Transfer for ramified covering G-maps

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Abstract Let G be a finite group. The main objective of this paper is to study ramified covering G-maps and to construct a transfer for them in Bredon-Illman equivariant homology with coefficients in a homological Mackey functor M. We show that this transfer has the usual properties of a transfer.

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Contents

0 INTRODUCTION

Let G be a finite group. The main objective of this paper is to study ramified covering G-maps and to construct a transfer for them in Bredon-Illman equivariant homology with coefficients in a homological Mackey functor M. We show that this transfer has many of the properties of other known transfers. Notice that in order to have the property that the composite of the transfer with the

*Corresponding author, Phone: ++5255-56224489, Fax ++5255-56160348. This author was partially supported by PAPIIT grant IN105106-3 and by CONACYT grant 58049. Webpage: http://www.matem.unam.mx/cprieto projection is multiplication by the multiplicity of the ramified covering map, M must be a homological Mackey functor. The transfer for any ramified covering G-map will be given by a homomorphism between certain topological abelian groups. It cannot be given by a stable transfer map (see Remark 5.4).

To construct the transfer, we shall use the homotopical definition of Bredon-Illman homology $H_*^G(-;L)$ given in [4], when L is a G-module, and of $H_*^G(X;M)$ given in [3] when M is a Mackey functor. Namely, to each pointed G-space X and a G-module L and a Mackey functor M for G, we associate topological abelian groups $F^G(X,L)$ and $F^G(X,M)$ such that $\pi_q(F^G(X,L)) \cong \widetilde{H}^G(X;L)$ and $\pi_q(F^G(X,M)) \cong \widetilde{H}^G(X;M)$. The topology in $F^G(X,L)$ is a generalization of the usual topology of the infinite symmetric product $\mathrm{SP}^\infty X$. The topology in $F^G(X,M)$ is defined using the singular simplicial set $\mathcal{S}(X)$ associated to X. With these topologies, the homomorphisms induced by any pointed G-map $f : X \longrightarrow Y$ turn out to be continuous. For a ramified covering G-map $p : E \longrightarrow X$ we define transfer homomorphisms $t_p^G : F^G(X,L) \longrightarrow F^G(E,L)$ and $t_p^G : F^G(X,M) \longrightarrow F^G(E,M)$. The first one is always continuous. When we take coefficients in a homological Mackey functor, we prove that the transfer is continuous provided that the spaces involved are ρ-spaces. The category of ρ-spaces is a subcategory of the category of k-spaces that contains all CW-complexes.

This approach to the transfer was already used by the authors in [1] in the nonequivariant case for singular homology.

The paper is organized as follows. In Section 1, we define a transfer $t_p^G : F^G(X,M) \longrightarrow F^G(E,M)$ for certain finite-to-one G-functions $p : E \longrightarrow X$ between G-sets, which we call n-fold G-functions with multiplicity (1.1). The reader should think of them as ramified covering G-maps without topology. We show that this transfer has all the usual properties, namely the pullback property (2.14), normalization (2.16), additivity (2.19), and that its composite with p_*^G is multiplication by n (2.23). In Section 3, we define the concept of a ramified covering G-map $p : E \longrightarrow X$. This is an n-fold G-function with multiplicity and some topological properties. This generalizes to the equivariant case the definition in [8]. For any G-module L, using the topology on $F^G(X,L)$ described in [4], we prove that the transfer constructed in the previous section is continuous for any p (3.5). In Section 4, we recall a theorem of Thevenaz and Webb that relates a homological Mackey functor with the Mackey functor associated to a G-module L. Using this result and the continuity of the transfer in the case of coefficients in a G-module L, we prove the continuity of the transfer with coefficients in a homological Mackey functor M, provided that E and X are ρ-spaces (4.6).

Finally, in Section 5, we pass to Bredon-Illman homology (applying the homotopy-group functors) and give the transfer and its properties in homology.

1 The transfer in the category of G-sets

In this section we shall define the transfer for a certain family of G-functions.

Definition 1.1 By an *n-fold G-function with multiplicity* we understand a G-function $p : E \longrightarrow X$ between G-sets with finite fibers, together with a G-invariant function $\mu : E \longrightarrow \mathbb{N}$, called *multiplicity function*, such that for each $x \in X$,

$$\sum_{a \in p^{-1}(x)} \mu(a) = n\,.$$

We say that the n-fold G-function with multiplicity $p : E \longrightarrow X$ is *pointed* if the sets E and X have base points, which are fixed under the G-action, and p is a pointed function.

Definition 1.2 Given a G-function $p : E \longrightarrow X$ with multiplicity $\mu : E \longrightarrow \mathbb{N}$, one may define the the G-function

$$\varphi_p : X \longrightarrow \mathrm{SP}^n E$$

by

$$\varphi_p(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu(a_r)} \rangle\,.$$

This function will play an important role in Section 3.

Example 1.3 Let X be a G-set and consider the G-function $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ given by $\pi\langle x_1, \ldots, x_n; i\rangle = \langle x_1, \ldots, x_n\rangle$, where G acts on both sets diagonally and trivially on the set $\overline{n} = \{1, 2, \ldots, n\}$. Define $\mu : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N}$ by

$$\mu\langle x; i\rangle = \# x^{-1}(x(i))\,,$$

where one regards x as a function $\overline{n} \longrightarrow X$. Then p is an n-fold G-function with multiplicity, since the sets $x^{-1}x(i)$ form a partition of the set $\overline{n}$. Furthermore, μ is clearly G-invariant. The function $\varphi_\pi : \mathrm{SP}^n X \longrightarrow \mathrm{SP}^n(X^n \times_{\Sigma_n} \overline{n})$ is given in this case by

$$\varphi_\pi\langle x_1, \ldots, x_n\rangle = \langle \langle x_1, \ldots, x_n; 1\rangle, \ldots, \langle x_1, \ldots, x_n; n\rangle\rangle\,.$$

Remark 1.4 We can always assume that an n-fold G-function with multiplicity is pointed by adding isolated points $*$ to E and to X which remain fixed under the G-action and by defining $\mu(*) = n$. Therefore we shall always consider pointed n-fold G-functions with multiplicity without saying it explicitly.

We now recall the definition of the groups $F(Y,M)$ and $F^G(Y,M)$ for a pointed G-set Y and a Mackey functor M for the group G. First one has the set $\widehat{M} = \cup_{H\subset G} M(G/H)$, which has a G-action given for $g \in G$ and $l \in M(G/H)$ by $g \cdot l = M_*(R_{g^{-1}})(l) \in M(G/gHg^{-1})$. Then $F(Y,M)$ consists of functions $u : Y \longrightarrow M$ such that $u(y) \in M(G/G_y)$, $u(*) = 0$, and $u(y) = 0$ for all but a finite number of elements $y \in Y$. The canonical generators of this group are functions denoted by ly given by

$$(ly)(y') = \begin{cases} l & \text{if } y' = y\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $l \in M(G/G_y)$ and $y \in Y - \{*\}$. The group $F(Y,M)$ has a natural (left) action of G given by defining $(gu)(y) = g \cdot u(g^{-1}y)$. Define $F^G(Y,M)$ as the subgroup of the functions u that are G-equivariant or, equivalently, the fixed points of $F(X,M)$ under the described G-action. The canonical generators of $F^G(Y,M)$ are functions denoted by $\gamma_y^G(l)$ given by

$$\gamma_y^G(l) = \sum_{j=1}^{m} M_*(R_{g_j^{-1}})(l)(g_j y)\,,$$

where $l \in M(G/G_y)$, $y \in Y - \{*\}$, and $G/G_y = \{[g_j] \mid j = 1,\ldots,m\}$. Given a pointed G-function $f : X \longrightarrow Y$, the homomorphism $f_*^G : F^G(X,M) \longrightarrow F^G(Y,M)$ is given on the generators by

$$f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G M_*(\widehat{f}_x)(l)\,,$$

where $\widehat{f}_x : G/G_x \longrightarrow G/G_{f(x)}$ is the canonical quotient function (see [3, 5] for details).

Definition 1.5 Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ, and let M be a Mackey functor. Define a homomorphism

$$t_p : F(X,M) \longrightarrow F(E,M)\,,$$

by

$$t_p(u)(a) = \mu(a) M^*(\widehat{p}_a) u(p(a))\,,$$

where $u \in F(X,M)$ and $a \in E$. If we assume that $u \in F^G(X,M)$, i.e., that $u(gx) = g \cdot u(x)$, then

$$\begin{aligned} t_p(u)(ga) &= \mu(ga) M^*(\widehat{p}_{ga})(u(p(ga))) \\ &= \mu(a) M^*(\widehat{p}_{ga})(g \cdot u(p(a))) \\ &= \mu(a) M^*(\widehat{p}_{ga}) M_*(R_{g^{-1}})(u(p(a))) \\ &= \mu(a) M_*(R_{g^{-1}}) M^*(\widehat{p}_a)(u(p(a))) \\ &= g \cdot (t_p(u)(a)) \,, \end{aligned}$$

where the next to the last equality follows from the pullback property of the Mackey functor. Thus $t_p(u) \in F^G(E,M)$. Therefore, the homomorphism t_p restricts to a *transfer* homomorphism

$$t_p^G : F^G(X,M) \longrightarrow F^G(E,M) \,.$$

REMARK 1.6 Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ. The isotropy group G_x acts on $p^{-1}(x)$ and the inclusion $p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$. Let $\{a_\iota\} \subset p^{-1}(x)$ be a set of representatives one for each G_x-orbit. Let $\gamma_x^G(l)$ be a generator of $F^G(X,M)$. Since the value of this function is zero on points which do not belong to the orbit Gx, and $\gamma_x^G(l)(x) = l$. One can give the transfer t_p^G on the generators $\gamma_x^G(l)$ by the formula

$$t_p^G(\gamma_x^G(l)) = \sum_{[a_\iota] \in p^{-1}(x)/G_x} \mu(a_\iota) \gamma_{a_\iota}^G(M^*(\widehat{p}_{a_\iota})(l)) \,. \tag{1.5}$$

2 Properties of the transfer

In this section we shall give all general properties of the transfer that do not depend on the topology. We start with a definition.

Definition 2.1 Let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X'$ be n-fold G-functions with multiplicity functions μ and μ', respectively. A *morphism* from p to p' is a pair of G-functions $(\widetilde{f}, f)$ such that

(a) the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \,, \end{array}$$

(b) for each $x \in X$, the restriction $\widetilde{f}|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$ is surjective,

(c) for each $x \in X$ and $a' \in p'^{-1}(f(x))$, one has the equality

$$\mu'(a') = \sum_{p(a)=x\,,\ \widetilde{f}(a)=a'} \mu(a)\,, \quad \text{and} \tag{2.2}$$

(d) for each $a \in E$ one has the formula

$$G_a = G_{p(a)} \cap G_{\widetilde{f}(a)}\,. \tag{2.3}$$

We have the next useful characterization of a morphism of ramified covering G-maps.

Proposition 2.4 *Let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X'$ be n-fold G-functions with multiplicity, and let $f : X \longrightarrow X$ and $\widetilde{f} : E \longrightarrow E'$ be G-functions such that $\widetilde{f} \circ p' = p \circ f$ and for $a \in E$, $G_a = G_{p(a)} \cap G_{\widetilde{f}(a)}$. Then $(\widetilde{f}, f)$ is a morphism from p to p' if and only if*

$$\varphi_{p'} \circ f = \mathrm{SP}^n \widetilde{f} \circ \varphi_p : X \longrightarrow \mathrm{SP}^n E'\,,$$

where the φs are as given in 1.2.

Proof: Assume that $\varphi_{p'} \circ f = \mathrm{SP}^n \widetilde{f} \circ \varphi_p$. This clearly implies that $\widetilde{f}$ is surjective on fibers. Take $x \in X$ and let $\{a'_1, \ldots, a'_{k'}\} = p'^{-1}(f(x))$. Moreover, let $\{a_{1\,1}, \ldots, a_{1\,r_1}\} = \widetilde{f}^{-1}(a'_1)$,...,$\{a_{k'\,1}, \ldots, a_{k'\,r_{k'}}\} = \widetilde{f}^{-1}(a'_{k'})$. Hence $p^{-1}(x) = \{a_{1\,1}, \ldots, a_{1\,r_1}, \ldots, a_{k'\,1}, \ldots, a_{k'\,r_{k'}}\}$. Writing down $\mathrm{SP}^n \widetilde{f}(\varphi_p(x))$ using this description of $p^{-1}(x)$, which is equal to $\varphi_{p'}(f(x))$, one easily obtains the equality (2.2).

Assuming now (b) and (c) in the definition and using the same labels for the elements of the fibers as in the first part, one obtains the desired equality. ■

Examples 2.5 There are two interesting examples of morphisms between G-functions with multiplicity:

(a) Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ, and let $f : Y \longrightarrow X$ be a G-function. Consider the pullback diagram

$$\begin{tikzcd} f^*E \arrow[r, "\widetilde{f}"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X\,, \end{tikzcd} \tag{2.5}$$

where $f^*E = Y \times_X E = \{(y,a) \mid f(y) = p(a)\}$. Clearly, q is also an n-fold G-function with multiplicity μ' given by $\mu'(y,a) = \mu(a)$, since $\mu'(g(y,a)) = \mu'(gy,ga) = \mu(ga) = \mu(a) = \mu'(y,a)$. Consider the restriction of $\widetilde{f}$ from the fiber $(p')^{-1}(y)$ to the fiber $p^{-1}(f(y))$. This function induces a surjective function

$$q : (p')^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}\,.$$

Clearly, conditions (a), (b), and (c) in the previous definition hold. Moreover, clearly $G_{(y,a)} = G_y \cap G_a$, thus contition (d) also holds. Hence $(\widetilde{f}, f)$ is a morphism from p to p'.

(b) Let X and Y be G-sets and let $f : X \longrightarrow Y$ be G-equivariant. We say that f is *n-permutable* if the equality

$$G_{\langle x_1,\ldots,x_n;i\rangle} = G_{\langle x_1,\ldots,x_n\rangle} \cap G_{\langle f(x_1),\ldots,f(x_n);i\rangle} \tag{2.6}$$

holds in terms of isotropy groups, where $\langle x_1,\ldots,x_n;i\rangle \in X^n \times_{\Sigma_n} \overline{n}$, $\langle x_1,\ldots,x_n\rangle \in \mathrm{SP}^n X$, and $\langle f(x_1),\ldots,f(x_n);i\rangle \in Y^n \times_{\Sigma_n} \overline{n}$. If $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ and $\pi' : Y^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n Y$ are as in 1.3, then the pair of G-functions $(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}}, \mathrm{SP}^n f)$ is a morphism from π to π'. To see this, we use Proposition 2.4 above. Namely, we have to show that

$$\varphi_{\pi'} \circ \mathrm{SP}^n f = \mathrm{SP}^n(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}}) \circ \varphi_\pi : \mathrm{SP}^n X \longrightarrow \mathrm{SP}^n(Y^n \times_{\Sigma_n} \overline{n})\,.$$

For any $\langle y_1,\ldots,y_n\rangle \in \mathrm{SP}^n Y$, we have,

$$\varphi_{\pi'}\langle y_1,\ldots,y_n\rangle = \langle\langle y_1,\ldots,y_n;1\rangle,\ldots,\langle y_1,\ldots,y_n;n\rangle\rangle\,,$$

thus, if we take $\langle x_1,\ldots,x_n\rangle \in \mathrm{SP}^n X$, we have

$$\varphi_{\pi'}(\mathrm{SP}^n f\langle x_1,\ldots,x_n\rangle) = \langle\langle f(x_1),\ldots,f(x_n);1\rangle,\ldots,\langle f(x_1),\ldots,f(x_n);n\rangle\rangle\,.$$

On the other hand, we have

$$\varphi_\pi\langle x_1,\ldots,x_n\rangle = \langle\langle x_1,\ldots,x_n;1\rangle,\ldots,\langle x_1,\ldots,x_n;n\rangle\rangle$$

and so

$$\mathrm{SP}^n(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}})(\varphi_\pi\langle x_1,\ldots,x_n\rangle) = \langle\langle f(x_1),\ldots,f(x_n);1\rangle,\ldots,\langle f(x_1),\ldots,f(x_n);n\rangle\rangle\,.$$

Thus both are equal.

The following is the **naturality property** of the transfer.

Proposition 2.7 *Let $(\widetilde{f}, f)$ be a morphism from $p : E \longrightarrow X$ to $p' : E' \longrightarrow X'$. Then the following diagram commutes:*

$$\begin{tikzcd} F^G(X,M) \arrow[r, "f_*^G"] \arrow[d, "t_p^G"'] & F^G(X',M) \arrow[d, "t_{p'}^G"] \\ F^G(E,M)) \arrow[r, "\widetilde{f}_*^G"'] & F^G(X',M)\,. \end{tikzcd}$$

Proof: First note that the function $\widetilde{f}$ induces a surjection

$$p^{-1}(\langle x\rangle)/G_{\langle x\rangle} \twoheadrightarrow q^{-1}(\overline{f}(\langle x\rangle))/G_{\overline{f}(x)}\,,$$

This surjection can be written as the composite

$$p^{-1}(x)/G_x \twoheadrightarrow p'^{-1}(f(x))/G_x \overset{q}{\twoheadrightarrow} p'^{-1}(f(x))/G_{f(x)}\,.$$

This allows us to write the elements of these quotient sets as follows. Let

$$p'^{-1}(f(x))/G_{f(x)} = \{[a'_j] \mid j = 1, \dots s\}\,.$$

By [5, Lemma (6.6)], one can write

$$p'^{-1}(f(x))/G_x \cong \sqcup_{j=1}^{s} G_x \backslash G_{f(x)}/G_{a'_j}\,.$$

If $G_x\backslash G_{f(x)}/G_{a'_j} = \{[g_j^\nu] \mid \nu = 1, \dots, s_j\}$, then we can write

$$p'^{-1}(f(x))/G_x = \{[g_j^\nu a'_j] \mid \nu = 1, \dots, s_j\,,\ j = 1, \dots s\}\,.$$

Therefore,

$$p^{-1}(x)/G_x = \{[a_\alpha^{\nu j}] \mid \alpha = 1, \dots, r_j^\nu, \nu = 1, \dots, s_j\,,\ j = 1, \dots s\}\,,$$

where $a_\alpha^{\nu j} = \langle x_1, \dots, x_n; k_\alpha^{\nu j}\rangle \in p^{-1}(x) \cap \widetilde{f}^{-1}(g_j^\nu a'_j)$ are such that

$$\{[a_\alpha^{\nu j}] \mid \alpha = 1, \dots, r_j^\nu\} = (\widetilde{f})^{-1}([g_j^\nu a'_j])\,.$$

Hence we have

$$\widetilde{f}_*^G t_p^G(\gamma_x^G(l)) = \sum_{j,\nu,\alpha=1,1,1}^{s,s_j,r_j^\nu} \mu(a_\alpha^{\nu j})\gamma_{g_j^\nu a'_j}^G M_*(\widehat{\widetilde{f}}_{a_\alpha^{\nu j}})M^*(\widehat{p}_{a_\alpha^{\nu j}})(l)\,. \tag{2.8}$$

Since $\gamma_{g_j^\nu a'_j}^G = \gamma_{a'_j}^G \circ M_*(R_{g_j^\nu})$, we can rewrite (2.8) as

$$\widetilde{f}_*^G t_p^G(\gamma_x^G(l)) = \sum_{j,\nu,\alpha=1,1,1}^{s,s_j,r_j^\nu} \mu(a_\alpha^{\nu j})\gamma_{a'_j}^G M_*(R_{g_j^\nu})M_*(\widehat{\widetilde{f}}_{a_\alpha^{\nu j}})M^*(\widehat{p}_{a_\alpha^{\nu j}})(l)\,. \tag{2.9}$$

On the other hand, we have

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j=1}^{s} \mu'(a'_j)\gamma_{a'_j}^G M^*(\widehat{p'}_{a'_j})M_*(\widehat{\overline{f}}_x)(l)\,. \tag{2.10}$$

In order to compare these two sums, consider the pullback diagram

$$\begin{tikzcd}
G/G_x \times_{G/G_{f(x)}} G/G_{a'_j} \arrow[r, "\tau"] \arrow[d, "\pi"'] & G/G_{a'_j} \arrow[d, "\widehat{p'}_{a'_j}"] \\
G/G_x \arrow[r, "\widehat{f}_x"'] & G/G_{f(x)}\,.
\end{tikzcd}$$

By [5, Lemma (6.1)], there is a bijection

$$\varphi : \bigsqcup_{\nu=1}^{s^j} G/G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1} \xrightarrow{\approx} G/G_x \times_{G/G_{f(x)}} G/G_{a'_j}\,,$$

where the elements g_j^ν are as above. Denote $\varphi_j = \varphi|_{G/G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1}}$.

By [5, Lemma (6.3)], any element $w \in M(G/G_{\langle x\rangle} \times_{G/G_{\overline{f}(\langle x\rangle)}} G/G_{b_j})$ can be written as

$$w = \sum_{\nu=1}^{s^j} M_*(\varphi_j^\nu)M^*(\varphi_j^\nu)(w)\,.$$

Set $\pi_j^\nu = \pi \circ \varphi_j^\nu$ and $\tau_j^\nu = \tau \circ \varphi_j^\nu$. Consequently,

$$M^*(\widehat{p'}_{a'_j})M_*(\widehat{f}_x)(l) = M_*(\tau)M^*(\pi)(l) = \sum_{\nu=1}^{s^j} M_*(\tau_j^\nu)M^*(\pi_j^\nu)(l)\,.$$

Replacing this in (2.10), we obtain

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j,\nu=1,1}^{s,s_j} \mu'(b_j)\gamma_{a'_j}^G M_*(\tau_j^\nu)M^*(\pi_j^\nu)(l)\,. \tag{2.11}$$

Let $\rho_j^\nu : G/G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1} \twoheadrightarrow G/g_j^\nu G_{a'_j}(g_j^\nu)^{-1}$ be the quotient function. Since $\tau_j^\nu = R_{g_j^\nu} \circ \rho_j^\nu$, we can rewrite (2.11) as

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j,\nu=1,1}^{s,s_j} \mu'(a'_j)\gamma_{a'_j}^G M_*(R_{g_j^\nu})M_*(\rho_j^\nu)M^*(\pi_j^\nu)(l)\,. \tag{2.12}$$

By (2.3), we have that $G_x \cap g_j^\nu G_{a'_j}(g_j^\nu)^{-1} = G_{a_\alpha^{\nu j}}$, and therefore, $\pi_j^\nu = \widehat{p}_{a_\alpha^{\nu j}}$ and $\rho_j^\nu = \widehat{\widetilde{f}}_{a_\alpha^{\nu j}}$. Hence (2.12) becomes

$$t_{p'}^G f_*^G(\gamma_x^G(l)) = \sum_{j=1,1}^{s,s_j} \mu'(a'^j)\gamma_{a'_j}^G M_*(R_{g_j^\nu})M_*(\widehat{\widetilde{f}}_{a_\alpha^{\nu j}})M^*(\widehat{p}_{a_\alpha^{\nu j}})(l)\,. \tag{2.13}$$

By Proposition 1.3 (b) and the G-invariance of μ' we have that

$$\mu'(a'_j) = \mu'(g_j^\nu a'_j) = \sum_{\alpha=1}^{r_j^\nu} \mu_m(a_\alpha^{\nu j})\,.$$

Replacing this in (2.13), we obtain (2.9). Therefore,

$$\widetilde{f}_*^G \circ t_p^G = t_{p'}^G \circ f_*^G : F^G(X,M) \longrightarrow F^G(E',M)\,.$$

■

By Example 2.5(a), the **pullback property** is now a consequence of the naturality property.

Proposition 2.14 *Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ and let $f : Y \longrightarrow X$ be a G-function. Then*

$$t_p^G \circ f_*^G = \widetilde{f}_*^G \circ t_q^G : F^G(Y,M) \longrightarrow F^G(E,M)\,,$$

where $\widetilde{f}$ and q are as in the pullback diagram (2.5). ■

From Example 2.5(b), we obtain another consequence of the naturality property as follows.

Proposition 2.15 *Let $f : X \longrightarrow Y$ be an n-permutable G-function. Then*

$$(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}})_*^G \circ t_\pi^G = t_{\pi'}^G \circ (\mathrm{SP}^n f)_*^G : F^G(\mathrm{SP}^n X, M) \longrightarrow F^G(Y^n \times_{\Sigma_n} \overline{n}, M)\,.$$

■

The **normalization property** is elementary and is as follows.

Proposition 2.16 *If $p : E = X \longrightarrow X$ is the identity function with multiplicity function μ constant equal to 1, then $t_p^G : F^G(X,M) \longrightarrow F^G(X,M)$ is the identity too.* ■

The next is needed for the **additivity property**.

Proposition 2.17 *Let X_α, $\alpha \in \mathcal{A}$, be a family of pointed G-sets. Then there is an isomorphism of abelian groups*

$$F^G(\bigvee_{\alpha\in\mathcal{A}} X_\alpha, M) \cong \bigoplus_{\alpha\in\mathcal{A}} F^G(X_\alpha, M)\,.$$

Proof: Let $i_\alpha : X_\alpha \longrightarrow \bigvee X_\alpha$ be the inclusion into the wedge of the pointed sets X_α. By the universal property of the direct sum, the homomorphisms $i^G_{\alpha *}$ induce a homomorphism $\varphi : \bigoplus F^G(X_\alpha, M) \longrightarrow F^G(\bigvee X_\alpha, M)$.

Take now $x \in \bigvee X_\alpha$ and let $x_\alpha \in X_\alpha$ be such that $i_\alpha(x_\alpha) = x$. Consider the family of homomorphisms $\iota_\alpha \circ \gamma^G_{x_\alpha} : M(G/G_x) = M(G/G_{x_\alpha}) \longrightarrow \bigoplus F^G(X_\alpha, M)$, where ι_α is the canonical monomorphism int the direct sum. Then by the universal property of $F^G(\bigvee X_\alpha, M)$ (see [3]), there is a unique homomorphism $\psi : F^G(\bigvee X_\alpha, M) \longrightarrow \bigoplus F^G(X_\alpha, M)$, such that $\psi \circ \gamma^G_x = \iota_\alpha \circ \gamma^G_{x_\alpha}$. Checking on generators of the form $\gamma^G_x(l)$, resp. $\iota_\alpha \gamma^G_{x_\alpha}(l)$, one easily verifies that the composite $\varphi \circ \psi$, resp. $\psi \circ \varphi$, is the identity. ∎

Definition 2.18 Let $p_\alpha : E_\alpha \longrightarrow X$, $\alpha = 1, \ldots, r$, be a family of n_α-fold G-functions with multiplicity functions $\mu_\alpha : E_\alpha \longrightarrow \mathbb{N}$. Define $p : E = \bigvee_{\alpha=1}^r E_\alpha \longrightarrow X$ by $p|_{X_\alpha} = p_\alpha$. If $n = \sum_{\alpha=1}^r n_\alpha$, then clearly p is an n-fold G-function with multiplicity function $\mu : E \longrightarrow \mathbb{N}$ given by

$$\mu(a) = \begin{cases} \mu_\alpha(a) & \text{if } a \in E_\alpha - \{*_\alpha\}\,, \\ \sum_{\alpha=1}^r \mu_\alpha(*_\alpha) & \text{if } a = *\,, \end{cases}$$

where $* \in E$, resp. $*_\alpha \in E_\alpha$, denotes the base point. We call p the *sum* of the p_αs, and denote it by $\sum_{\alpha=1}^r p_\alpha$.

The transfer has the following additivity property.

Proposition 2.19 *If* $p = \sum_{\alpha=1}^r p_\alpha : E = \bigvee_{\alpha=1}^r E_\alpha \longrightarrow X$, *then* $t^G_{\Sigma p_\alpha} = \sum t^G_{p_\alpha}$. *More precisely, the following diagram commutes:*

$$\begin{tikzcd} F^G(X,M) \arrow[r, "t^G_p"] \arrow[dr, "(t^G_{p_\alpha})"'] & F^G(E,M) \\ & \bigoplus_{\alpha=1}^r F^G(E_\alpha, M) \arrow[u, "\cong"', "(i^G_{\alpha*})"] \end{tikzcd} \quad ,$$

where the isomorphism is as given in 2.17.

Proof: Notice first that for any $x \in X$, not the base point, $p^{-1}(x) = \sqcup_{\alpha=1}^r p_\alpha^{-1}(x)$. Since each $p_\alpha^{-1}(x)$ is G_x-invariant, then $p^{-1}(x)/G_x = \sqcup_{\alpha=1}^r p_\alpha^{-1}(x)/G_x$. Let

$\gamma_x^G(l) \in F^G(X,M)$ be a generator. Then

$$\begin{aligned} t_p^G(\gamma_x^G(l)) &= \sum_{[a]\in p^{-1}(x)/G_x} \gamma_a^G M^*(\widehat{p}_a)(l) \\ &= \sum_{\alpha=1}^r i_{\alpha *}^G \left(\sum_{[a]\in p_\alpha^{-1}(x)/G_x} \gamma_a^G M^*(\widehat{p_\alpha}_a)(l) \right) \\ &= \sum_{\alpha=1}^r i_{\alpha *}^G t_{p_\alpha}^G(\gamma_x^G(l)) \end{aligned}$$

■

An immediate consequence of the normalization property 2.16 and the additivity property 2.19 is the **quasiadditivity property**, namely the following.

Proposition 2.20 *Let $p: E \longrightarrow X$ be a G-function with multiplicity, and let $q: E \vee X \longrightarrow X$ be given by $q|_E = p$ and $q|_X = \mathrm{id}_X$ with the corresponding multiplicity function μ'. Then the next is a commutative diagram:*

$$\begin{tikzcd} F^G(X,M) \arrow[r, "t_q^G"] \arrow[dr, "(t_p^G{,}1)"'] & F^G(E\vee X, M) \\ & F^G(E,M)\oplus F^G(X,M)\,, \arrow[u, "\cong"'] \end{tikzcd}$$

where the isomorphism is as given in 2.17. ■

To show that the transfer has a **functoriality property**, we need the following.

Proposition 2.21 *Let $q: E' \longrightarrow E$ be an n'-fold G-function with multiplicity function μ', and let $p: E \longrightarrow X$ be an n-fold G-function with multiplicity function μ. If one defines $\nu: E' \longrightarrow \mathbb{N}$ by*

$$\nu(a') = \mu'(a')\mu(q(a))\,,$$

then the composite $p\circ q: E' \longrightarrow X$ is an (nn')-fold G-function with multiplicity function ν.

Proof: We only have to compute the sum

$$\begin{aligned} \sum_{a'\in p\circ q^{-1}(x)} \nu(a') &= \sum_{a\in p^{-1}(x)} \sum_{a'\in q^{-1}(a)} \mu'(a')\mu(a) \\ &= \sum_{a\in p^{-1}(x)} \mu(a) \sum_{a'\in q^{-1}(a)} \mu'(a') \\ &= nn'\,. \end{aligned}$$

■

The functoriality property is the following.

Proposition 2.22 *Let $q : E' \longrightarrow E$ be an n'-fold G-function with multiplicity function μ', and let $p : E \longrightarrow X$ be an n-fold G-function with multiplicity function μ. Then*

$$t^G_{q\circ p} = t^G_p \circ t^G_q : F^G(X, M) \longrightarrow F^G(E', M)\,.$$

Proof: Take $u \in F^G(X, M)$. By definition of the transfer, we have

$$\begin{aligned} t^G_q(t^G_p(u))(a') &= \mu'(a')M^*(\widehat{q}_{a'})(t^G_p(u)(q(a'))) \\ &= \mu'(a')M^*(\widehat{q}_{a'})(\mu(q(a'))M^*(\widehat{p}_{q(a')})(u(p(q(a'))))) \\ &= \nu(a')M^*(\widehat{p\circ q}_{a'})(u((p\circ q)(a'))) \\ &= t^G_{p\circ q}(u)(a)\,. \end{aligned}$$

■

Recall that a Mackey functor M for G is said to be *homological* if whenever $H \subset K \subset G$ and $q : G/H \longrightarrow G/K$ is the quotient function, then one has

$$M_*(q)M^*(q) = [K : H]\,,$$

that is, this composite is multiplication by the index of H in K in the group $M(G/K)$. We have the following result.

Proposition 2.23 *Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ and let M be a homological Mackey functor for G. Then one has that the composite*

$$p^G_* \circ t^G_p : F^G(X, M) \longrightarrow F^G(X, M)$$

is multiplication by n.

Proof: Let $\gamma_x(l) \in F^G(X, M)$ be a generator, and let $\{[a_\iota] \mid \iota \in \mathfrak{I}\} = p^{-1}(x)/G_x$. Since M is homological, $M_*(\widehat{p}_{a_\iota})M^*(\widehat{p}_{a_\iota})$ is multiplication by the index $[G_x : G_{a_\iota}]$. Moreover, the orbit of $a_\iota \in p^{-1}(x)$ under the action G_x has exactly $[G_x : G_{a_\iota}]$ elements. Since the multiplicity function μ is G-invariant, we have that $\sum_{\iota\in\mathfrak{I}} \mu(a_\iota)[G_x : G_{a_\iota}] = n$. Hence

$$\begin{aligned} p^G_* t^G_p(\gamma^G_x(l)) &= p^G_* \left(\sum_{\iota\in\mathfrak{I}} \mu(a_\iota)\gamma^G_{a_\iota} M^*(\widehat{p}_{a_\iota})(l) \right) \\ &= \sum_{\iota\in\mathfrak{I}} \mu(a_\iota)\gamma^G_x M_*(\widehat{p}_{a_\iota})M^*(\widehat{p}_{a_\iota})(l) \\ &= \left(\sum_{\iota\in\mathfrak{I}} \mu(a_\iota)[G_x : G_{a_\iota}] \right) \gamma^G_x(l) = n\gamma^G_x(l)\,. \end{aligned}$$

■

3 The transfer for coefficients in a G-module

In this section we shall define the concept of ramified covering G-map and study its transfer in the topological abelian groups $F^G(X,L)$ with coefficients in a G-module L. We shall work here in the category of k-*spaces* (see e.g. [10]), which we briefly described in Section 3 of [3]. The next definition, puts in the the topological setting the concept of an n-fold G-function with multiplicity.

Definition 3.1 Let E and X be G-spaces. An *n-fold ramified covering G-map* is a continuous G-map $p : E \longrightarrow X$ together with a *multiplicity function* $\mu : E \longrightarrow N$, such that the following hold:

(i) The fibers $p^{-1}(x)$ are finite for each $x \in X$.

(ii) For each $x \in X$, $\sum_{a\in p^{-1}(x)} \mu(a) = n$.

(ii) The map $\varphi_p : X \longrightarrow \mathrm{SP}^n E = E^n/\Sigma_n$, given by

$$\varphi_p(x) = \langle \underbrace{a_1,\dots,a_1}_{\mu(a_1)},\dots,\underbrace{a_m,\dots,a_m}_{\mu(a_m)} \rangle ,$$

where $p^{-1}(x) = \{a_1,\dots,a_m\}$, is continuous.

(iv) μ is G-invariant.

Notice that by (iv), the map φ_p is G-equivariant. We can always assume that the ramified covering G-map is pointed (see Remark 1.4). This definition in the nonequivariant case was given by Smith [8] and it includes ordinary covering maps with finitely many leaves.

Proposition 3.2 *The family of ramified covering G-maps has the following properties:*

(a) *If $p_\alpha : E_\alpha \longrightarrow X$, $\alpha = 1,\dots,k$, are ramified covering G-maps with multiplicity functions μ_α, then*

$$p : \bigvee_{\alpha=1}^{k} E_\alpha \longrightarrow X$$

given by $p|_{E_\alpha} = p_\alpha$ is an $\sum_{\alpha=1}^{k} n_\alpha$-fold ramified covering G-map with multiplicity function μ given by $\mu|_{E_\alpha} = \mu_\alpha$.

(b) *If $q : E' \longrightarrow E$ is an n'-fold ramified covering G-map with multiplicity function μ', and $p : E \longrightarrow X$ is an n-fold ramified covering G-map with multiplicity function μ, then $p \circ q : E' \longrightarrow X$ is an (nn')-fold ramified covering G-map with multiplicity function ν, where $\nu(a') = \mu'(a')\mu(q(a'))$.*

(c) *If X is a G-space, then the projection $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ is an n-fold ramified covering G-map.*

Proof: By 2.18 p is a $(\sum_{\alpha=1}^{k} n_\alpha)$-fold G-function with multiplicity. Thus we only have to prove that $\varphi_p : X \longrightarrow \mathrm{SP}^{\Sigma_{\alpha=1}^{k} n_\alpha}$ is continuous. This follows from the commutativity of the next diagram:

$$\begin{tikzcd}
X \arrow[r, "(\varphi_{p_\alpha})"] \arrow[dr, dashed, "\varphi_p"'] & \prod_{\alpha=1}^{k} \mathrm{SP}^{n_\alpha} E_\alpha \arrow[d, dashed] & \prod_{\alpha=1}^{k} E_\alpha^{n_\alpha} \arrow[l, two heads, "q"'] \arrow[d, hook, "\rho"] \\
& \mathrm{SP}^{\Sigma n_\alpha}(\bigvee_{\alpha=1}^{k} E_\alpha) & (\coprod_{\alpha=1}^{k} E_\alpha)^{\Sigma n_\alpha} \arrow[l]
\end{tikzcd},$$

where ρ is the inclusion in the corresponding summand, and q is an identification.

By [2, 4.20], the composite $p \circ q$ in an (nn')-fold ramified covering map. Thus (b) follows from this and 2.21.

By [8], $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ is an n-fold ramified covering map. Thus (c) follows from 1.3. ■

Remark 3.3 Note that in [8], the setting is the category of topological spaces. By [1, Prop. 3.4], its definitions are equivalent to the ones herein, which are given in the setting of k-spaces.

Since an n-fold ramified covering G-map is in particular an n-fold G-function with multiplicity, we have by the previous section, a transfer homomorphism

$$t_p^G : F^G(X, M) \longrightarrow F^G(E, M)\,.$$

Let M be a Mackey functor. When Y is a k-space, the groups $F(Y, M)$ and $F^G(Y, M)$ defined in Section 1 have a topology given by making use of the geometric realization of the singular simplicial set $\mathcal{S}(Y)$ of Y, as done in [3]. With this topology, these are topological groups in the category of k-spaces. In what follows, we shall analyze under which conditions, the transfer t_p^G is continuous. We first consider the case when the Mackey functor is determined by a G-module.

Given a G-module L, one may define a Mackey functor M_L as follows:

$$M_L(G/H) = L^H\,.$$

If $H \subset K$ and $q : G/H \longrightarrow G/K$ is the quotient map, then

$$M_{L*}(q) : L^H \longrightarrow L^K \quad \text{is given by} \quad M_{L*}(q)(l) = \sum_{i=1}^{r} k_i l\,,$$

where $K/H = \{[k_i] \mid i = 1, \ldots, r\}$. Furthermore,

$$M_L^*(q) : L^K \longrightarrow L^H \quad \text{is the inclusion.}$$

On the other hand, let $R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$ by right translation by g^{-1}. Then

$$M_{L*}(R_{g^{-1}}) : L^H \longrightarrow L^{gHg^{-1}} \quad \text{is given by} \quad M_{L*}(R_{g^{-1}})(l) = gl\,.$$

Moreover,

$$M_L^*(R_{g^{-1}}) : L^{gHg^{-1}} \longrightarrow L^H \quad \text{is given by} \quad M_L^*(R_{g^{-1}})(l) = g^{-1}l\,.$$

To study the continuity of $t_p^G : F^G(X, M_L) \longrightarrow F^G(E, M_L)$, we shall consider an alternative topology for the groups, as was studied in [4].

Let Y be a pointed G-space. Notice that the G-set $\widehat{M_L} = L$. Thus the group $F^G(Y, M_L)$ consists of G-equivariant functions $u : Y \longrightarrow L$ such that $u(*) = 0$ and $u(y) = 0$ for all but a finite number of elements $y \in Y$. To simplify the notation, we denote this group by $F^G(Y, L)$. Given a pointed map $f : X \longrightarrow Y$, the homomorphism $f_*^G : F^G(X, M_L) \longrightarrow F^G(Y, M_L)$ has a very simple description when it is seen as a homomorphism $F^G(X, L) \longrightarrow F^G(Y, L)$; namely we have the following.

Lemma 3.4 *Let $f_\bullet^G : F^G(X, L) \longrightarrow F^G(Y, L)$ be given by $f_\bullet^G(\sum_{x\in X} l_x x) = \sum_{x\in X} l_x f(x)$. Then $f_\bullet^G = f_*^G$.*

Proof: Let $G/G_{f(x)} = \{[g_1], \ldots, [g_m]\}$ and $G_{f(x)}/G_x = \{[h_1], \ldots, [h_r]\}$. Hence $G/G_x = \{[g_i h_j] \mid i = 1, \ldots, m;\ j = 1, \ldots, r\}$. Recall that $M_*(G/H) = L^H$ for all H and that $M_*(R_{g^{-1}})(l) = gl$. Since $h_j \in G_{f(x)}$, we have that

$$\begin{aligned} f_\bullet^G(\gamma_x^G(l)) &= f_\bullet^G\left(\sum_{i,j} M_*(R_{(g_i h_j)^{-|}})(l)(g_i h_j x)\right) \\ &= f_\bullet^G\left(\sum_{i,j} (g_i h_j l)(g_i h_j x)\right) \\ &= \sum_{i,j} ((g_i h_j) l)(g_i f(x))\,. \end{aligned}$$

On the other hand,

$$\begin{aligned}
f_*^G(\gamma_x^G(l)) &= \gamma_{f(x)}^G M_{L*}(\widehat{f}_x)(l) \\
&= \gamma_{f(x)}^G \left(\sum_{j=1}^{r} h_j l \right) \\
&= \sum_{j=1}^{r} \left(\sum_{i=1}^{m} M_{L*}(R_{g_i^{-1}})(h_j l)(g_i f(x)) \right) \\
&= \sum_{i,j} (g_i(h_j l))(g_i f(x))\,.
\end{aligned}$$

Thus the result follows. ∎

As already remarked, the group $F^G(Y,L)$ has a natural topology that was studied in [4]. This topology is defined as follows. Consider the group $F(Y,L)$ of functions $u : Y \longrightarrow L$ such that $u(*) = 0$ and $u(y) = 0$ for all but a finite number of elements $y \in Y$. This is filtered by subsets $F_r(Y,L)$ of those functions which are nonzero in at most r points of Y. We give to these sets the quotient topology induced by the map $(L \times Y)^r \twoheadrightarrow F_r(Y,L)$ given by $(l_1, y_1, \ldots, l_r, y_r) \mapsto l_1 y_1 + \cdots + l_r y_r$, where the product $(L \times Y)^r$ has the k-topology. Then $F(Y,L)$ has the union topology and $F^G(Y,L) \subset F(Y,L)$ has the subspace topology. From now on the notation $F^G(Y,L)$ will mean the group with this topology. The topology of $F^G(Y, M_L)$ is in general finer than this topology, as will be shown in 4.5 below.

We have the following.

Proposition 3.5 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map with multiplicity function $\mu : E \longrightarrow \mathbb{N}$. Then the transfer $t_p^G : F^G(X,L) \longrightarrow F^G(E,L)$, is continuous.*

Proof: First notice that if $l \in M_L(G/G_a) = L^{G_a} \subset L$, $a \in E$, then $M_L^*(\widehat{p}_a)(l) = l$. Hence we have the homomorphism $t_p : F(X,L) \longrightarrow F(E,L)$ given by

$$t_p(lx) = \sum_{\iota=1}^{m} \mu(a_\iota) l a_\iota\,,$$

where $p^{-1}(x) = \{a_\iota \mid \iota = 1, \ldots, m\}$. The transfer is, as before, the restriction

$$t_p^G = t_p|_{F^G(X,L)} : F^G(X,L) \longrightarrow F^G(E,L)\,.$$

Consider the map $\delta : L \times X \longrightarrow F_n(E,L)$ given by $\delta(l,x) = t_p(lx)$ and $\alpha : L \times X \longrightarrow (L \times E^n)/\Sigma_n$ given by

$$\alpha(l,x) = \langle \underbrace{(l,a_1),\ldots,(l,a_1)}_{\mu(a_1)},\ldots,\underbrace{(l,a_m),\ldots,(l,a_m)}_{\mu(a_m)} \rangle\,,$$

where $p^{-1}(x) = \{a_1,\ldots,a_m\}$. Let $j_l : E^n/\Sigma_n \longrightarrow (L \times E)^n/\Sigma_n$ be given by $j_l\langle a_1,\ldots,a_n\rangle = \langle (l,a_1),\ldots,(l,a_n)\rangle$. If $i_l : X \longrightarrow L \times X$ is the inclusion at level l, then $\alpha \circ i_l = j_l \circ \varphi_p$. Hence α is continuous. Notice that the identification $(L \times E)^n \longrightarrow F_n(E,L)$ factors as the composite

$$(L \times E)^n \twoheadrightarrow (L \times E)^n/\Sigma_n \overset{\rho_n}{\twoheadrightarrow} F_n(E,L)\,,$$

where $\rho_n\langle (l_1,a_1),\ldots,(l_n,a_n)\rangle = \sum_{i=1}^n l_i a_i$. Therefore, ρ_n is continuous.

Since $\rho_n \circ \alpha = \delta$, δ is continuous. In order to see that $t_p|_{F_r(X,L)}$ is continuous, consider the diagram

$$\begin{tikzcd}
(L \times X)^r \arrow[r, "\delta^r"] \arrow[d, two heads] & (F_n(E,L))^r \arrow[d, "\mathrm{sum}"] \\
F_r(X,L) \arrow[r, "t_p|_{F_r(X,L)}"'] & F(E,L)\,,
\end{tikzcd}$$

where sum is given by the operation on $F(E,L)$, which is continuous. Hence t_p is continuous, and since $F^G(X,L)$ has the subspace topology, then t_p^G is also continuous, as desired. ■

4 Change of coefficients and the transfer for homological Mackey functors

In this section we show that a morphism of Mackey functors $\xi : M' \longrightarrow M$ induces a continuous homomorphism of topological groups $\xi_\diamond : F^G(X;M') \longrightarrow F^G(X;M)$ for any G-space X.

Definition 4.1 Recall that a *morphism* $\xi : M \longrightarrow M'$ is a natural transformation of both the covariant and the contravariant parts of M and M'; it is an *epimorphism* if for each object it is a group epimorphism. Define

$$\xi_\diamond : F^G(X;M) \longrightarrow F^G(X;M')$$

by $\xi_\diamond(u)(x) = \xi_{G/G_x}(u(x)) \in M'(G/G_x)$. Note that since u is equivariant and ξ is natural, then $\xi_\diamond(u)$ is also equivariant, and therefore it is well defined.

Proposition 4.2 *The homomorphism $\xi_\diamond : F^G(X,M) \longrightarrow F^G(X,M')$ is natural in X.*

Proof: From the definition it is clear that $\xi_\diamond(\gamma_x^G(l)) = \gamma_x'^G(\xi_{G/G_x}(l))$ for any generator $\gamma_x^G(l) \in F^G(X,M)$. Let now $f : X \longrightarrow Y$ be a pointed G-function. Then

$$f_*^G \xi_\diamond(\gamma_x^G(l)) = f_*^G \gamma_x'^G(\xi_{G/G_x}(l)) = \gamma_{f(x)}'^G M_*(\widehat{f}_x)(\xi_{G/G_x}(l)) = \xi_\diamond f_*^G(\gamma_x^G(l))\,.$$

■

The next follows immediately from the previous proposition.

Corollary 4.3 *Let K be a simplicial G-set. Then $\xi_\diamond : F^G(K,M) \longrightarrow F^G(K,M)$ is a homomorphism of simplicial groups.* ■

Proposition 4.4 *The homomorphism $\xi_\diamond$ is continuous.*

Proof: Recall [3] that for any Mackey functor, the topology of $F^G(X,M)$ is the identification topology given by the epimorphism

$$|F^G(\mathcal{S}(X),M)| \cong F^G(|\mathcal{S}(X)|,M) \overset{\rho_{X*}^G}{\twoheadrightarrow} F^G(X,M)\,,$$

where $\mathcal{S}(X)$ is the singular simplicial set associated to X and $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is given by $\rho_X[\sigma,t] = \sigma(t)$. Here $\sigma \in \mathcal{S}_k(X)$ and $t \in \Delta^k$.

Consider the following commutative diagram

$$\begin{tikzcd}
|F^G(\mathcal{S}(X),M)| \arrow[r, "|\xi_\diamond|"] \arrow[d, two heads] & |F^G(\mathcal{S}(X),M')| \arrow[d, two heads] \\
F^G(X,M) \arrow[r, "\xi_\diamond"'] & F^G(X,M')\,.
\end{tikzcd}$$

Since by 4.2, $\xi_\diamond : F^G(\mathcal{S}(X),M) \longrightarrow F^G(\mathcal{S}(X),M')$ is a homomorphism of simplicial groups, $|\xi_\diamond|$ on the top is continuous. The vertical arrows are identifications, thus $\xi_\diamond$ on the bottom is also continuous. ■

Given a G-module L, in what follows, we shall compare the topologies of the identical groups $F^G(X,L)$ (defined in [4]) and $F^G(X,M_L)$ (defined in [3]), where M_L is the Mackey functor associated to L.

Proposition 4.5 *The identity $F^G(X,M_L) \longrightarrow F^G(X,L)$ is continuous. Thus the topology on the left-hand side is finer than that on the right.*

Proof: Let first K be any simplicial G-set. Notice that by Lemma 3.4, the simplicial groups $F^G(K,L)$ and $F^G(K,M_L)$ are identical. Hence the realizations $|F^G(K,L)|$ and $|F^G(K,M_L)|$ are identical topological groups. In [4, Cor. 2.6] one proves that the topological groups $F^G(|K|,L)$ and $|F^G(K,L)|$ are (topologically) isomorphic, and in [3, Prop. 2.6 (b)] it is proved that the topological groups $F^G(|K|,M)$ and $|F^G(K,M)|$ are also (topologically) isomorphic, in particular for $M = M_L$. Let X be any G-space and let $\mathcal{S}(X)$ be its asociated singular simplicial G-set and $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ the canonical surjection introduced above. There is a commutative diagram

$$\begin{array}{ccc} F^G(|\mathcal{S}(X)|, M_L) & \xrightarrow{\cong} & F^G(|\mathcal{S}(X)|, L) \\ {\scriptstyle \rho^G_{X*}} \downarrow & & \downarrow {\scriptstyle \rho^G_{X*}} \\ F^G(X, M_L) & \longrightarrow & F^G(X, L)\,, \end{array}$$

where the top arrow is a homeomorphism, the vertical arrow on the left is an identification (by definition) and the vertical arrow on the right is continuous. Hence the identity arrow on the bottom is continuous, as desired. ■

In the rest of this section, we analyze the continuity of the transfer for ramified covering G-maps in a convenient category of topological spaces.

Definition 4.6 Let X be a topological space. We say that X is a *ρ-space* if the map $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is an identification. Since quotients of k-spaces are k-spaces, any ρ-space is a k-space. There is a functor from $\mathcal{T}\mathrm{op}$ to the (full) subcategory of ρ-spaces as follows. For any topological space X give X the identification topology induced by ρ_X and denote this new space by $\rho(X)$. Thus the identity $\mathrm{id} : \rho(X) \longrightarrow X$ is continuous. We denote by ρ-$\mathcal{T}\mathrm{op}$ the full subcategory of $\mathcal{T}\mathrm{op}$ whose objects are the ρ-spaces.

We have the following.

Proposition 4.7 *The assignment $X \mapsto \rho(X)$ is a functor $\rho : \mathcal{T}\mathrm{op} \longrightarrow \rho$-$\mathcal{T}\mathrm{op}$, such that if X is a ρ-space, then $\rho(X) = X$. Thus ρ-$\mathcal{T}\mathrm{op}$ is a reflective subcategory of $\mathcal{T}\mathrm{op}$.*

Proof: Let $f : X \longrightarrow Y$ be a continuous map in $\mathcal{T}\mathrm{op}$. The continuity of $\rho(f) = f$ follows from the commutativity of the next diagram

$$\begin{array}{ccc} |\mathcal{S}(X)| & \xrightarrow{|\mathcal{S}(f)|} & |\mathcal{S}(Y)| \\ {\scriptstyle \rho_X} \downarrow & & \downarrow {\scriptstyle \rho_Y} \\ \rho(X) & \xrightarrow[f]{} & \rho(Y)\,, \end{array}$$

since ρ_X is an identification. The second part is clear. ∎

In the following result we give properties of the ρ-spaces that we shall need.

Proposition 4.8 *The category ρ-$\mathcal{T}\mathrm{op}$ has the following properties:*

(a) *Identification spaces of ρ-spaces are ρ-spaces.*

(b) *It has arbitrary coproducts.*

(c) *It has finite products.*

(d) *The product of two identifications is an identification.*

Proof: (a) follows from the commutative diagram

$$\begin{tikzcd} |\mathcal{S}(X)| \arrow[r, "|\mathcal{S}(q)|"] \arrow[d, two heads, "\rho_X"'] & |\mathcal{S}(X')| \arrow[d, "\rho_{X'}"] \\ X \arrow[r, two heads, "q"'] & X', \end{tikzcd}$$

since if q and ρ_X are identifications, so is $\rho_{X'}$.

To see (b), let $\{X_\alpha\}$ be a family of ρ-spaces. Let $i_\alpha : X_\alpha \longrightarrow \coprod X_\alpha$ be the canonical inclusion. By 4.7, we have a continuous map $X_\alpha = \rho(X_\alpha) \xrightarrow{\rho(i_\alpha)} \rho(\coprod X_\alpha)$, and by the universal property of the coproduct, the identity $\coprod X_\alpha \longrightarrow \rho(\coprod X_\alpha)$ is continuous. Therefore $\rho(\coprod X_\alpha) = \coprod X_\alpha$.

(c) is a consequence of the following two facts: (i) $\mathcal{S}(X \times Y) = \mathcal{S}(X) \times \mathcal{S}(Y)$ and (ii) $|\mathcal{S}(X) \times \mathcal{S}(Y)| \approx |\mathcal{S}(X)| \times |\mathcal{S}(Y)|$.

(d) follows from the corresponding fact in k-spaces. ∎

Proposition 4.9 *The category of ρ-spaces contains the CW-complexes.*

Proof: First notice that since Δ^n is the realization of the simplicial set $\Delta_n = \mathrm{Mor}_\Delta(-, \overline{n})$, then $\rho_{\Delta^n} : |\mathcal{S}(\Delta^n)| \longrightarrow \Delta^n$ is an identification (see [3, Lem. 3.5]). Hence Δ^n is a ρ-space.

Let X be a CW-complex. Define

$$\varphi : \coprod_{\alpha,n} \Delta^n_\alpha \twoheadrightarrow X$$

by $\varphi|_{\Delta^n_\alpha} = \varphi^n_\alpha : \Delta^n_\alpha \longrightarrow X$, the characteristic map of each cell of X. Then φ is an identification, and since Δ^n_α is a ρ-space for each α and each n, so is also $\coprod \Delta^n_\alpha$ by property (b) of 4.8. Hence, by property (a), X is a ρ-space. ∎

Consider the following.

Lemma 4.10 *Let X be a G-space in the category of ρ-spaces and let L be a G-module. Then the topological groups $F^G(X, M_L)$ and $F^G(X, L)$ are equal.*

Proof: Since $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is an identification, then $\coprod_k (L \times |\mathcal{S}(X)|)^k \longrightarrow \coprod_k (L \times |X|)^k$ is an identification too. Therefore $F^G(|\mathcal{S}(X)|, L) \longrightarrow F^G(X, L)$ is also an identification. As already remarked in the proof of 4.5, $F^G(|\mathcal{S}(X)|, M_L)$ and $F^G(|\mathcal{S}(X)|, L)$ are equal. Hence the topological groups $F^G(X, M_L)$ and $F^G(X, L)$ are equal too. ■

Lemma 4.11 *Let X be a G-space. If $\xi : M' \longrightarrow M$ is an epimorphism of Mackey functors, then $\xi_\diamond : F^G(X, M) \longrightarrow F^G(X, M')$ is an identification.*

Proof: We start by proving that $\xi_\diamond : F^G(X, M) \longrightarrow F^G(X, M')$ is surjective. To see this recall first the epimorphisms $\beta_X : F(X, M) \longrightarrow F^G(X, M)$ given on generators by $\beta_X(lx) = \gamma_x^G(l)$ in [3]. One has the following commutative diagram

$$\begin{tikzcd} F(X,M) \arrow[r, "\xi_\diamond"] \arrow[d, two heads, "\beta_X"'] & F(X,M') \arrow[d, two heads, "\beta'_X"] \\ F^G(X,M) \arrow[r, "\xi_\diamond"'] & F^G(X,M')\,. \end{tikzcd}$$

Since the horizontal arrow on the top is obviously surjective, so is the one on the bottom.

In order to see that $\xi_\diamond : F^G(X, M) \longrightarrow F^G(X, M')$ is an identification, note that the commutative diagram

$$\begin{tikzcd} \coprod_k F^G(\mathcal{S}_k(X), M) \times \Delta^k \arrow[r, two heads, "\sqcup\xi_\diamond\times\mathrm{id}"] \arrow[d, two heads] & \coprod_k F^G(\mathcal{S}_k(X), M') \times \Delta^k \arrow[d, two heads] \\ |F^G(\mathcal{S}(X), M)| \arrow[r, "|\xi_\diamond|"'] & |F^G(\mathcal{S}(X), M')| \end{tikzcd}$$

implies that $|\xi_\diamond|$ on the bottom is an identification, because $\sqcup\xi_\diamond\times\mathrm{id}$ on the top is an identification, since for each k, by the first part of this proof, $\xi_\diamond$ is a surjective map (of discrete spaces). Therefore, $\xi_\diamond : F^G(|\mathcal{S}(X)|, M) \twoheadrightarrow F^G(|\mathcal{S}(X)|, M')$ is an identification as well. Finally, the commutativity of

$$\begin{tikzcd} F^G(|\mathcal{S}(X)|, M) \arrow[r, two heads, "\xi_\diamond"] \arrow[d, two heads] & F^G(|\mathcal{S}(X)|, M') \arrow[d, two heads] \\ F^G(X,M) \arrow[r, "\xi_\diamond"'] & F^G(X,M') \end{tikzcd}$$

implies that $\xi_\diamond$ on the bottom is an identification. ■

The main result in this section is the next.

Theorem 4.12 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map in the category of ρ-spaces, and let M be a homological Mackey functor. Then $t_p^G : F^G(X, M) \longrightarrow F^G(E, M)$ is continuous.*

Proof: Since M is homological, by [9, Thm. (16.5)(i)], there exists a G-module L and an epimorphism of Mackey functors $\xi : M_L \longrightarrow M$. By Lemma 4.11, the induced epimorphism $\xi_\diamond : F^G(Y, M_L) \longrightarrow F^G(Y, M)$ is an identification for any G-space Y. Moreover, by Lemma 4.10, for any ρ-space Y, the groups $F^G(Y, M_L)$ and $F^G(Y, L)$ are equal. Thus we have a commutative diagram

$$\begin{tikzcd} F^G(X,L) \arrow[r, "t_p^G"] \arrow[d] & F^G(E,L) \arrow[d] \\ F^G(X,M) \arrow[r, dashed, "t_p^G"'] & F^G(E,M)\,, \end{tikzcd}$$

where the vertical arrows are identifications and the horizontal arrow on the top is continuous by Proposition 3.5. Therefore the horizontal arrow on the bottom is also continuous. ■

The following is a **homotopy invariance** property, whose assumptions depend on the kind of coefficients, namely, if one has coefficients in a G-module L, then X, Y, and E may be any k-spaces; otherwise, if M is an arbitrary homological Mackey functor, then X, Y, and E must be ρ-spaces.

Proposition 4.13 *Let $p : E \longrightarrow X$ be a ramified covering G-map. If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps and one has the following two pullback diagrams*

$$\begin{tikzcd} f_0^*(E) \arrow[r, "\widetilde{f}_0"] \arrow[d, "p_0"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f_0"'] & X \end{tikzcd} \qquad \begin{tikzcd} f_1^*(E) \arrow[r, "\widetilde{f}_1"] \arrow[d, "p_1"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f_1"'] & X\,, \end{tikzcd}$$

then

$$\widetilde{f}_{0*}^G \circ t_{p_0}^G \simeq \widetilde{f}_{1*}^G \circ t_{p_1}^G : F^G(Y, M) \longrightarrow F^G(E, M)\,.$$

Proof: By the pullback property one has that

$$\widetilde{f}^G_{0*} \circ t^G_{p_0} = t^G_p \circ f^G_{0*} \quad \text{and} \quad t^G_p \circ f^G_{1*} = \widetilde{f}^G_{1*} \circ t^G_{p_1}\,.$$

Moreover, from [3, Prop. 3.14(b)], one has $f^G_{0*} \simeq f^G_{1*}$. Thus the assertion follows. ■

Finally, we have the following **invariance under change of coefficients**.

Proposition 4.14 *Let $p : E \longrightarrow X$ be a ramified covering G-map, where E and X are ρ-spaces. If $\xi : M \longrightarrow M'$ is a morphism of homological Mackey functors, then one has the following commutative diagram:*

$$\begin{tikzcd} F^G(X,M) \arrow[r, "\xi_\diamond"] \arrow[d, "t^G_p"'] & F^G(X,M') \arrow[d, "t^G_p"] \\ F^G(E,M) \arrow[r, "\xi_\diamond"'] & F^G(E,M')\,. \end{tikzcd}$$

Proof: Take $u \in F^G(X,M)$, Then by the naturality of ξ with respect to M^*, we have

$$\begin{aligned} \xi_\diamond t^G_p(u)(a) &= \xi_{G/G_a}(\mu(a)M^*(\widehat{p}_a)(u(p(a)))) \\ &= \mu(a)\xi_{G/G_a}(M^*(\widehat{p}_a)(u(p(a)))) \\ &= \mu(a)M'^*((\widehat{p}_a)\xi_{G/G_a}(u(p(a)))) = t^G\xi_\diamond(u)(a) \end{aligned}$$

■

5 Transfers in Bredon-Illman homology

In this section we shall define the transfer τ_p in Bredon-Illman homology with coefficients in a homological Mackey functor. The transfer has the following properties:

- Naturality (see 2.7),
- Pullback (see 2.14),
- Normalization (see 2.16),
- Additivity (see 2.19),
- Quasiadditivity (see 2.20),
- Functoriality (see 2.22),

- Homotopy invariance (see 4.13),
- Invariance under change of coefficients (see 4.14), and
- The composite

$$p_* \circ \tau_p : \widetilde{H}_*^G(X; M) \longrightarrow \widetilde{H}_*^G(X; M)$$

  is multiplication by n (see 2.23).

Theorem 5.1 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map in the category of ρ-spaces of the same homotopy type of a G-CW-complex, and let M be a homological Mackey functor for G. Then there exists a transfer*

$$\tau_p : \widetilde{H}_*^G(X; M) \longrightarrow \widetilde{H}_*^G(E, M)$$

with all properties given above.

Proof: By [4, Thm. 1.2], for all pointed G-spaces Y of the same homotopy type of a G-CW-complex, there is a natural isomorphism

$$\widetilde{H}_q^G(Y; M) \longrightarrow \pi_q(F^G(Y, M)) .$$

Therefore, by Theorem 4.12, there is a transfer homomorphism

$$\tau_p : \widetilde{H}_*^G(X; M) \longrightarrow \widetilde{H}_*^G(E, M)$$

corresponding to the homomorphism induced by t_p^G in the homotopy groups. The properties follow immediately from the corresponding results for t_p^G. ■

Remark 5.2 By Proposition 3.5, if we take Bredon-Illman homology with coefficients in a G-module L, then the previous result holds in the more general category of k-spaces.

Corollary 5.3 *Let $p : E \longrightarrow X$ be an n-fold ramified covering G-map in the category of ρ-spaces, such that G acts freely on the total space E, and let M be a homological Mackey functor for G such that multiplication by n is an isomorphism in each of the groups $M(G/G_x)$ for $x \in X$. Then the transfer*

$$\tau_p : H_*^G(X; M) \longrightarrow H_*^G(E; M(G))$$

is a split monomorphism and thus the homology group $H_*^G(X; M)$ is a direct summand of $H_*^G(X; M(G))$.

Proof: Since the action of G on E is free, $F^G(E^+, M) = F^G(E^+, M(G))$. Therefore

$$\widetilde{H}_*^G(X^+, M) = \widetilde{H}_*^G(E^+, M(G))\,.$$

By (b) in the previous theorem, the composite

$$\widetilde{H}_*^G(X^+, M) \xrightarrow{\tau_{|p|}} \widetilde{H}_*^G(E^+, M(G)) \xrightarrow{|p|_*} \widetilde{H}_*^G(X^+, M)$$

is multiplication by n, and thus an isomorphism. Hence the result follows. ∎

Remark 5.4 The transfer for any ramified covering G-map cannot be given by a stable transfer map, which has the naturality, the normalization, and the quasiadditivity (see 2.20) properties, because otherwise there would be a transfer for ramified covering G-maps in any representable (cohomology) theory. But by [2, Thm. 4.8], if there is such a transfer, then the theory must be given by a product of Eilenberg-Mac Lane spaces. (One can construct such a stable transfer for n-fold ramified covering maps in the nonequivariant case [7], provided that one inverts $n!$.)

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