(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}}) \circ \varphi_\pi : \mathrm{SP}^n C \longrightarrow \mathrm{SP}^n(D^n \times_{\Sigma_n} \overline{n})\,.$$

For any $\langle y_1, \dots, y_n \rangle \in \mathrm{SP}^n D$, we have,

$$\varphi_{\pi'}\langle y_1, \dots, y_n \rangle = \langle \langle y_1, \dots, y_n; 1 \rangle, \dots, \langle y_1, \dots, y_n; n \rangle \rangle\,,$$

thus, if we take $\langle x_1, \dots, x_n \rangle \in \mathrm{SP}^n C$, we have

$$\varphi_{\pi'}(\mathrm{SP}^n f \langle x_1, \dots, x_n \rangle) = \langle \langle f(x_1), \dots, f(x_n); 1 \rangle, \dots, \langle f(x_1), \dots, f(x_n); n \rangle \rangle\,.$$

On the other hand, we have

$$\varphi_{\pi}\langle x_1, \dots, x_n \rangle = \langle \langle x_1, \dots, x_n; 1 \rangle, \dots, \langle x_1, \dots, x_n; n \rangle \rangle$$

and so

$$\mathrm{SP}^n(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}})(\varphi_{\pi}\langle x_1, \dots, x_n \rangle) = \langle \langle f(x_1), \dots, f(x_n); 1 \rangle, \dots, \langle f(x_1), \dots, f(x_n); n \rangle \rangle\,.$$

Thus both are equal.

The following is the **naturality property** of the transfer.

Proposition 2.8 *Let* $(\widetilde{f}, f)$ *be a morphism from* $p : A \longrightarrow C$ *to* $p' : A' \longrightarrow C'$. *Then the following diagram commutes:*

$$\begin{array}{ccc}
F^G(C, M) & \xrightarrow{f_*^G} & F^G(C', M) \\
{\scriptstyle t_p^G}\big\downarrow & & \big\downarrow{\scriptstyle t_{p'}^G} \\
F^G(A, M) & \xrightarrow[\widetilde{f}_*^G]{} & F^G(A', M)\,.
\end{array}$$

Proof: First note that the function $\widetilde{f}$ induces a surjection

$$p^{-1}(x)/G_x \twoheadrightarrow q^{-1}(f(x))/G_{f(x)}\,,$$

This surjection can be written as the composite

$$p^{-1}(x)/G_x \overset{[\widetilde{f}]}{\twoheadrightarrow} p'^{-1}(f(x))/G_x \overset{q}{\twoheadrightarrow} p'^{-1}(f(x))/G_{f(x)}\,.$$

This allows us to write the elements of these quotient sets as follows. Let

$$p'^{-1}(f(x))/G_{f(x)} = \{[a'_j] \mid j = 1, \dots s\}\,.$$

By [5, Lemma (6.6)], one can write

$$p'^{-1}(f(x))/G_x \cong \sqcup_{j=1}^{s} G_x \backslash G_{f(x)}/G_{a'_j}\,,$$

where $G_x \backslash G_{f(x)}/G_{a'_j} = \{[g_j^\nu] \mid \nu = 1, \dots, s_j\}$ and

$$p'^{-1}(f(x))/G_x = \{[g_j^\nu a'_j] \mid \nu = 1, \dots, s_j\,,\ j = 1, \dots s\}\,.$$

Therefore,

$$p^{-1}(x)/G_x = \{[a_\alpha^{\nu j}] \mid \alpha = 1, \ldots, r_j^\nu\,,\ \nu = 1, \ldots, s_j\,,\ j = 1, \ldots s\}\,,$$

where the elements $a_\alpha^{\nu j} \in p^{-1}(x) \cap \widetilde{f}^{-1}(g_j^\nu a'_j)$ are such that

$$\{[a_\alpha^{\nu j}] \mid \alpha = 1, \ldots, r_j^\nu\} = [\widetilde{f}]^{-1}([g_j^\nu a'_j])\,.$$

Hence we have

$$\widetilde{f}^G_* t^G_p(\gamma^G_x(l)) = \sum_{j,\nu,\alpha=1,1,1}^{s,s_j,r_j^\nu} \mu(a_\alpha^{\nu j})\gamma^G_{g_j^\nu a'_j} M_*(\widehat{\widetilde{f}}_{a_\alpha^{\nu j}}) M^*(\widehat{p}_{a_\alpha^{\nu j}})(l)\,. \tag{2.9}$$

Since $\gamma^G_{g_j^\nu a'_j} = \gamma^G_{a'_j} \circ M_*(R_{g_j^\nu})$, we can rewrite (2.9) as

$$\widetilde{f}^G_* t^G_p(\gamma^G_x(l)) = \sum_{j,\nu,\alpha=1,1,1}^{s,s_j,r_j^\nu} \mu(a_\alpha^{\nu j})\gamma^G_{a'_j} M_*(R_{g_j^\nu}) M_*(\widehat{\widetilde{f}}_{a_\alpha^{\nu j}}) M^*(\widehat{p}_{a_\alpha^{\nu j}})(l)\,. \tag{2.10}$$

On the other hand, we have

$$t^G_{p'} f^G_*(\gamma^G_x(l)) = \sum_{j=1}^{s} \mu'(a'_j)\gamma^G_{a'_j} M^*(\widehat{p'}_{a'_j}) M_*(\widehat{f}_x)(l)\,. \tag{2.11}$$

In order to compare these two sums, consider the pullback diagram

$$\begin{array}{ccc}
G/G_x \times_{G/G_{f(x)}} G/G_{a'_j} & \xrightarrow{\ \tau\ } & G/G_{a'_j} \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \widehat{p'}_{a'_j}} \\
G/G_x & \xrightarrow[\ \widehat{f}_x\ ]{} & G/G_{f(x)}\,.
\end{array}$$

By [5, Lemma (6.1)], there is a bijection

$$\varphi : \coprod_{\nu=1}^{s^j} G/G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1} \xrightarrow{\approx} G/G_x \times_{G/G_{f(x)}} G/G_{a'_j}\,,$$

where the elements g_j^ν are as above. Set $\varphi_j^\nu = \varphi|_{G/G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1}}$.

By [5, Lemma (6.3)], any element $w \in M(G/G_x \times_{G/G_{\overline{f}(x)}} G/G_{b_j})$ can be written as

$$w = \sum_{\nu=1}^{s^j} M_*(\varphi_j^\nu) M^*(\varphi_j^\nu)(w)\,.$$

Set $\pi_j^\nu = \pi \circ \varphi_j^\nu$ and $\tau_j^\nu = \tau \circ \varphi_j^\nu$. Consequently,

$$M^*(\widehat{p'}_{a'_j}) M_*(\widehat{f}_x)(l) = M_*(\tau) M^*(\pi)(l) = \sum_{\nu=1}^{s^j} M_*(\tau_j^\nu) M^*(\pi_j^\nu)(l)\,.$$

Replacing this in (2.11), we obtain

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j,\nu=1,1}^{s,s_j} \mu'(a'_j)\gamma_{a'_j}^G M_*(\tau_j^\nu)M^*(\pi_j^\nu)(l)\,. \tag{2.12}$$

Let $\rho_j^\nu : G/G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1} \twoheadrightarrow G/g_j^\nu G_{a'_j}(g_j^\nu)^{-1}$ be the quotient function. Since $\tau_j^\nu = R_{g_j^\nu} \circ \rho_j^\nu$, we can rewrite (2.12) as

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j,\nu=1,1}^{s,s_j} \mu'(a'_j)\gamma_{a'_j}^G M_*(R_{g_j^\nu})M_*(\rho_j^\nu)M^*(\pi_j^\nu)(l)\,. \tag{2.13}$$

By (2.3), we have that $G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1} = G_{a_\alpha^{\nu j}}$, and therefore, $\pi_j^\nu = \widehat{p}_{a_\alpha^{\nu j}}$ and $\rho_j^\nu = \widehat{\widetilde{f}}_{a_\alpha^{\nu j}}$. Hence (2.13) becomes

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j,\nu=1,1}^{s,s_j} \mu'(a'_j)\gamma_{a'_j}^G M_*(R_{g_j^\nu})M_*(\widehat{\widetilde{f}}_{a_\alpha^{\nu j}})M^*(\widehat{p}_{a_\alpha^{\nu j}})(l)\,. \tag{2.14}$$

By Definition 2.1 (c) and the G-invariance of μ' we have that

$$\mu'(a'_j) = \mu'(g_j^\nu a'_j) = \sum_{\alpha=1}^{r_j^\nu} \mu(a_\alpha^{\nu j})\,.$$

Replacing this in (2.14), we obtain (2.10). Therefore,

$$\widetilde{f}_*^G \circ t_p^G = t_{p'}^G \circ f_*^G : F^G(C,M) \longrightarrow F^G(A',M)\,.$$

■

By Example 2.5(a), the **pullback property** is now a consequence of the naturality property.

Proposition 2.15 *Let $p : A \longrightarrow C$ be a n-fold G-function with multiplicity μ and let $f : D \longrightarrow C$ be a G-function. Then*

$$t_p^G \circ f_*^G = \widetilde{f}_*^G \circ t_q^G : F^G(D,M) \longrightarrow F^G(A,M)\,,$$

where $\widetilde{f}$ and q are as in the pullback diagram (2.6). ■

From Example 2.5(b), we obtain another consequence of the naturality property as follows.

Proposition 2.16 *Let $f : C \longrightarrow D$ be an n-permutable G-function. Then*

$$(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}})_*^G \circ t_\pi^G = t_{\pi'}^G \circ (\mathrm{SP}^n f)_*^G : F^G(\mathrm{SP}^n C, M) \longrightarrow F^G(D^n \times_{\Sigma_n} \overline{n}, M)\,.$$

■

The **normalization property** is elementary and it is as follows.

Proposition 2.17 *If $p : A = C \longrightarrow C$ is the identity function with multiplicity function μ constant equal to 1, then $t_p^G : F^G(C, M) \longrightarrow F^G(C, M)$ is the identity too.* ■

The **additivity property** is based on the following.

Proposition 2.18 *Let C_α, $\alpha \in \mathcal{A}$, be a family of pointed G-sets. Then there is an isomorphism of abelian groups*

$$F^G(\bigvee_{\alpha\in\mathcal{A}} C_\alpha, M) \cong \bigoplus_{\alpha\in\mathcal{A}} F^G(C_\alpha, M)\,.$$

Proof: Let $i_\alpha : C_\alpha \longrightarrow \bigvee C_\alpha$ be the inclusion into the wedge of the pointed sets C_α. By the universal property of the direct sum, the homomorphisms $i_{\alpha *}^G$ induce a homomorphism $\varphi : \bigoplus F^G(C_\alpha, M) \longrightarrow F^G(\bigvee C_\alpha, M)$.

Take now $x \in \bigvee C_\alpha$ and let $x_\alpha \in C_\alpha$ be such that $i_\alpha(x_\alpha) = x$. Consider the family of homomorphisms $\iota_\alpha \circ \gamma_{x_\alpha}^G : M(G/G_x) = M(G/G_{x_\alpha}) \longrightarrow \bigoplus F^G(C_\alpha, M)$, where ι_α is the canonical monomorphism into the direct sum. Then by the universal property of $F^G(\bigvee C_\alpha, M)$ (see [3]), there is a unique homomorphism $\psi : F^G(\bigvee C_\alpha, M) \longrightarrow \bigoplus F^G(C_\alpha, M)$, such that $\psi \circ \gamma_x^G = \iota_\alpha \circ \gamma_{x_\alpha}^G$. Checking on generators of the form $\gamma_x^G(l)$ and $\iota_\alpha \gamma_{x_\alpha}^G(l)$, one easily verifies that the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity. ■

Definition 2.19 Let $p_\alpha : A_\alpha \longrightarrow C$, $\alpha = 1, \ldots, r$, be a family of n_α-fold G-functions with multiplicity functions $\mu_\alpha : A_\alpha \longrightarrow \mathbb{N}$. Define $p : A = \bigvee_{\alpha=1}^r A_\alpha \longrightarrow C$ by $p|_{C_\alpha} = p_\alpha$. If $n = \sum_{\alpha=1}^r n_\alpha$, then clearly p is an n-fold G-function with multiplicity function $\mu : A \longrightarrow \mathbb{N}$ given by

$$\mu(a) = \begin{cases} \mu_\alpha(a) & \text{if } a \in A_\alpha - \{*_\alpha\}\,, \\ \sum_{\alpha=1}^r \mu_\alpha(*_\alpha) & \text{if } a = *\,, \end{cases}$$

where $* \in A$ and $*_\alpha \in A_\alpha$ denote the corresponding base points. We call p the *sum* of the p_αs, and we denote it by $\sum_{\alpha=1}^r p_\alpha$.

The transfer has the following additivity property.

Proposition 2.20 *If $p = \sum_{\alpha=1}^{r} p_\alpha : A = \bigvee_{\alpha=1}^{r} A_\alpha \longrightarrow C$, then $t^G_{\Sigma p_\alpha} = \sum t^G_{p_\alpha}$. More precisely, the following diagram commutes:*

$$\begin{array}{ccc} F^G(C,M) & \xrightarrow{t^G_p} & F^G(A,M) \\ & {\scriptstyle (t^G_{p_\alpha})}\searrow & \uparrow {\scriptstyle \cong\ (i^G_{\alpha *})} \\ & & \bigoplus_{\alpha=1}^{r} F^G(A_\alpha, M) \end{array} \quad ,$$

where the isomorphism is as given in 2.18.

Proof: Notice first that for any $x \in C$ different from the base point, $p^{-1}(x) = \sqcup_{\alpha=1}^{r} p_\alpha^{-1}(x)$. Since each $p_\alpha^{-1}(x)$ is G_x-invariant, then $p^{-1}(x)/G_x = \sqcup_{\alpha=1}^{r} p_\alpha^{-1}(x)/G_x$. Let $\gamma_x^G(l) \in F^G(C,M)$ be a generator. Then

$$\begin{aligned} t^G_p(\gamma^G_x(l)) &= \sum_{[a]\in p^{-1}(x)/G_x} \gamma^G_a M^*(\widehat{p}_a)(l) \\ &= \sum_{\alpha=1}^{r} i^G_{\alpha *}\left(\sum_{[a]\in p_\alpha^{-1}(x)/G_x} \gamma^G_a M^*(\widehat{p_{\alpha a}})(l) \right) \\ &= \sum_{\alpha=1}^{r} i^G_{\alpha *} t^G_{p_\alpha}(\gamma^G_x(l)) \end{aligned}$$

■

An immediate consequence of the normalization property 2.17 and the additivity property 2.20 is the **quasiadditivity property**, namely the following.

Proposition 2.21 *Let $p : A \longrightarrow C$ be a G-function with multiplicity, and let $q : A \vee C \longrightarrow C$ be given by $q|_A = p$ and $q|_C = \mathrm{id}_C$ with the corresponding multiplicity function μ'. Then the next is a commutative diagram:*

$$\begin{array}{ccc} F^G(C,M) & \xrightarrow{t^G_q} & F^G(A\vee C,M) \\ & {\scriptstyle (t^G_p,1)}\searrow & \uparrow {\scriptstyle \cong} \\ & & F^G(A,M) \oplus F^G(C,M)\,, \end{array}$$

where the isomorphism is as given in 2.18. ■

To show that the transfer has a **functoriality property**, we need the following.

Proposition 2.22 *Let $q : A' \longrightarrow A$ be an n'-fold G-function with multiplicity function μ', and let $p : A \longrightarrow C$ be an n-fold G-function with multiplicity function μ. If one defines $\nu : A' \longrightarrow \mathbb{N}$ by*

$$\nu(a') = \mu'(a')\mu(q(a))\,,$$

then the composite $p\circ q : A' \longrightarrow C$ is an (nn')-fold G-function with multiplicity function ν.

Proof: We only have to compute the sum

$$\begin{aligned}\sum_{a'\in p\circ q^{-1}(x)} \nu(a') &= \sum_{a\in p^{-1}(x)} \sum_{a'\in q^{-1}(a)} \mu'(a')\mu(a) \\ &= \sum_{a\in p^{-1}(x)} \mu(a) \sum_{a'\in q^{-1}(a)} \mu'(a') \\ &= nn'\,.\end{aligned}$$

■

The functoriality property is the following.

Proposition 2.23 *Let $q : A' \longrightarrow A$ be an n'-fold G-function with multiplicity function μ', and let $p : A \longrightarrow C$ be an n-fold G-function with multiplicity function μ. Then*

$$t^G_{p\circ q} = t^G_q \circ t^G_p : F^G(C, M) \longrightarrow F^G(A', M)\,.$$

Proof: Take $u \in F^G(C, M)$. By definition of the transfer, we have

$$\begin{aligned}t^G_q(t^G_p(u))(a') &= \mu'(a')M^*(\widehat{q}_{a'})(t^G_p(u)(q(a'))) \\ &= \mu'(a')M^*(\widehat{q}_{a'})\Big(\mu(q(a'))M^*(\widehat{p}_{q(a')})\big(u(p(q(a')))\big)\Big) \\ &= \nu(a')M^*(\widehat{p\circ q}_{a'})(u((p\circ q)(a'))) \\ &= t^G_{p\circ q}(u)(a)\,.\end{aligned}$$

■

Recall that a Mackey functor M for G is said to be *homological* if whenever $H \subset K \subset G$ and $q : G/H \longrightarrow G/K$ is the quotient function, then

$$M_*(q)M^*(q) = [K\,{:}\,H]\,,$$

that is, this composite is multiplication by the index of H in K in the group $M(G/K)$. We have the following result.

Proposition 2.24 *Let $p : A \longrightarrow C$ be a n-fold G-function with multiplicity μ and let M be a homological Mackey functor for G. Then the composite*

$$p_*^G \circ t_p^G : F^G(C, M) \longrightarrow F^G(C, M)$$

is multiplication by n.

Proof: Let $\gamma_x(l) \in F^G(C, M)$ be a generator, and let $\{[a_\iota] \mid \iota \in \mathfrak{I}\} = p^{-1}(x)/G_x$. Since M is homological, $M_*(\widehat{p}_{a_\iota})M^*(\widehat{p}_{a_\iota})$ is multiplication by the index $[G_x : G_{a_\iota}]$. Moreover, the orbit of $a_\iota \in p^{-1}(x)$ under the action G_x has exactly $[G_x : G_{a_\iota}]$ elements. Since the multiplicity function μ is G-invariant, we have that $\sum_{\iota \in \mathfrak{I}} \mu(a_\iota)[G_x : G_{a_\iota}] = n$. Hence

$$\begin{aligned} p_*^G t_p^G(\gamma_x^G(l)) &= p_*^G \left(\sum_{\iota \in \mathfrak{I}} \mu(a_\iota) \gamma_{a_\iota}^G M^*(\widehat{p}_{a_\iota})(l) \right) \\ &= \sum_{\iota \in \mathfrak{I}} \mu(a_\iota) \gamma_x^G M_*(\widehat{p}_{a_\iota}) M^*(\widehat{p}_{a_\iota})(l) \\ &= \left(\sum_{\iota \in \mathfrak{I}} \mu(a_\iota)[G_x : G_{a_\iota}] \right) \gamma_x^G(l) = n\gamma_x^G(l) \,. \end{aligned}$$

■

3 The transfer for coefficients in a G-module

In this section we shall define the concept of a ramified covering G-map and study its transfer in the topological abelian groups $F^G(X, L)$ with coefficients in a G-module L. We shall work here in the category of k-spaces. We understand by a k-*space* a topological space X with the property that a set $C \subset X$ is closed if and only if $f^{-1}C \subset K$ is closed for any continuous map $f : K \longrightarrow X$, where K is any compact Hausdorff space (see [13]). There is a functor that associates to every topological space X a k-space $k(X)$ with the same underlying set and a finer topology defined as before. Thus the identity $k(X) \longrightarrow X$ is continuous and a weak homotopy equivalence. Instead of the usual topological product, we shall take its image under the functor k; we shall use the same notation $\times$ for it. This category has two useful properties([13]):

1. If X is a k-space and $p : X \longrightarrow X'$ is an identification, then X' is a k-space; and
2. if $p : X \longrightarrow X'$ and $q : Y \longrightarrow Y'$ are identifications between k-spaces, then $p \times q : X \times Y \longrightarrow X' \times Y'$ is an identification.

The next definition, puts in the the topological setting the concept of an n-fold G-function with multiplicity.

Definition 3.1 Let E and X be G-spaces. An *n-fold ramified covering G-map* is a continuous G-map $p : E \longrightarrow X$ together with a *multiplicity function* $\mu : E \longrightarrow N$, such that the following hold:

(i) The fibers $p^{-1}(x)$ are finite for each $x \in X$.

(ii) For each $x \in X$, $\sum_{a \in p^{-1}(x)} \mu(a) = n$.

(ii) The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E = E^n/\Sigma_n$, given by

$$\varphi_p(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu(a_1)}, \ldots, \underbrace{a_m, \ldots, a_m}_{\mu(a_m)} \rangle ,$$

where $p^{-1}(x) = \{a_1, \ldots, a_m\}$, is continuous.

(iv) μ is G-invariant.

Notice that by (iv), the map φ_p is G-equivariant. We can always assume that the ramified covering G-map is pointed (see Remark 1.4). This definition in the nonequivariant case was given by Smith [11] and it includes ordinary covering maps with finitely many leaves.

Proposition 3.2 *The family of ramified covering G-maps has the following properties:*

(a) *If $p_\alpha : E_\alpha \longrightarrow X$, $\alpha = 1, \ldots, k$, are ramified covering G-maps with multiplicity functions μ_α, then*

$$p : \bigvee_{\alpha=1}^{k} E_\alpha \longrightarrow X$$

given by $p|_{E_\alpha} = p_\alpha$ is an $\sum_{\alpha=1}^{k} n_\alpha$-fold ramified covering G-map with multiplicity function μ given by $\mu|_{E_\alpha} = \mu_\alpha$.

(b) *If $q : E' \longrightarrow E$ is an n'-fold ramified covering G-map with multiplicity function μ', and $p : E \longrightarrow X$ is an n-fold ramified covering G-map with multiplicity function μ, then $p \circ q : E' \longrightarrow X$ is an (nn')-fold ramified covering G-map with multiplicity function ν, where $\nu(a') = \mu'(a')\mu(q(a'))$.*

(c) *If X is a G-space, then the projection $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ is an n-fold ramified covering G-map.*

Proof: By 2.19, p is a $(\sum_{\alpha=1}^{k} n_\alpha)$-fold G-function with multiplicity. Thus we only have to prove that $\varphi_p : X \longrightarrow \mathrm{SP}^{\Sigma_{\alpha=1}^{k} n_\alpha}(\bigvee_{\alpha=1}^{k} E_\alpha)$ is continuous. This follows from the commutativity of the next diagram:

$$\begin{array}{ccccc} X & \xrightarrow{(\varphi_{p_\alpha})} & \prod_{\alpha=1}^{k} \mathrm{SP}^{n_\alpha} E_\alpha & \xleftarrow{\;q\;} & \prod_{\alpha=1}^{k} E_\alpha^{n_\alpha} \\ & \searrow^{\varphi_p} & \downarrow & & \downarrow \rho \\ & & \mathrm{SP}^{\Sigma n_\alpha}(\bigvee_{\alpha=1}^{k} E_\alpha) & \longleftarrow & (\coprod_{\alpha=1}^{k} E_\alpha)^{\Sigma n_\alpha}, \end{array}$$

where ρ is the inclusion in the corresponding summand, and q is an identification.

By [2, 4.20], the composite $p \circ q$ in an (nn')-fold ramified covering map. Thus (b) follows from this and 2.22.

By [11], $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ is an n-fold ramified covering map. Thus (c) follows from 1.3. ■

Remark 3.3 Note that in [11], the setting is the category of topological spaces. By [1, Prop. 3.4], its definitions are equivalent to the ones herein, which are given in the setting of k-spaces.

Given a G-module L, one defines a Mackey functor M_L as follows:

$$M_L(G/H) = L^H \, .$$

If $H \subset K$ and $q : G/H \longrightarrow G/K$ is the quotient map, then

$$M_{L*}(q) : L^H \longrightarrow L^K \quad \text{is given by} \quad M_{L*}(q)(l) = \sum_{i=1}^{r} k_i l \, ,$$

where $K/H = \{[k_i] \mid i = 1, \ldots, r\}$. Furthermore,

$$M_L^*(q) : L^K \longrightarrow L^H \quad \text{is the inclusion.}$$

On the other hand, let $R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$ by right translation by g^{-1}. Then

$$M_{L*}(R_{g^{-1}}) : L^H \longrightarrow L^{gHg^{-1}} \quad \text{is given by} \quad M_{L*}(R_{g^{-1}})(l) = gl \, .$$

Moreover,

$$M_L^*(R_{g^{-1}}) : L^{gHg^{-1}} \longrightarrow L^H \quad \text{is given by} \quad M_L^*(R_{g^{-1}})(l) = g^{-1} l \, .$$

We now recall the definition given in [4] of the functor $F^G(-, L) : G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$.

Definition 3.4 Let L be a G-module. If C is a pointed G-set, then $F^G(C,L)$ consists of G-equivariant functions $u : C \longrightarrow L$ such that $u(*) = 0$ and $u(x) = 0$ for all but a finite number of elements $x \in C$. Furthermore, if $f : C \longrightarrow D$ is a pointed G-function, then the induced homomorphism $f_\bullet^G : F^G(C,L) \longrightarrow F^G(D,L)$ is given by $f_\bullet^G(\sum_{x\in C} l_x x) = \sum_{x\in C} l_x f(x)$.

Proposition 3.5 *The functors $F^G(-,L)$ and $F^G(-,M_L)$ are equal.*

Proof: Notice that $\widehat{M}_L = L$. Thus $F^G(C,M_L) \subset F^G(C,L)$. Since an element $u \in F^G(C,L)$ is a G-function, then $u(x) \in L^{G_x} = M_L(G/G_x)$. Hence the abelian groups $F^G(C,L)$ and $F^G(C,M_L)$ are equal.

Let $f : C \longrightarrow D$ be a pointed G-function. To show that $f_\bullet^G = f_*^G$, let $G/G_{f(x)} = \{[g_1],\ldots,[g_m]\}$ and $G_{f(x)}/G_x = \{[h_1],\ldots,[h_r]\}$. Hence $G/G_x = \{[g_ih_j] \mid i = 1,\ldots,m;\ j = 1,\ldots,r\}$. By definition, $M_{L*}(G/H) = L^H$ for all H and $M_{L*}(R_{g^{-1}})(l) = gl$. Since $h_j \in G_{f(x)}$, we have

$$\begin{aligned} f_\bullet^G(\gamma_x^G(l)) &= f_\bullet^G\left(\sum_{i,j} M_{L*}(R_{(g_ih_j)^{-1}})(l)(g_ih_jx)\right)\\ &= f_\bullet^G\left(\sum_{i,j}(g_ih_jl)(g_ih_jx)\right)\\ &= \sum_{i,j}((g_ih_j)l)(g_if(x))\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} f_*^G(\gamma_x^G(l)) &= \gamma_{f(x)}^G M_{L*}(\widehat{f}_x)(l)\\ &= \gamma_{f(x)}^G\left(\sum_{j=1}^r h_jl\right)\\ &= \sum_{j=1}^r\left(\sum_{i=1}^m M_{L*}(R_{g_i^{-1}})(h_jl)(g_if(x))\right)\\ &= \sum_{i,j}(g_i(h_jl))(g_if(x))\,. \end{aligned}$$

Thus the result follows. ∎

Let Y be a pointed G-space. As already remarked, the group $F^G(Y,L)$ has a natural topology that was studied in [4]. This topology is defined as follows.

Consider the group $F(Y,L)$ of functions $u : Y \longrightarrow L$ such that $u(*) = 0$ and $u(y) = 0$ for all but a finite number of elements $y \in Y$. This is filtered by subsets $F_r(Y,L)$ of those functions which are nonzero in at most r points of Y. We give to these sets the quotient topology induced by the map $(L \times Y)^r \twoheadrightarrow F_r(Y,L)$ given by $(l_1, y_1, \ldots, l_r, y_r) \mapsto l_1 y_1 + \cdots + l_r y_r$, where the product $(L \times Y)^r$ has the k-topology. Then $F(Y,L)$ has the union topology and $F^G(Y,L) \subset F(Y,L)$ has the subspace topology. From now on the notation $F^G(Y,L)$ will mean the group with this topology.

We have the following.

Proposition 3.6 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$. Then the transfer $t_p^G : F^G(X,L) \longrightarrow F^G(E,L)$, is continuous.*

Proof: First notice that if $l \in M_L(G/G_a) = L^{G_a} \subset L$, $a \in E$, then $M_L^*(\widehat{p}_a)(l) = l$. Hence we have the homomorphism $t_p : F(X,L) \longrightarrow F(E,L)$ given by

$$t_p(lx) = \sum_{\iota=1}^{m} \mu(a_\iota) l a_\iota \,,$$

where $p^{-1}(x) = \{a_\iota \mid \iota = 1, \ldots, m\}$. The transfer is, as before, the restriction

$$t_p^G = t_p|_{F^G(X,L)} : F^G(X,L) \longrightarrow F^G(E,L)\,.$$

Consider the map $\delta : L \times X \longrightarrow F_n(E,L)$ given by $\delta(l,x) = t_p(lx)$ and $\alpha : L \times X \longrightarrow (L \times E^n)/\Sigma_n$ given by

$$\alpha(l,x) = \langle \underbrace{(l,a_1), \ldots, (l,a_1)}_{\mu(a_1)}, \ldots, \underbrace{(l,a_m), \ldots, (l,a_m)}_{\mu(a_m)} \rangle \,,$$

where $p^{-1}(x) = \{a_1, \ldots, a_m\}$. Let $j_l : E^n/\Sigma_n \longrightarrow (L \times E)^n/\Sigma_n$ be given by $j_l\langle a_1, \ldots, a_n\rangle = \langle (l,a_1), \ldots, (l,a_n)\rangle$. If $i_l : X \longrightarrow L \times X$ is the inclusion at level l, then $\alpha \circ i_l = j_l \circ \varphi_p$. Hence α is continuous. Notice that the identification $(L \times E)^n \longrightarrow F_n(E,L)$ factors as the composite

$$(L \times E)^n \twoheadrightarrow (L \times E)^n/\Sigma_n \overset{\rho_n}{\twoheadrightarrow} F_n(E,L)\,,$$

where $\rho_n\langle (l_1,a_1), \ldots, (l_n,a_n)\rangle = \sum_{i=1}^n l_i a_i$. Therefore, ρ_n is continuous.

Since $\rho_n \circ \alpha = \delta$, δ is continuous. In order to see that $t_p|_{F_r(X,L)}$ is continuous, consider the diagram

$$\begin{array}{ccc}
(L \times X)^r & \xrightarrow{\ \delta^r\ } & (F_n(E,L))^r \\
\big\downarrow & & \big\downarrow \text{sum} \\
F_r(X,L) & \xrightarrow[t_p|_{F_r(X,L)}]{} & F(E,L)\,,
\end{array}$$

where sum is given by the operation on $F(E,L)$, which is continuous. Hence t_p is continuous, and since $F^G(X,L)$ has the subspace topology, then t_p^G is also continuous, as desired. ∎

4 Change of coefficients and the transfer for homological Mackey functors

Let K be a simplicial pointed G-set. Then the composite

$$\Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F^G(-,M)} \mathcal{A}\mathrm{b}$$

is a simplicial abelian group, which will be denoted by $F^G(K,M)$. On the other hand, for any simplicial set S, we denote by $|S|$ its geometric realization (see [10]).

Let X be a pointed G-space. Recall from [5] that for a homological Mackey functor, we can give a topology to the abelian group $F^G(X,M)$. This topological group is denoted by $\mathbb{F}^G(X,M)$ and its topology is the identification topology given by the epimorphism

$$\pi_X^G : |F^G(\mathcal{S}(X),M)| \overset{\psi_M^G}{\longrightarrow} F^G(|\mathcal{S}(X)|,M) \overset{\rho_{X*}^G}{\twoheadrightarrow} \mathbb{F}^G(X,M)\,.$$

Here $\mathcal{S}(X)$ is the singular simplicial set associated to X. The group isomorphism ψ_M^G is defined on a generator by

$$\psi_M^G([\gamma_\sigma^G(l),t]) = \gamma_{[\sigma,t]}^G M_*(\widehat{q}_{\sigma,t})(l)\,,$$

as in [3, 2.6], where $\widehat{q}_{\sigma,t} : G/G_{\sigma,t} \twoheadrightarrow G/G_{[\sigma,t]}$ is the quotient function. The surjection $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is given by $\rho_X[\sigma,t] = \sigma(t)$. And $\sigma \in \mathcal{S}_k(X)$ and $t \in \Delta^k$.

Recall [4] that one has an equivariant isomorphism of topological groups $\psi_L : |F(\mathcal{S}(X),L)| \longrightarrow F(|\mathcal{S}(X)|,L)$ given by $\psi_L([l\sigma,t]) = l[\sigma,t]$, so that it restricts to an isomorphism $\psi_L^G : |F^G(\mathcal{S}(X),L)| \longrightarrow F^G(|\mathcal{S}(X)|,L)$. We have the next.

Lemma 4.1 *The following is a commutative diagram*

$$\begin{array}{ccc} |F^G(\mathcal{S}(X),L)| & \xrightarrow{\mathrm{id}} & |F^G(\mathcal{S}(X),M_L)| \\ \Big\downarrow{\scriptstyle \psi_L^G} & & \Big\downarrow{\scriptstyle \psi_{M_L}^G} \\ F^G(|\mathcal{S}(X)|,L) & \xrightarrow[\mathrm{id}]{} & \mathbb{F}^G(|\mathcal{S}(X)|,M_L) \end{array}$$

Proof: By Proposition 3.5, the functors $F^G(-,L)$ and $F^G(-,M_L)$ are the same. Therefore, the simplicial groups $F^G(\mathcal{S}(X),L)$ and $F^G(\mathcal{S}(X),M_L)$ are also the same. If we write $G/G_{[\sigma,t]} = \{[g_i] \mid i=1,\ldots,r\}$ and $G_{[\sigma,t]}/G_\sigma = \{[h_j] \mid j=1,\ldots,s\}$, then $G/G_\sigma = \{[g_ih_j] \mid (i,j)=(1,1),\ldots,(r,s)\}$. Thus we can write

$$\gamma_\sigma^G(l) = \sum_{(i,j)=(1,1)}^{(r,s)} (g_ih_jl)(g_ih_j\sigma) \in F^G(\mathcal{S}(X),L)\,.$$

Using this description, one can check as in the proof of 3.5, that $\psi_L^G([\gamma_\sigma^G(l),t]) = \psi_{M_L}^G([\gamma_\sigma^G(l),t])$. ■

In the rest of this section we show that a morphism of (homological) Mackey functors $\xi : M' \longrightarrow M$ induces a continuous homomorphism of topological groups $\xi_\diamond : \mathbb{F}^G(X;M') \longrightarrow \mathbb{F}^G(X;M)$ for any G-space X.

Definition 4.2 Recall that a *morphism* $\xi : M \longrightarrow M'$ is a natural transformation of both the covariant and the contravariant parts of M and M'; it is an *epimorphism* if for each object it is a group epimorphism. Define

$$\xi_\diamond : \mathbb{F}^G(X;M) \longrightarrow \mathbb{F}^G(X;M')$$

by $\xi_\diamond(u)(x) = \xi_{G/G_x}(u(x)) \in M'(G/G_x)$. Note that since u is equivariant and ξ is natural, then $\xi_\diamond(u)$ is also equivariant, and therefore it is well defined. Notice that by the naturality of ξ, $\xi_\diamond$ is given on generators by

$$\xi_\diamond(\gamma_x^G(l)) = \gamma_x'^G(\xi_{G/G_x}(l))\,.$$

Lemma 4.3 *The homomorphism $\xi_\diamond : \mathbb{F}^G(X,M) \longrightarrow \mathbb{F}^G(X,M')$ is natural in X.*

Proof: Let $f : X \longrightarrow Y$ be a pointed G-function. Then

$$f_*^G\xi_\diamond(\gamma_x^G(l)) = f_*^G\gamma_x'^G(\xi_{G/G_x}(l)) = \gamma_{f(x)}'^G M_*(\widehat{f}_x)(\xi_{G/G_x}(l)) = \xi_\diamond f_*^G(\gamma_x^G(l))\,.$$

■

The next follows immediately from the previous lemma.

Corollary 4.4 *Let K be a simplicial G-set. Then $\xi_\diamond : F^G(K,M) \longrightarrow F^G(K,M)$ is a homomorphism of simplicial groups.* ■

Proposition 4.5 *The homomorphism $\xi_\diamond$ is continuous.*

Proof: Consider the following commutative diagram

$$\begin{array}{ccc} |F^G(\mathcal{S}(X),M)| & \xrightarrow{|\xi_\diamond|} & |F^G(\mathcal{S}(X),M')| \\ {\scriptstyle \pi_X^G}\downarrow & & \downarrow{\scriptstyle \pi_X^G} \\ \mathbb{F}^G(X,M) & \xrightarrow[\xi_\diamond]{} & \mathbb{F}^G(X,M')\,. \end{array}$$

Since by 4.3, $\xi_\diamond : F^G(\mathcal{S}(X),M) \longrightarrow F^G(\mathcal{S}(X),M')$ is a homomorphism of simplicial groups, $|\xi_\diamond|$ on the top is continuous. The vertical arrows are identifications, thus $\xi_\diamond$ on the bottom is also continuous. ■

Given a G-module L, in what follows, we shall compare the topologies of the identical groups $F^G(X,L)$ and $\mathbb{F}^G(X,M_L)$, where M_L is the Mackey functor associated to L.

Proposition 4.6 *The identity $\mathbb{F}^G(X,M_L) \longrightarrow F^G(X,L)$ is continuous. Thus the topology on the left-hand side is finer than that on the right.*

Proof: Let X be any G-space and let $\mathcal{S}(X)$ be its associated singular simplicial G-set and $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ the canonical surjection introduced above. Notice that by Proposition 3.5, the simplicial groups $F^G(\mathcal{S}(X),L)$ and $F^G(\mathcal{S}(X),M_L)$ are identical. Hence the realizations $|F^G(\mathcal{S}(X),L)|$ and $|F^G(\mathcal{S}(X),M_L)|$ are identical topological groups. In [4, Cor. 2.6] one proves that the topological groups $F^G(|\mathcal{S}(X)|,L)$ and $|F^G(\mathcal{S}(X),L)|$ are (topologically) isomorphic, and in [5, Prop. (5.17)] it is proved that the topological groups $\mathbb{F}^G(|\mathcal{S}(X)|,M)$ and $|F^G(\mathcal{S}(X),M)|$ are also (topologically) isomorphic, when M is homological, in particular for $M = M_L$. Consider the following diagram:

$$\begin{array}{ccccc} |F^G(\mathcal{S}(X),M_L)| & = & |F^G(\mathcal{S}(X),L)| & \xrightarrow[\cong]{\psi_L^G} & F^G(|\mathcal{S}(X)|,L) \\ {\scriptstyle \pi_X^G}\downarrow & & \downarrow{\scriptstyle \widetilde{\pi}_X^G} & \swarrow{\scriptstyle \rho_{X\bullet}^G} & \\ \mathbb{F}^G(X,M_L) & \xrightarrow[\mathrm{id}]{} & F^G(X,L)\,. & & \end{array}$$

By definition, $\pi_X^G = \rho_{X*}^G \circ \psi_{M_L}^G$ and $\widetilde{\pi}_X^G = \rho_{X\bullet}^G \circ \psi_L^G$. By 3.5, $\rho_{X*}^G = \rho_{X\bullet}^G$, and by 4.1, $\psi_{M_L}^G = \psi_L^G$. Therefore, the diagram commutes. Since by definition, π_X^G is an identification, the result follows. ■

In the rest of this section, we analyze the continuity of the transfer for ramified covering G-maps in a convenient category of topological spaces.

Definition 4.7 A G-space X is called a *strong ρ-space* if the map $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is a G-retraction.

We have the following.

Lemma 4.8 *Let K be a simplicial G-set. Then $|K|$ is a strong ρ-space.*

Proof: Define $\delta : K \longrightarrow \mathbb{S}(|K|)$ by $\delta(a)(t) = [a, t]$, where $a \in K_n$ and $t \in \Delta^n$. One can easily check that δ is a simplicial G-function. Now take $|\delta| : |K| \longrightarrow |\mathbb{S}(|K|)|$. Since δ is equivariant, $|\delta|$ is equivariant too. Clearly $\rho_{|K|} \circ |\delta| = \mathrm{id}_{|K|}$. ■

Proposition 4.9 *Every simplicial G-complex is a strong ρ-space.*

Proof: Let C be a simplicial G-complex. We can always assume that it is ordered and that the action of G preserves the order. This can be achieved by passing to the barycentric subdivision. Now let us define a simplicial G-set $K(C)$, given by $K(C)_n = \{(v_0, \ldots, v_n) \mid v_0 \leq \cdots \leq v_n \text{ and } \{v_0, \ldots, v_n\} \in C\}$ with the obvious face and degeneracy operators. Then $|K(C)| \approx |C|$, and the result follows by the previous lemma. ■

Lemma 4.10 *Let Y be a strong ρ-space and let Z be a G-retract of Y. Then Z is a strong ρ-space.*

Proof: Let $r : Y \longrightarrow Z$ be a G-retraction with left G-inverse $i : Z \longrightarrow Y$. Let $\iota_Y : Y \longrightarrow |\mathbb{S}(Y)|$ be a left G-inverse of ρ_Y, and consider $|\mathbb{S}(r)| : |\mathbb{S}(Y)| \longrightarrow |\mathbb{S}(Z)|$. Then the G-map $\iota_Z : Z \longrightarrow |\mathbb{S}(Z)|$ given by $\iota_Z = |\mathbb{S}(r)| \circ \iota_Y \circ i$ is clearly a left inverse of ρ_Z, since by the naturality of ρ, $\rho_Z \circ |\mathbb{S}(r)| = r \circ \rho_Y$. ■

Proposition 4.11 *The class of strong ρ-spaces contains the class of G-ENRs.*

Proof: By a result of Illman [8], any finite-dimensional smooth G-manifold is G-triangulable, i.e. it is G-homeomorphic to the geometric realization of a G-simplicial complex.

Since a G-ENR is a retract of an open G-invariant set in a finite-dimensional G-representation, then the result follows from 4.9 and 4.10. ■

The following two results describe two classes of G-CW-complexes that belong to the class of strong ρ-spaces.

Proposition 4.12 *The class of strong ρ-spaces contains the class of locally compact, countable, finite-dimensional G-CW-complexes.*

Proof: First of all, one has that a CW-complex is metrizable if and only if it is locally compact [7]. On the other hand, a theorem of Jaworowski [9] guarantees that a metrizable, separable, and finite-dimensional G-space X is a G-ENR if and only if it is locally compact and its fixed-point sets X^H are ANRs. The fixed-point sets of any G-CW-complex X are subcomplexes, thus they are ANRs. Since X is countable, it is separable, and thus it is a G-ENR. By the previous proposition the result follows. ■

The next is the G-equivariant version of a result in [7].

Proposition 4.13 *Let X be a regular G-CW-complex. Then X is G-homeomorphic to a G-simplicial complex. Therefore, the class of strong ρ-spaces contains the class of regular G-CW-complexes.*

Proof: Consider the simplicial complex $T(X)$, whose vertices are the cells of X, and whose q-simplexes are sequences of closed cells $\{\overline{e}_0 \subsetneq \overline{e}_1 \subsetneq \cdots \subsetneq \overline{e}_q\}$. $T(X)$ is a G-simplicial complex, where the G-action on the vertices is given by $g \cdot e = g(e)$. There is a homeomorphism $h : |T(X)| \longrightarrow X$, which is constructed inductively on the skeleta as follows (see [7, 3.4.1]).

The complex $T(X^0)$ is the set of vertices of X. Then $h^0 : |T(X^0)| \longrightarrow X^0$ is the identity, which is G-equivariant.

We assume inductively that we have a G-homeomorphism $h^{n-1} : T(X^{n-1}) \xrightarrow{\approx} X^{n-1}$. We extend h^{n-1} to $h^n : |T(X^n)| \xrightarrow{\approx} X^n$ cell by cell as follows. Take a characteristic map of an n-cell

$$\varphi_{e^n} : \Delta^n \longrightarrow X^n\,,$$

whose image is the closed n-cell $\overline{e}^n$. Take the restriction to the boundary $\dot{\varphi} : \dot{\Delta}^n \longrightarrow \dot{e}^n$, and let C be the subcomplex of $T(X^{n-1})$ that triangulates $\dot{e}^n$. Define $\widetilde{\psi} : |\mathrm{Cone}(C)| \longrightarrow \Delta^n$ by $\widetilde{\psi}(\lambda e^n + (1-\lambda)\beta) = \lambda b_n + (1-\lambda)\psi(\beta)$, where $\beta \in |C|$ and $b_n \in \Delta^n$ is the barycenter. Then extend h^{n-1} to $h^n| : |\mathrm{Cone}(C)| \longrightarrow e^n$ by $h^n| = \varphi_{e^n} \circ \widetilde{\psi}$. Doing this for each n-cell of X, we obtain $h^n : |T(X^n)| \longrightarrow X^n$. In order to see that h^n is G-equivariant, consider the equalities

$$h^n(g\cdot\gamma) = \varphi_{g\cdot e^n}\widetilde{\psi}(\lambda(g\cdot e^n) + (1-\lambda)g\cdot\beta) = \varphi_{g\cdot e^n}(\lambda b_n + (1-\lambda)\varphi_{ge^n}^{-1} g\cdot h^{n-1}(\beta))\,,$$

$$g\cdot h^n(\gamma) = g\cdot(\varphi_{e^n}\widetilde{\psi}(\lambda e^n + (1-\lambda)\beta)) = g\cdot(\varphi_{e^n}(\lambda b_n + (1-\lambda)\varphi_{e^n}^{-1} h^{n-1}(\beta))\,,$$

which are clearly the same, since $g\cdot\varphi_{e^n} = \varphi_{g\cdot e^n}$. This completes the induction. ■

The relevance of the strong ρ-spaces is shown in the following two lemmas.

Lemma 4.14 *Let X be a pointed strong ρ-space and let L be a G-module. Then $\mathrm{id} : \mathbb{F}^G(X, M_L) \longrightarrow F^G(X, L)$ is a homeomorphism.*

Proof: Consider the following diagram

$$\begin{array}{ccccc}
|F^G(\mathcal{S}(X), M_L)| & = & |F^G(\mathcal{S}(X), L)| & \xrightarrow[\cong]{\psi_L^G} & F^G(|\mathcal{S}(X)|, L) \\
\downarrow{\scriptstyle \pi_X^G} & & \downarrow{\scriptstyle \widetilde{\pi}_X^G} & \swarrow{\scriptstyle \rho_{X\bullet}^G} & \\
\mathbb{F}^G(X, M_L) & \xrightarrow[\mathrm{id}]{} & F^G(X, L)\,. & &
\end{array}$$

By definition, $\pi_X^G = \rho_{X*}^G \circ \psi_{M_L}^G$ and $\widetilde{\pi}_X^G = \rho_{X\bullet}^G \circ \psi_L^G$. By 3.5, $\rho_{X*}^G = \rho_{X\bullet}^G$, and by 4.1, $\psi_{M_L}^G = \psi_L^G$. Therefore, the diagram commutes. By [4], ψ_L^G is a homeomorphism, and since X is a strong ρ-space, $\rho_{X\bullet}^G$ is an identification. Since by definition, π_X^G is also an identification, the result follows. ■

Lemma 4.15 *Let X be a strong ρ-space such that the action of G is free, and let M be a Mackey functor. Then there is a natural isomorphism of topological groups $\eta_X : \mathbb{F}^G(X^+, M) \longrightarrow F^G(X^+, M(G))$, where $M(G) = M(G/e)$ is a G-module with the action given by $g \cdot l = M_*(R_{g^{-1}})(l)$.*

Proof: We define η_X as follows:

$$\eta_X(u)(x) = \begin{cases} u(x) \in M(G) & \text{if } x \neq * \\ 0 \in M(G) & \text{if } x = *\,, \end{cases}$$

where $*$ is the isolated base point. Its inverse is given by

$$\eta_X^{-1}(v)(x) = \begin{cases} v(x) \in M(G) & \text{if } x \neq * \\ 0 \in M(G/G) & \text{if } x = *\,. \end{cases}$$

We now prove that η_X is natural assuming that X is G-set. Namely, let $f : X \longrightarrow Y$ be a G-function between G-sets with a free action. We shall see that the following diagram commutes:

$$\begin{array}{ccc}
F^G(X^+, M) & \xrightarrow{\eta_X} & F^G(X^+, M(G)) \\
{\scriptstyle f_*^G}\downarrow & & \downarrow{\scriptstyle f_\bullet^G} \\
F^G(Y^+, M) & \xrightarrow[\eta_Y]{} & F^G(Y, M(G))\,.
\end{array}$$

Take a generator $\gamma_x(l) \in F^G(X^+, M)$. Then

$$f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G(l) = \sum_{g \in G} (g \cdot l)(gf(x))\,,$$

since $\widehat{f}_x : G/e \twoheadrightarrow G/e$ is the identity. On the other hand

$$f_\bullet^G(\gamma_x^G(l)) = f_\bullet^G\left(\sum_{g\in G}(g\cdot l)(gx)\right) = \sum_{g\in G}(g\cdot l)f(gx)\,.$$

Hence the diagram commutes. Therefore, η defines an isomorphism of simplicial groups $F^G(K^+, M) \longrightarrow F^G(K^+, M(G))$ for any simplicial G-set K with a free action.

We now show that η is a homeomorphism. Consider the singular simplicial set $\mathcal{S}(X)$ and notice that $\mathcal{S}_n(X^+) = \mathcal{S}_n(X)^+$ for each n. Since X has a free G-action, so does $\mathcal{S}(X)$ and, by the previous considerations, the map of topological groups

$$|\eta_{\mathcal{S}(X)}| : |F^G(\mathcal{S}(X)^+, M)| \longrightarrow |F^G(\mathcal{S}(X)^+, M(G))|$$

is a homeomorphism. Consider the diagram

$$\begin{array}{ccc}
|F^G(\mathcal{S}(X)^+, M)| & \xrightarrow{|\eta_{\mathcal{S}(X)}|} & |F^G(\mathcal{S}(X)^+, M(G))| \\
\downarrow{\scriptstyle \psi_M^G} & & \cong\downarrow{\scriptstyle \psi_{M(G)}^G} \\
F^G(|\mathcal{S}(X)|^+, M) & \xrightarrow{\eta_{|\mathcal{S}(X)|}} & F^G(|\mathcal{S}(X)|^+, M(G)) \\
\downarrow{\scriptstyle \rho_{X^+ *}^G} & & \downarrow{\scriptstyle \rho_{X^+\bullet}^G} \\
\mathbb{F}^G(X^+, M) & \xrightarrow[\eta_X]{} & F^G(X^+, M(G))\,.
\end{array}
\qquad (\text{curved arrow } \pi_{X^+}^G \text{ from top left to bottom left})$$

The square at the top commutes because $G_{(\sigma,t)} = G_{[\sigma,t]} = e$ and hence $\widehat{q}_{\sigma,t} = \mathrm{id}_G$, and the square at the bottom commutes by the naturality of η.

The map $\pi_{X^+}^G$ is an identification by definition. The isomorphism $\psi_{M(G)}^G$ is a homeomorphism as proved in [4]. Since ρ_{X^+} is a retraction, so is $\rho_{X^+\bullet}^G$ and hence the vertical composite on the right is also an identification. Therefore, η_X is a homeomorphism. ■

Lemma 4.16 *Let X be a pointed G-space. If $\xi : M' \longrightarrow M$ is an epimorphism of Mackey functors, then $\xi_\diamond : \mathbb{F}^G(X, M) \longrightarrow \mathbb{F}^G(X, M')$ is an identification.*

Proof: To show that $\xi_\diamond : F^G(C, M) \longrightarrow F^G(C, M')$ is surjective for any pointed G-set C, take a generator $\gamma_x'^G(l') \in F^G(C, M')$. Since ξ_{G/G_x} is surjective, we can take $l \in M(G/G_x)$ such that $\xi_{G/G_x}(l) = l'$. Then $\xi_\diamond(\gamma_x^G(l)) = \gamma_x'^G(l')$. Therefore, the simplicial map $\xi_\diamond : F^G(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(X), M')$ is surjective, and by [7, 4.3.11], its geometric realization $|\xi_\diamond| : |F^G(\mathcal{S}(X), M)| \longrightarrow |F^G(\mathcal{S}(X), M')|$ is an identification.

Hence the commutativity of

$$\begin{array}{ccc} |F^G(\mathcal{S}(X),M)| & \xrightarrow{|\xi_\diamond|} & |F^G(\mathcal{S}(X),M')| \\ {\scriptstyle \pi_X^G}\big\downarrow & & \big\downarrow{\scriptstyle \pi_X^G} \\ \mathbb{F}^G(X,M) & \xrightarrow[\xi_\diamond]{} & \mathbb{F}^G(X,M') \end{array}$$

implies that $\xi_\diamond$ on the bottom is an identification too. ■

The main result in this section is the next.

Theorem 4.17 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map such that E and X are strong ρ-spaces, and let M be a homological Mackey functor. Then $t_p^G : \mathbb{F}^G(X,M) \longrightarrow \mathbb{F}^G(E,M)$ is continuous.*

Proof: Since M is homological, by [12, Thm. (16.5)(i)], there exists a G-module L and an epimorphism of Mackey functors $\xi : M_L \longrightarrow M$. By Lemma 4.16, the induced epimorphism $\xi_\diamond : \mathbb{F}^G(Y,M_L) \longrightarrow \mathbb{F}^G(Y,M)$ is an identification for any G-space Y. Moreover, by Lemma 4.14, for any strong ρ-space Y, the topological groups $\mathbb{F}^G(Y,M_L)$ and $F^G(Y,L)$ are equal. We have a commutative diagram

$$\begin{array}{ccccccc} \mathbb{F}^G(X,M_L) & = & F^G(X,L) & \xrightarrow{t_p^G} & F^G(E,L) & = & \mathbb{F}^G(X,M_L) \\ & {\scriptstyle \xi_\diamond}\searrow & \big\downarrow & & \big\downarrow & \swarrow{\scriptstyle \xi_\diamond} & \\ & & \mathbb{F}^G(X,M) & \overset{}{\underset{t_p^G}{\dashrightarrow}} & \mathbb{F}^G(E,M)\,. & & \end{array}$$

Since the horizontal arrow on the top is continuous by Proposition 3.6, the horizontal arrow on the bottom is also continuous. ■

The following is a **homotopy invariance** property, whose assumptions depend on the kind of coefficients. Namely, if one has coefficients in a G-module L, then X, Y, and E may be any k-spaces. And if M is an arbitrary homological Mackey functor, then X, Y, and E must be strong ρ-spaces.

Proposition 4.18 *Let $p : E \longrightarrow X$ be a ramified covering G-map. If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps and one has the following two pullback diagrams*

$$\begin{array}{ccc} f_0^*(E) & \xrightarrow{\widetilde{f}_0} & E \\ {\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow[f_0]{} & X \end{array} \qquad\qquad \begin{array}{ccc} f_1^*(E) & \xrightarrow{\widetilde{f}_1} & E \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow[f_1]{} & X\,, \end{array}$$

then

$$\widetilde{f}^G_{0*} \circ t^G_{p_0} \simeq \widetilde{f}^G_{1*} \circ t^G_{p_1} : \mathbb{F}^G(Y, M) \longrightarrow \mathbb{F}^G(E, M)\,.$$

Proof: By the pullback property one has that

$$\widetilde{f}^G_{0*} \circ t^G_{p_0} = t^G_p \circ f^G_{0*} \quad \text{and} \quad t^G_p \circ f^G_{1*} = \widetilde{f}^G_{1*} \circ t^G_{p_1}\,.$$

Moreover, from [5, Prop. (4.13)(c)], one has $f^G_{0*} \simeq f^G_{1*}$. Thus the assertion follows. ■

Finally, we have the following **invariance under change of coefficients**.

Proposition 4.19 *Let $p : E \longrightarrow X$ be a ramified covering G-map, where E and X are strong ρ-spaces. If $\xi : M \longrightarrow M'$ is a morphism of homological Mackey functors, then one has the following commutative diagram:*

$$\begin{array}{ccc} \mathbb{F}^G(X,M) & \xrightarrow{\xi_\diamond} & \mathbb{F}^G(X,M') \\ \downarrow {\scriptstyle t^G_p} & & \downarrow {\scriptstyle t^G_p} \\ \mathbb{F}^G(E,M) & \xrightarrow[\xi_\diamond]{} & \mathbb{F}^G(E,M')\,. \end{array}$$

Proof: Take $u \in \mathbb{F}^G(X, M)$, then by the naturality of ξ with respect to M^*, we have

$$\begin{aligned} \xi_\diamond t^G_p(u)(a) &= \xi_{G/G_a}\Big(\mu(a) M^*(\widehat{p}_a)\big(u(p(a))\big)\Big) \\ &= \mu(a)\xi_{G/G_a}\Big(M^*(\widehat{p}_a)\big(u(p(a))\big)\Big) \\ &= \mu(a)M'^*\Big((\widehat{p}_a)\xi_{G/G_a}\big(u(p(a))\big)\Big) = t^G_p \xi_\diamond(u)(a) \end{aligned}$$

■

5 Transfers in Bredon-Illman homology

In this section we shall define the transfer τ_p in Bredon-Illman homology with coefficients in a homological Mackey functor. The transfer has the following properties:

- Naturality (2.8),
- Pullback (2.15),
- Normalization (2.17),
- Additivity (2.20),

- Quasiadditivity (2.21),
- Functoriality (2.23),
- Homotopy invariance (4.18),
- Invariance under change of coefficients (4.19), and
- If $p : E \longrightarrow X$ is an n-fold ramified covering G-map, then the composite

$$p_* \circ \tau_p : \widetilde{H}_*^G(X; M) \longrightarrow \widetilde{H}_*^G(X; M)$$

is multiplication by n (2.24).

Theorem 5.1 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map such that E and X are strong ρ-spaces of the homotopy type of G-CW-complexes, and let M be a homological Mackey functor for G. Then there exists a transfer*

$$\tau_p : \widetilde{H}_*^G(X; M) \longrightarrow \widetilde{H}_*^G(E, M)$$

with all properties given above.

Proof: By [5], for all pointed G-spaces Y which have the homotopy type of G-CW-complexes, there is a natural isomorphism

$$\widetilde{H}_q^G(Y; M) \longrightarrow \pi_q\big(\mathbb{F}^G(Y, M)\big)\,.$$

Therefore, by Theorem 4.17, there is a transfer homomorphism

$$\tau_p : \widetilde{H}_*^G(X; M) \longrightarrow \widetilde{H}_*^G(E, M)$$

corresponding to the homomorphism induced by t_p^G in the homotopy groups. The properties follow immediately from the corresponding results for t_p^G. ■

Corollary 5.2 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map with E and X strong ρ-spaces of the homotopy type of G-CW-complexes, such that G acts freely on the total space E, and let M be a homological Mackey functor for G such that multiplication by n is an isomorphism in each of the groups $M(G/G_x)$ for $x \in X$. Then the transfer*

$$\tau_p : H_*^G(X; M) \longrightarrow H_*^G(E; M(G))$$

is a split monomorphism and thus the homology group $H_*^G(X; M)$ is a direct summand of $H_*^G(X; M(G))$.

Proof: Since the action of G on E is free, by Lemma 4.15, $\mathbb{F}^G(E^+, M) \cong F^G(E^+, M(G))$. On the other hand, by [4] there is a natural isomorphism

$$\widetilde{H}_q^G(E^+; M(G)) \cong \pi_q(F^G(E^+, M(G)))\,.$$

Therefore there is a natural isomorphism

$$\widetilde{H}^G_*(E^+, M) \cong \widetilde{H}^G_*(E^+, M(G))\,.$$

By 2.24, the composite

$$\begin{array}{ccccc}
\widetilde{H}^G_*(X^+, M) & \xrightarrow{\tau_p} & \widetilde{H}^G_*(E^+, M) & \xrightarrow{p_*} & \widetilde{H}^G_*(X^+, M) \\
& \searrow & \downarrow \cong & \nearrow & \\
& & \widetilde{H}^G_*(E^+, M(G)) & &
\end{array}$$

is multiplication by n, and thus an isomorphism. Hence the result follows. ∎

REMARK 5.3 The transfer for any ramified covering G-map cannot be given by a stable transfer map, which has the naturality, the normalization, and the quasiadditivity (see 2.21) properties, because otherwise there would be a transfer for ramified covering G-maps in any representable (cohomology) theory. But by [2, Thm. 4.8], if there is such a transfer, then the theory must be given by a product of Eilenberg-Mac Lane spaces. (One can construct such a stable transfer for n-fold ramified covering maps in the nonequivariant case [6], but provided that one inverts $n!$.)

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