Simplicial ramified covering G-maps and equivariant homology

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Abstract Let G be a finite group. The main objective of this paper is to study ramified covering G-maps of simplicial sets and of simplicial complexes, and to construct a transfer for them in Bredon-Illman equivariant homology with coefficients in an arbitrary Mackey functor M. We show that this transfer has the usual properties of a transfer.

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Contents

0 Introduction

Let G be a finite group. In [6] we constructed a transfer for ramified covering G-maps in Bredon-Illman equivariant homology with coefficients in a homological Mackey functor M. The main objective of this paper is to study G-equivariant ramified covering maps of simplicial sets and of simplicial complexes, and to

*Corresponding author, Phone: ++5255-56224489, Fax ++5255-56160348. This author was partially supported by PAPIIT grant IN105106-3 and by CONACYT grant 58049. Webpage: http://www.matem.unam.mx/cprieto construct a transfer for them in Bredon-Illman equivariant homology with coefficients in an arbitrary Mackey functor M. We show that this transfer has the usual properties of a transfer.

In order to define the transfer, we shall use the homotopical definition of Bredon-Illman homology $H_*^G(-;M)$ given in [2]. Namely, to each pointed G-space X and each Mackey functor M for G, we associate a topological abelian group $F^G(X,M)$ such that $\pi_q(F^G(X,M)) \cong \widetilde{H}^G(X;M)$.

The paper is organized as follows. In Section 1, we recall the transfer $t_p : F^G(X,M) \longrightarrow F^G(E,M)$ for the so-called n-fold G-functions with multiplicity (in the category of G-sets) defined in [6], and recall its main properties, namely the pullback property, additivity, normalization, and that when M is homological, then its composite with p_*^G is multiplication by n. In order to deal with the topological case, one needs to study G-equivariant ramified covering maps in the category of simplicial sets. This we do in Section 2, where we introduce the concept of G-equivariant simplicial (special) ramified covering map p (2.1) and prove several properties of the class of those p, like the fact that it is closed under pullbacks, that their geometric realization is a topological ramified covering G-map, etc. In Section 3, we define the transfer in the simplicial context and show that it is a simplicial map (3.2). This is the core of the paper, since as a consequence, we obtain the continuity of the topological transfer for the geometric realization $|p|$ of a special simplicial ramified covering map (3.4). Next, in Section 4, we give the definition of a (special) G-equivariant ramified covering map of simplicial complexes p (4.1), and in a natural way, we associate a ramified covering map of simplicial sets $K(p)$ (4.4) in such a way that the geometric realizations $|p|$ and $|K(p)|$ coincide. This allows us to prove that the transfer of the geometric realization of a special G-equivariant ramified covering map of simplicial complexes is continuous (4.6). At the end of this section, we pass to Bredon-Illman homology (applying the homotopy-group functors) and give in the homology setting the transfer and its properties.

1 The transfer in the category of G-sets

In this section we recall all the facts about the transfer at the level of G-sets, as we did in Section 1 of [6].

Definition 1.1 By an *n-fold G-function with multiplicity* we understand a G-function $p : E \longrightarrow X$ between G-sets with finite fibers, together with a G-invariant function $\mu : E \longrightarrow \mathbb{N}$, called *multiplicity function*, such that for each

$x \in X$,

$$\sum_{a \in p^{-1}(x)} \mu(a) = n\,.$$

We say that the n-fold G-function with multiplicity $p : E \longrightarrow X$ is *pointed* if the spaces E and X have base points, which are fixed under the G-action, and p is a pointed function. Associated to $p : E \longrightarrow X$ one has the G-function

$$\varphi_p : X \longrightarrow \mathrm{SP}^n E$$

given by

$$\varphi_p(x) = \langle \underbrace{a_1, \ldots, a_1}_{\mu(a_1)}, \ldots, \underbrace{a_r, \ldots, a_r}_{\mu(a_r)} \rangle\,.$$

Definition 1.2 Let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X'$ be n-fold G-functions with multiplicity functions μ and μ', respectively. A *morphism* from p to p' is a pair of G-functions $(\widetilde{f}, f)$ such that

(a) the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd},$$

(b) for each $x \in X$, the restriction $\widetilde{f}|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow p'^{-1}(f(x))$ is surjective,

(c) for each $x \in X$ and $a' \in p'^{-1}(f(x))$, one has the equality

$$\mu'(a') = \sum_{p(a)=x\,,\ \widetilde{f}(a)=a'} \mu(a)\,, \quad \text{and} \tag{1.3}$$

(d) for each $a \in E$ one has the formula

$$G_a = G_{p(a)} \cap G_{\widetilde{f}(a)} \tag{1.4}$$

for the isotropy groups.

There is a category whose objects are G-functions with multiplicity and its morphisms are just defined.

We have the next useful characterization of a morphism of G-functions with multiplicity [6, Prop. 2.4].

Proposition 1.5 *Let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X'$ be n-fold G-functions with multiplicity, and let $f : X \longrightarrow X$ and $\widetilde{f} : E \longrightarrow E'$ be G-functions such that $\widetilde{f} \circ p' = p \circ f$ and for $a \in E$, $G_a = G_{p(a)} \cap G_{\widetilde{f}(a)}$. Then $(\widetilde{f}, f)$ is a morphism from p to p' if and only if*

$$\varphi_{p'} \circ f = \mathrm{SP}^n \widetilde{f} \circ \varphi_p : X \longrightarrow \mathrm{SP}^n E'\,,$$

where the φs are as given in Definition 1.1. ■

EXAMPLES 1.6 The following are useful examples of morphisms.

(a) Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ, and let $f : Y \longrightarrow X$ be a G-function. Consider the pullback diagram

(1.7)
$$\begin{tikzcd} f^*E \arrow[r, "\widetilde{f}"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X\,, \end{tikzcd}$$

where $f^*E = Y \times_X E = \{(y, a) \mid f(y) = p(a)\}$, and q and $\widetilde{f}$ are the projections. Clearly, q is also an n-fold G-function with multiplicity μ' given by $\mu'(y, a) = \mu(a)$, since $\mu'(g(y, a)) = \mu'(gy, ga) = \mu(ga) = \mu(a) = \mu'(y, a)$. Consider the restriction of $\widetilde{f}$ from the fiber $q^{-1}(y)$ to the fiber $p^{-1}(f(y))$. This bijective function induces a surjective function

$$q^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}\,.$$

Clearly, conditions (a), (b), and (c) in the previous definition hold. Moreover, clearly $G_{(y,a)} = G_y \cap G_a$, thus condition (d) also holds. Hence $(\widetilde{f}, f)$ is a morphism from q to p.

(b) Let X be a G-set and consider the G-function $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ given by $\pi\langle x_1, \ldots, x_n; i\rangle = \langle x_1, \ldots, x_n\rangle$, where G acts on both sets diagonally and trivially on the set $\overline{n} = \{1, 2, \ldots, n\}$. Define $\mu : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathbb{N}$ by

$$\mu\langle x; i\rangle = \# x^{-1}(x(i))\,,$$

where one regards the ordered n-tuple $(x_1, \ldots, x_n)$ as a function $x : \overline{n} \longrightarrow X$. Then p is an n-fold G-function with multiplicity, since the sets $x^{-1}x(i)$ form a partition of the set $\overline{n}$. Furthermore, μ is clearly G-invariant. The function $\varphi_\pi : \mathrm{SP}^n X \longrightarrow \mathrm{SP}^n(X^n \times_{\Sigma_n} \overline{n})$ is given in this case by

$$\varphi_\pi\langle x_1, \ldots, x_n\rangle = \langle\langle x_1, \ldots, x_n; 1\rangle, \ldots, \langle x_1, \ldots, x_n; n\rangle\rangle\,.$$

Let X and Y be G-sets and let $f : X \longrightarrow Y$ be G-equivariant. We say that f is *n-permutable* if the equality

$$G_{\langle x_1,\ldots,x_n;i\rangle} = G_{\langle x_1,\ldots,x_n\rangle} \cap G_{\langle f(x_1),\ldots,f(x_n);i\rangle} \tag{1.8}$$

holds in terms of isotropy groups, where $\langle x_1,\ldots,x_n;i\rangle \in X^n \times_{\Sigma_n} \overline{n}$, $\langle x_1,\ldots,x_n\rangle \in \mathrm{SP}^n X$, and $\langle f(x_1),\ldots,f(x_n);i\rangle \in Y^n \times_{\Sigma_n} \overline{n}$. If $\pi : X^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n X$ and $\pi' : Y^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n Y$ are as above, then the pair of G-functions $(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}}, \mathrm{SP}^n f)$ is a morphism from π to π'. To see this, we use Proposition 1.5 above. Namely, one has to show that

$$\varphi_{\pi'} \circ \mathrm{SP}^n f = \mathrm{SP}^n(f^n \times_{\Sigma_n} \mathrm{id}_{\overline{n}}) \circ \varphi_\pi : \mathrm{SP}^n X \longrightarrow \mathrm{SP}^n(Y^n \times_{\Sigma_n} \overline{n}) .$$

This is done in [6, 2.5(b)] and we omit it here.

We now recall the definition of the groups $F(Y,M)$ and $F^G(Y,M)$ for a pointed G-set Y and a Mackey functor M for the group G. First one has the set $\widehat{M} = \cup_{H \subset G} M(G/H)$, which has a G-action given for $g \in G$ and $l \in M(G/H)$ by $g \cdot l = M_*(R_{g^{-1}})(l) \in M(G/gHg^{-1})$. Then $F(Y,M)$ consists of functions $u : Y \longrightarrow M$ such that $u(y) \in M(G/G_y)$, $u(*) = 0$, and $u(y) = 0$ for all but a finite number of elements $y \in Y$. The canonical generators of this group are functions denoted by ly given by

$$(ly)(y') = \begin{cases} l & \text{if } y' = y\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $l \in M(G/G_y)$ and $y \in Y - \{*\}$. The group $F(Y,M)$ has a natural (left) action of G given by defining $(gu)(y) = g \cdot u(g^{-1}y)$. Define $F^G(Y,M)$ as the subgroup of the functions u that are G-equivariant or, equivalently, the fixed points of $F(X,M)$ under the described G-action. The canonical generators of $F^G(Y,M)$ are functions denoted by $\gamma_y^G(l)$ given by

$$\gamma_y^G(l) = \sum_{j=1}^{m} M_*(R_{g_j^{-1}})(l)(g_j y)\,,$$

where $l \in M(G/G_y)$, $y \in Y - \{*\}$, and $G/G_y = \{[g_j] \mid j = 1,\ldots,m\}$. Given a pointed G-function $f : X \longrightarrow Y$, the homomorphism $f_*^G : F^G(X,M) \longrightarrow F^G(Y,M)$ is given on the generators by

$$f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G M_*(\widehat{f}_x)(l)\,,$$

where $\widehat{f}_x : G/G_x \longrightarrow G/G_{f(x)}$ is the canonical quotient function (see [2, 4] for details).

Definition 1.9 Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ, and let M be a Mackey functor. Define a homomorphism

$$t_p : F(X,M) \longrightarrow F(E,M)\,,$$

by

$$t_p(u)(a) = \mu(a) M^*(\widehat{p}_a) u(p(a))\,,$$

where $u \in F(X,M)$ and $a \in E$. If we assume that $u \in F^G(X,M)$, i.e., that $u(gx) = g \cdot u(x)$, then

$$\begin{aligned} t_p(u)(ga) &= \mu(ga) M^*(\widehat{p}_{ga})(u(p(ga))) \\ &= \mu(a) M^*(\widehat{p}_{ga})(g \cdot u(p(a))) \\ &= \mu(a) M^*(\widehat{p}_{ga}) M_*(R_{g^{-1}})(u(p(a))) \\ &= \mu(a) M_*(R_{g^{-1}}) M^*(\widehat{p}_a)(u(p(a))) \\ &= g \cdot (t_p(u)(a))\,, \end{aligned}$$

where the next to the last equality follows from the pullback property of the Mackey functor. Thus $t_p(u) \in F^G(E,M)$. Therefore, the homomorphism t_p restricts to a *transfer* homomorphism

$$t_p^G : F^G(X,M) \longrightarrow F^G(E,M)\,.$$

Remark 1.10 Let $p : E \longrightarrow X$ be a n-fold G-function with multiplicity μ. The isotropy group G_x acts on $p^{-1}(x)$ and the inclusion $p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$. Let $\{a_\iota\} \subset p^{-1}(x)$ be a set of representatives one for each G_x-orbit. Let $\gamma_x^G(l)$ be a generator of $F^G(X,M)$. Since the value of this function is zero on points which do not belong to the orbit Gx, and $\gamma_x^G(l)(x) = l$. One can give the transfer t_p^G on the generators $\gamma_x^G(l)$ by the formula

$$t_p^G(\gamma_x^G(l)) = \sum_{[a_\iota] \in p^{-1}(x)/G_x} \mu(a_\iota) \gamma_{a_\iota}^G(M^*(\widehat{p}_{a_\iota})(l))\,. \tag{1.5}$$

The next follows from what was done in [6, Section 2].

Proposition 1.6 *The transfer has the following properties: Naturality (with respect to morphisms of G-functions with multiplicity), Pullback, Normalization, Additivity, Quasiadditivity, Functoriality, Invariance under change of coefficients, and if M is homological, then the composite*

$$p_*^G \circ t_p^G : F^G(X,M) \longrightarrow F^G(X,M)$$

is multiplication by n. ∎

2 Equivariant ramified covering maps of simplicial sets

In this section, we shall understand by a G-equivariant simplicial ramified covering map. Our definition is based on the concept of a weighted map given by Friedlander and Mazur [8]. We show that these simplicial ramified covering G-maps have properties analogous to those proved by Smith [11] and Dold [7] for topological ramified covering maps.

Definition 2.1 Let $p : K \longrightarrow Q$ be a pointed simplicial function between pointed simplicial sets. We say that p is an *n-fold G-equivariant simplicial ramified covering map* if the following conditions hold:

1. For each m, $p_m : K_m \longrightarrow Q_m$ has finite fibers.
2. The function $d_i^K|_{p_m^{-1}(x)} : p_m^{-1}(x) \longrightarrow p_{m-1}^{-1}(d_i^Q(x))$ is surjective for all i.
3. There is a family of G-invariant *multiplicity functions* $\mu_m : K_m \longrightarrow \mathbb{N}$, such that:
   (a) For all $x \in Q_m$, one has $\sum_{a \in p_m^{-1}(x)} \mu_m(a) = n$.
   (b) $\mu_{m+1} \circ s_i^K = \mu_m : K_m \longrightarrow \mathbb{N}$.
   (c) For all $x \in Q_m$ and $a \in p_m^{-1}(x)$ one has $\mu_{m-1}(d_i^K(a)) = \sum_{\alpha=1}^{r} \mu_m(a_\alpha)$, where $\{a_1, \ldots, a_r\} = (d_i^K)^{-1}(d_i^K(a)) \cap p_m^{-1}(x)$.

We call p *special* if, furthermore, the following condition holds.

4. For each $a \in K_m$, one has $G_a = G_{p_m(a)} \cap G_{d_i^K(a)}$ for all i.

Remark 2.2 Properties 3 (a) and (b) imply that the restrictions $s_i^K| : p_m^{-1}(x) \longrightarrow p_{m+1}^{-1}(s_i^Q(x))$ are bijective.

Remark 2.3 The corresponding definition in the nonequivariant setting is given in [5].

Proposition 2.4 *Let $p : K \longrightarrow Q$ be a map of simplicial G-sets. Then p is a special n-fold G-equivariant simplicial ramified covering map if and only if p_m is an n-fold G-function with multiplicity μ_m and the pairs (d_i^K, d_i^Q), (s_i^K, s_i^Q) are morphisms of G-functions with multiplicity for all face and degeneracy operators.*

Proof: The conditions on the face operators d_i in Definition 2.1 are exactly the same as the conditions for the pair (d_i^K, d_i^Q) to be a morphism of G-functions with multiplicity. Moreover, since the degeneracy operators s_i are always injective, then condition 3(b) is equivalent to the condition of the pair (s_i^K, s_i^Q) being a morphism. ∎

Proposition 2.5 *Let $p : K \longrightarrow Q$ be an n-fold G-equivariant simplicial ramified covering map. If p_m is isovariant for all m, then p is special.*

Proof: If p_m is isovariant, then $G_a = G_{p_m(a)}$ for all a. Since d_i^K is G-equivariant, $G_a \subset G_{d_i^K(a)}$. Thus condition 4 holds. ∎

Corresponding to [5, Thm. 3.5], we have the following.

Proposition 2.6 *Let $p : K \longrightarrow Q$ be an n-fold G-equivariant (special) simplicial ramified covering map, and let $f : Q' \longrightarrow Q$ be a simplicial map. Then the pullback of p over f, $p' : K' = Q' \times_Q K \longrightarrow Q'$, is an n-fold (special) simplicial G-equivariant ramified covering map.*

Proof: By [5, Prop. 1.4], we have that p' is an n-fold simplicial ramified covering map. Since the map p' is clearly G-equivariant, we only have to prove that if p is special, then also p' is special. To see this, take $a' = (b', a) \in Q'_m \times_{Q_m} K_m$. Since p is special, we have $G_a = G_{p_m(a)} \cap G_{d_i^K(a)}$. Notice that since $f_m(b') = p_m(a)$, one has $G_{b'} \subset G_{p_m(a)}$. Hence

$$G_{a'} = G_{b'} \cap G_a = G_{b'} \cap G_{p_m(a)} \cap G_{d_i^K(a)} = G_{b'} \cap G_{d_i^K(a)} .$$

On the other hand,

$$G_{p'_m(a')} \cap G_{d_i^{K'}(a')} = G_{b'} \cap G_{d_i^{Q'}(b')} \cap G_{d_i^K(a)} = G_{b'} \cap G_{d_i^K(a)} .$$

Hence p' is special. ∎

The following is the equivariant version of [5, Prop. 1.6].

Proposition 2.7 *Let T be a simplicial G-set. Then the simplicial function $\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow T^n/\Sigma_n$, where $\overline{n} = \{1, 2, \ldots, n\}$, is a G-equivariant simplicial n-fold ramified covering map.*

Proof: T is a contravariant functor $\Delta \longrightarrow G\text{-}\mathcal{S}\mathrm{et}$, where Δ is the category whose objects are the sets $\mathbf{n} = \{0, 1, 2, \ldots, n\}$, $n \geq 0$, and whose morphisms are order-preserving functions (see [9] or [10]). Consider the functors

$$E_n : G\text{-}\mathcal{S}\mathrm{et} \longrightarrow G\text{-}\mathcal{S}\mathrm{et} \quad \text{and} \quad B_n : G\text{-}\mathcal{S}\mathrm{et} \longrightarrow G\text{-}\mathcal{S}\mathrm{et}$$

given by $S \longmapsto S^n \times_{\Sigma_n} \overline{n}$ and $S \longmapsto S^n/\Sigma_n$, respectively, where G acts diagonally on S^n. Then $T^n \times_{\Sigma_n} \overline{n} = E_n \circ T$ and $T^n/\Sigma_n = B_n \circ T$. The natural transformation $E_n \longrightarrow B_n$ that maps $\langle s_1, \ldots, s_n; j\rangle$ to $\langle s_1, \ldots, s_n\rangle$ determines the function of simplicial sets

$$\pi : T^n \times_{\Sigma_n} \overline{n} \longrightarrow T^n/\Sigma_n .$$

∎

Proposition 2.8 *The functions $\varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n K_m$ defined in 1.1 determine a map $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ of simplicial G-sets.*

Proof: Since the multiplicity functions μ_m are G-invariant, the functions φ_{p_m} are clearly G-equivariant. By [5, Prop. 1.8 and Thm. 1.9], these functions define a map of simplicial sets. ■

Theorem 2.9 *Let $p : K \longrightarrow Q$ be a simplicial G-equivariant ramified covering map and take $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$. Then p is the pullback over φ_p of the simplicial G-equivariant ramified covering map $\pi : K^n \times_{\Sigma_n} \overline{n} \longrightarrow \mathrm{SP}^n K$.*

Proof: By [5, Thm. 1.9], it follows that p is the pullback of π over φ. Moreover, since the face operators of K are isovariant, by 2.7, π is special, and by 2.6, p must be special. ■

The next is the equivariant version of [5, Thm. 3.1].

Theorem 2.10 *Let $p : K \longrightarrow Q$ be a map of simplicial G-sets. Then p is an n-fold simplicial G-equivariant ramified covering map with multiplicity functions μ_m if and only if there is a map of simplicial G-sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ such that for each m the following hold:*

1. *If $a \in K_m$, then $a \in \varphi_{p_m}(p_m(a))$.*
2. *The composition $\mathrm{SP}^n p_m \circ \varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n Q_m$ is the diagonal map.*

Proof: Assume first that $p : K \longrightarrow Q$ is an n-fold G-equivariant simplicial ramified covering map with multiplicity functions μ_m. Define $\varphi_{p_m} : Q_m \longrightarrow \mathrm{SP}^n K_m$ by

$$\varphi_{p_m}(x) = \langle \underbrace{a_1, \dots, a_1}_{\mu_m(a_1)}, \dots, \underbrace{a_r, \dots, a_r}_{\mu_m(a_r)} \rangle ,$$

where $p_m^{-1}(x) = \{a_1, \dots, a_r\}$. By [5, Prop. 1.8], the functions φ_{p_m} determine a map φ_p of simplicial sets. Since the functions μ_m are G-invariant, φ_p is a map of simplicial G-sets. By [5, Prop. 3.1], the functions φ_{p_m} satisfy conditions 1 and 2.

Conversely, suppose that there is a map of simplicial G-sets $\varphi_p : Q \longrightarrow \mathrm{SP}^n K$ which satisfies conditions 1 and 2. Take $a \in K_m$ and consider $\varphi_{p_m}(p_m(a)) \in \mathrm{SP}^n K_m$. If $(a_1, \dots, a_n) \in K_m^n$ is a representative of $\varphi_{p_m}(p_m(a))$. Then define $\mu_m : K_m \longrightarrow \mathbb{N}$ by

$$\mu_m(a) = \#\{i \in \overline{n} \mid a_i = a\} .$$

Again by [5, Prop. 3.1], $p : K \longrightarrow Q$ together with the family $\{\mu_m\}$ is an n-fold simplicial ramified covering map. Since φ_{p_m} is G-equivariant, the functions μ_m are G-invariant. ■

We have the following.

Theorem 2.11 *Let $p : K \longrightarrow Q$ be an n-fold G-equivariant simplicial ramified covering map. Then there exists a simplicial G-set W provided with a simplicial action of the symmetric group Σ_n such that there are simplicial equivariant isomorphisms $\alpha : W/\Sigma_{n-1} \longrightarrow K$ and $\beta : W/\Sigma_n \longrightarrow Q$ such that the following diagram commutes:*

$$\begin{array}{ccc} W/\Sigma_{n-1} & \xrightarrow{\alpha} & K \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ W/\Sigma_n & \xrightarrow[\beta]{} & Q\,, \end{array}$$

where $\pi : W/\Sigma_{n-1} \longrightarrow W/\Sigma_n$ is the canonical projection.

Proof: Define the simplicial set W as follows. Take

$$W_m = \{(x; a_1, \ldots, a_n) \in Q_m \times K_m^n \mid \varphi_{p_m}(x) = \langle a_1, \ldots, a_n \rangle\}\,.$$

If $f : \mathbf{k} \longrightarrow \mathbf{m}$ is a morphism in Δ, define $f^W : W_m \longrightarrow W_k$ by

$$f^W(x; a_1, \ldots, a_n) = (f^Q(x); f^K(a_1), \ldots, f^K(a_n))\,.$$

This is well defined, since φ_p is a map of simplicial sets.

We define a right action of Σ_n on W_m by

$$(x; a_1, \ldots, a_n)\sigma = (x; a_{\sigma(1)}, \ldots, a_{\sigma(n)})\,.$$

We consider Σ_{n-1} as the subgroup of those permutations that leave the first coordinate fixed. Let $\alpha_m : W_m/\Sigma_{n-1} \longrightarrow K_m$ and $\beta_m : W_m/\Sigma_n \longrightarrow Q_m$ be given by

$$\alpha_m([x; a_1, \ldots, a_n]_{\Sigma_{n-1}}) = a_1 \quad \text{and} \quad \beta_m([x; a_1, \ldots, a_n]_{\Sigma_n}) = x\,.$$

Let $\pi_m : W_m/\Sigma_{n-1} \longrightarrow W_m/\Sigma_n$ be the canonical surjection. Clearly $p_m \circ \alpha_m = \beta_m \circ \pi_m$. One can easily check that both α_m and β_m are bijective and determine maps α and β of simplicial sets. Observe that the Σ_n-action on W is simplicial.

Now we define a left action of G on each W_m by

$$g(x; a_1, \ldots, a_n) = (gx; ga_1, \ldots, ga_n)\,.$$

Since φ_{p_m} is G-equivariant, then this action is well defined. Notice that the left G-action and the right Σ_n-action satisfy the following associativity condition:

$$g((x; a_1, \ldots, a_n)\sigma) = (g(x; a_1, \ldots, a_n))\sigma\,.$$

This guarantees that the G-action passes to the quotients. One easily verifies that the isomorphisms α and β are G-equivariant. Observe that the G-action on W is simplicial. ■

Conversely, we have the following.

Theorem 2.12 *Let Γ be a finite group and $\Lambda \subset \Gamma$ be a subgroup of index n, and let W be a simplicial (left) G-set. Assume that Γ acts (simplicially) on the right on W, in such a way that*

$$g(w\gamma) = (gw)\gamma\,,$$

for all $w \in W$, $g \in G$, and $\gamma \in \Gamma$. Then the orbit map of simplicial sets

$$\pi : W/\Lambda \longrightarrow W/\Gamma$$

is an n-fold G-equivariant simplicial ramified covering map.

Proof: We shall prove that p satisfies conditions 1 and 2 of Theorem 2.10. Let $\varphi_\pi : W/\Gamma \longrightarrow \mathrm{SP}^n W/\Lambda$ be given for $[w]_\Gamma \in W_m/\Gamma$ by

$$\varphi_{\pi_m}([w]_\Gamma) = \langle [w\gamma_1]_\Lambda, \ldots, [w\gamma_n]_\Lambda \rangle \in \mathrm{SP}^n W_m/\Lambda\,,$$

where $\Gamma/\Lambda = \{[\gamma_1], \ldots, [\gamma_n]\}$ $(\gamma_1 = e \in \Gamma)$.

Since the action of Γ on W is simplicial, one easily verifies that φ_π is a map of simplicial sets. To see condition 1, take $a = [w]_\Lambda \in W_m/\Lambda$; since $\gamma_1 = e$, $a = [w\gamma_1]_\Lambda \in \varphi_{\pi_m}\pi_m(a) = \varphi_{\pi_m}([w]_\Gamma)$. To see condition 2, take $x = [w]_\Gamma \in W_m/\Gamma$. Then

$$\begin{aligned}
\mathrm{SP}^n \pi_m \varphi_{\pi_m}(x) &= \mathrm{SP}^n \pi_m(\langle [w\gamma_1]_\Lambda, \ldots, [w\gamma_1]_\Lambda \rangle) \\
&= \langle [w\gamma_1]_\Gamma, \ldots, [w\gamma_1]_\Gamma \rangle \\
&= \langle [w]_\Gamma, \ldots, [w]_\Gamma \rangle \\
&= \langle x, \ldots, x \rangle
\end{aligned}$$

■

Corollary 2.13 *Under the previous hypotheses, assume that the following condition holds for every $w \in W_m$ and all m:*

- *If $g \notin G_w$ and $gw = w\gamma$, then $\gamma \in \Lambda$.*

Then the n-fold G-equivariant simplicial ramified covering map $\pi : W/\Lambda \longrightarrow W/\Gamma$ is such that π_m is isovariant for all m; thus π is special. In particular, if for each $w \in W_m$ and $g \notin G_w$ we have that $gw \neq w\gamma$, $\gamma \in \Gamma$, then $\pi : W \longrightarrow W/\Gamma$ is special.

Proof: Take $w \in W_m$ and assume that $e \neq g \in G_{[w]_\Gamma}$, that is $g[w]_\Gamma = [w]_\Gamma$. Then $gw = w\gamma$, with $\gamma \in \Gamma$ and so, by the condition, $\gamma \in \Lambda$, and thus $g[w]_\Lambda = [w]_\Lambda$. Hence $G_{[w]_\Gamma} \subset G_{[w]_\Lambda}$ and so π_m is isovariant. The second part follows taking Λ to be the trivial subgroup of Γ. ■

Similarly to [5, Thm. 5.8], we have the next.

Theorem 2.14 *Let $p : K \longrightarrow Q$ be an n-fold simplicial G-equivariant ramified covering map. Then $|p| : |K| \longrightarrow |Q|$ is a topological n-fold G-equivariant ramified covering map.* ■

3 The transfer in the category of simplicial G-sets

Definition 3.1 Let $p : K \longrightarrow Q$ be a simplicial G-equivariant ramified covering map. We can define $t_{p_m}^G : F^G(Q_m, M) \longrightarrow F^G(K_m, M)$ on generators, as before, by

$$t_{p_m}^G(\gamma_x^G(l)) = \sum_{\{[a]\}=p_m^{-1}(x)/G_x} \mu_m(a)\gamma_a^G M^*(\widehat{p_m}_a)(l)\,.$$

The following is the main result of this section.

Theorem 3.2 *If $p : K \longrightarrow Q$ is a special simplicial G-equivariant ramified covering map, then the set of maps $\{t_{p_m}^G \mid m \in \mathbb{N}\}$ determines a morphism $t_p^G : F^G(Q, M) \longrightarrow F^G(K, M)$ of simplicial sets.*

Proof: The compatibility of t_p^G with the degeneracy functions s_i^K and s_i^Q and with the face functions d_i^K and d_i^Q is an immediate consequence of the naturality of the transfer with respect to G-functions with multiplicity and the fact that the pairs (d_i^K, d_i^Q), (s_i^K, s_i^Q) are morphisms of G-functions with multiplicity, as shown in 2.4. ■

In [2, 3.3(b)], for any simplicial pointed G-set S and a Mackey functor M for the group G, we defined an isomorphism of topological groups

$$\theta_S : F^G(|S|, M) \longrightarrow |F^G(S, M)|\,,$$

by $\theta_S(\gamma_{[\sigma,t]}^G(l)) = [\gamma_\sigma^G(l), t]$, where $(\sigma, t) \in S_m \times \Delta^m$ is a nondegenerate representative. We have the following.

Proposition 3.3 *Let $p : K \longrightarrow Q$ be a special simplicial G-equivariant ramified covering map. Then the following is a commutative diagram:*

$$\begin{tikzcd}
F^G(|Q|, M) \arrow[r, "t_{|p|}^G"] \arrow[d, "\theta_Q"', "\cong"] & F^G(|K|, M) \arrow[d, "\cong"', "\theta_K"] \\
|F^G(Q, M)| \arrow[r, "|t_p^G|"'] & |F^G(K, M)|\,.
\end{tikzcd}$$

Proof: Take an element $\gamma^G_{[x,t]}(l) \in F^G(|Q|, M)$ such that $(x,t) \in Q_m \times \Delta^m$ is a nondegenerate representative. Then $G_{[x,t]} = G_x$ (see [2, 2.4]). By [5, 4.1] and Remark 2.2, there is a bijection $p_m^{-1}(x) \approx |p|^{-1}[x,t]$ given by $a \mapsto [a,t]$. The representatives $(a,t) \in K_m \times \Delta^m$ are clearly nondegenerate. Since $G_{[a,t]} = G_a$, $\widehat{|p|}_{[a,t]} = \widehat{p_m}_a$. Hence

$$\begin{aligned}\theta_K t^G_{|p|}(\gamma^G_{[x,t]}(l)) &= \theta_K\Big(\sum_{[[a,t]] \in |p|^{-1}[x,t]/G_{[x,t]}} \mu[a,t]\gamma^G_{[a,t]} M^*(\widehat{|p|}_{[a,t]})(l)\Big) \\ &= \sum_{[a] \in p_m^{-1}(x)/G_x} \mu_m(a)[\gamma^G_a M^*(\widehat{p_m}_a)(l), t] \\ &= |t^G_p|([\gamma^G_x(l), t]) = |t^G_p|\theta_Q(\gamma^G_{[x,t]}(l)) \,.\end{aligned}$$

■

Corollary 3.4 *Let $p : K \longrightarrow Q$ be a special simplicial G-equivariant ramified covering map. Then $t^G_{|p|} : F^G(|Q|, M) \longrightarrow F^G(|K|, M)$ is continuous.* ■

Remark 3.5 By [3, Cor. 2.6], resp. [2, Prop. 3.3(b)], for any simplicial pointed G-set S and any G-module L, the topologies on $F^G(|S|, L)$ and on $F^G(|S|, M_L)$ coincide. Thus in the case $M = M_L$, the continuous transfer $t^G_{|p|}$ in the previous corollary coincides with the one in [6, Prop. 3.5]. More generally, since $|K|$ and $|Q|$ are ρ-spaces, the transfer with coefficients in a homological Mackey functor M also coincides with the transfer given in [6, Thm. 4.12].

4 The transfer in the category of simplicial G-complexes

Recall ([12]) that a *simplicial complex* C is a family of nonempty finite subsets of a set V_C, whose elements are the *vertices* of C and which have the following two properties:

(i) For each $v \in V_C$, the set $\{v\} \in C$.

(ii) Given $\sigma \in C$ and $\sigma' \subset \sigma$, then $\sigma' \in C$.

A map $f : C \longrightarrow D$ of simplicial complexes is given by a function $f : V_C \longrightarrow V_D$ such that if $\{v_0, \ldots, v_q\} \in C$, then $\{f(v_0), \ldots, f(v_q)\} \in D$.

In what follows, we shall assume that any simplicial complex C is *ordered*, that is, the vertices of C have a partial order such that each simplex is totally ordered. Moreover, we can also assume that any simplicial map $p : C \longrightarrow D$ preserves the order. This can always be achieved by considering the barycentric subdivision of each of the simplicial complexes, $\mathrm{sd}(C)$, $\mathrm{sd}(D)$, with the order given by inclusion. We denote by $\sigma^{(i)}$ the ith face of any ordered m-simplex $\sigma = (v_0 < \cdots < v_m)$ in a simplicial complex, which is defined by $\sigma^{(i)} = (v_0 < \cdots < \widehat{v_i} < \cdots < v_m)$, where we omit the ith vertex.

Definition 4.1 Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes. We say that p is an *n-fold ramified covering map of simplicial complexes* if there exists a *multiplicity function* $\mu : C \longrightarrow \mathbb{N}$ such that the following conditions hold:

1. For each vertex w of D, the fiber $p^{-1}(w)$ is a finite nonempty set and if $\sigma \in p^{-1}(\tau)$, then σ and τ have the same dimension.
2. For each simplex $\tau \in D$ and each simplex $\sigma' \in C$, such that $p(\sigma') = \tau^{(i)}$, there is a simplex $\sigma \in D$ such that $p(\sigma) = \tau$ and $\sigma^{(i)} = \sigma'$.
3. For each simplex τ in D,
$$\sum_{p(\sigma)=\tau} \mu(\sigma) = n\,.$$
4. For each simplex $\sigma \in C$,
$$\mu(\sigma^{(i)}) = \sum_{\substack{p(\sigma)=p(\sigma') \\ \sigma^{(i)}=\sigma'^{(i)}}} \mu(\sigma')\,.$$

Proposition 4.2 *Let $p : C \longrightarrow D$ be a simplicial map between simplicial complexes, and let $\mathrm{sd}(p) : \mathrm{sd}(C) \longrightarrow \mathrm{sd}(D)$ be the induced map between the barycentric subdivisions. Then, if p is a ramified covering map of simplicial complexes with multiplicity function μ, so also $\mathrm{sd}(p)$ is a ramified covering map of simplicial complexes with multiplicity function $\mathrm{sd}(\mu)$, where*
$$\mathrm{sd}(\mu)(\sigma_0 \subsetneq \cdots \subsetneq \sigma_m) = \mu(\sigma_m)\,.$$

Proof: Notice first that there is a bijection $\mathrm{sd}(p)^{-1}(\tau_0 \subsetneq \cdots \subsetneq \tau_m) \approx p^{-1}(\tau_m)$, since clearly $\sigma_m \in p^{-1}(\tau_m)$ determines σ_i such that $p(\sigma_i) = \tau_i$. Therefore, 1 holds for $\mathrm{sd}(p)$.

To see 2, let
$$\begin{aligned}\mathrm{sd}(p)(\sigma_0 \subsetneq \cdots \subsetneq \sigma_{i-1} \subsetneq \sigma_{i+1} \subsetneq \cdots \subsetneq \sigma_m) &= (\tau_0 \subsetneq \cdots \subsetneq \tau_{i-1} \subsetneq \tau_{i+1} \subsetneq \cdots \subsetneq \tau_m) \\ &= (\tau_0 \subsetneq \cdots \subsetneq \tau_m)^{(i)}\end{aligned}$$

Since $\tau_i \in D$ is a certain face of $\tau_{i+1} \in D$, one should take the corresponding face of σ_{i+1} and call it σ_i. Then, since p preserves faces, clearly
$$\mathrm{sd}(p)(\sigma_0 \subsetneq \cdots \subsetneq \sigma_m) = (\tau_0 \subsetneq \cdots \subsetneq \tau_m)$$

and

$$(\sigma_0 \subsetneq \cdots \subsetneq \sigma_m)^{(i)} = (\sigma_0 \subsetneq \cdots \subsetneq \sigma_{i-1} \subsetneq \sigma_{i+1} \subsetneq \cdots \subsetneq \sigma_m)\,.$$

Condition 3 follows immediately from the definition of $\mathrm{sd}(\mu)$ and the remark at the beginning of this proof.

Finally, to show condition 4, we have two cases:

Case 1. If $i < m$, then we have

$$\mathrm{sd}(\mu)((\tau_0 \subsetneq \cdots \subsetneq \tau_m)^{(i)}) = \mu(\tau_m)$$

and the condition follows immediately

Case 2. If $i = m$, then we have

$$\mathrm{sd}(\mu)((\tau_0 \subsetneq \cdots \subsetneq \tau_m)^{(i)}) = \mu(\tau_{m-1})\,.$$

τ_{m-1} is an iterated face of τ_m. We assume first that $\tau_{m-1} = \tau_m^{(j)}$. Then the condition follows immediately from condition 4 for p. In the general case, one can proceed inductively. ■

We understand by an *ordered simplicial G-complex* an ordered simplicial complex C together with an order-preserving action of G on the set of vertices V_C, such that each $g \in G$ induces a simplicial map. We can always assume that a simplical G-complex is ordered by passing to the barycentric subdivision, if necessary. A *simplicial G-map* $p : C \longrightarrow D$ of simplicial G-complexes is an order-preserving simplicial map which is G-equivariant. In what follows, we shall always consider ordered simplicial G-complexes.

Definition 4.3 We say that $p : C \longrightarrow D$ is an *n-fold G-equivariant ramified covering map of simplicial complexes* with multiplicity function μ, if p is an n-fold ramified covering map such that C and D are G-complexes, p is a G-map, and μ is G-invariant. We say that p is *special* if the following condition holds:

5. For each m-simplex $\sigma \in C$, one has $G_\sigma = G_{p(\sigma)} \cap G_{\sigma^{(i)}}$, for all $i = 0, \ldots, m$.

Proposition 4.2 allows us to assume that the G-actions in a G-equivariant ramified covering map of simplicial complexes preserve the orderings. From now on we shall assume that this is the case.

Recall ([5, Def. 5.4]) that given a simplicial complex C, one has a simplicial set $K(C)$ such that

$$K(C)_m = \{(v_0, \ldots, v_m) \mid \{v_0, \ldots, v_m\} \in C,\ v_0 \leq \cdots \leq v_m\}\,,$$

$d_i^{K(C)} : K(C)_m \longrightarrow K(C)_{m-1}$ is given by

$$d_i^{K(C)}(v_0, \dots, v_m) = (v_0, \dots, \widehat{v_i}, \dots, v_m)\,,$$

and $s_i^{K(C)} : K(C)_m \longrightarrow K(C)_{m+1}$ is given by

$$s_i^{K(C)}(v_0, \dots, v_m) = (v_0, \dots, v_i, v_i, \dots, v_m)\,.$$

If $p : C \longrightarrow D$ is an n-fold ramified covering map of simplicial complexes, call $K(p)_m : K(C)_m \longrightarrow K(D)_m$ the induced map of simplicial sets, given by $K(p)_m(v_0, \dots, v_m) = (p(v_0), \dots, p(v_m))$. Define $\mu_m : K(C)_m \longrightarrow \mathbb{N}$ by $\mu_m(\sigma) = \mu(\sigma')$, where $\sigma' \in K(C)_l$, $l \leq m$, is the unique nondegenerate simplex such that $s^{K(C)}(\sigma') = \sigma$.

Proposition 4.4 *Let $p : C \longrightarrow D$ be an n-fold G-equivariant ramified covering map of simplicial complexes. Then $K(p) : K(C) \longrightarrow K(D)$ is an n-fold simplicial G-equivariant ramified covering map. Furthermore, if p is special, then so is $K(p)$.*

Proof: By [5, Thm. 5.6], $K(p)$ is an n-fold simplicial ramified covering map and is clearly G-equivariant with the obvious actions. One easily verifies that μ_m is G-invariant for every m.

Now assume that p is special and take $a = (v_0 \leq \cdots \leq v_m) \in K(C)_m$. Let

$$v'_0 = v_0 = \cdots = v_{j_1 - 1} < v'_1 = v_{j_1} = \cdots = v_{j_2 - 1} < \cdots < v'_{m'} = v_{j_{m'}} = \cdots = v_m\,,$$

and take i such that $0 \leq i \leq m$. We have that $(v'_0 < \cdots < v'_{m'})$ is the unique nondegenerate simplex associated to $(v_0 \leq \cdots \leq v_m)$. There are two cases:

Case 1. The vertex v_i appears more than once in $a = (v_0 \leq \cdots \leq v_m)$. In this case, a and $a^{(i)}$ have the same nondegenerate associated simplex and condition 4 in Definition 2.1 follows trivially.

Case 2. The vertex v_i appears once in $a = (v_0 \leq \cdots \leq v_m)$. In this case, the nondegenerate associated simplex of $a^{(i)}$ is $(v'_0 < \cdots < v'_{m'})^{(j)}$, where $v_i = v'_j$ and condition 4 in Definition 2.1 follows from condition 5 in Definition 4.3 applied to the nondegenerate associated simplexes. ∎

Remark 4.5 Let us recall that given a simplicial complex C, its *geometric realization* is given by

$$|C| = \{\alpha : V_C \longrightarrow I \mid \alpha^{-1}(0,1] \in C \text{ and } \sum_{v \in V_C} \alpha(v) = 1\}\,.$$

It has the coherent topology with respect to the family of realizations $|\sigma|$ that are homeomorphic to Δ^n for some n. If $\sigma = \{v_0 < \cdots < v_m\}$, then the homeomorphism $|\sigma| \longrightarrow \Delta^n$ is given by $\alpha \mapsto (\alpha(v_0), \ldots, \alpha(v_m))$.

Given an n-fold ramified covering map of simplicial complexes $p : C \longrightarrow D$, by Property 1 in Definition 4.1, $p(\sigma) = \{p(v_0) < \cdots < p(v_m)\}$. Therefore we have the commutative square

$$\begin{tikzcd}
|\sigma| \arrow[r, "|p|"] \arrow[d, "\approx"'] & |p(\sigma)| \arrow[d, "\approx"] \\
\Delta^m \arrow[r, "\mathrm{id}"'] & \Delta^m .
\end{tikzcd}$$

Hence $|p|$ maps the realization of every simplex of C homeomorphically onto the realization of a simplex of D.

As a consequence of the previous result we can formulate the following.

Proposition 4.6 *Let $p : C \longrightarrow D$ be a special n-fold G-equivariant ramified covering map of simplicial complexes, and let M be a Mackey functor for G. Then $t^G_{|p|} : F^G(|D|, M) \longrightarrow F^G(|C|, M)$ is continuous.*

Proof: For any simplicial complex C, there is a homeomorphism $\varphi_C : |C| \longrightarrow |K(C)|$ given by $\varphi_C(\alpha) = [v_0, \ldots, v_m; \alpha(v_0), \ldots, \alpha(v_m)]$, where $\alpha : V_C \longrightarrow I$ is such that $\alpha^{-1}(0,1] = \{v_0, \ldots, v_m\}$ is a simplex of C and $\sum_{v\in V_C} \beta(v) = 1$ (see [5, 5.5]). Moreover, for any simplicial set K, there is an isomorphism of topological groups $\varphi'_K : F^G(|K|, M) \longrightarrow |F^G(K, M)|$ given by $\varphi_K(\gamma^G_{[\sigma,t]}(l)) = [\gamma^G_\sigma(l), t]$, where (σ, t) is a nondegenerate representative of $[\sigma, t] \in |K|$ (see [2, 2.6]). It will be enough to prove that the following diagram is commutative.

$$\begin{tikzcd}
F^G(|D|, M) \arrow[r, "\varphi^G_{D*}", "\cong"'] \arrow[d, "t^G_{|p|}"'] & F^G(|K(D)|, M) \arrow[r, "\varphi'_{K(D)}", "\cong"'] & {|F^G(K(D), M)|} \arrow[d, "|t^G_{K(p)}|"] \\
F^G(|C|, M) \arrow[r, "\cong", "\varphi^G_{C*}"'] & F^G(|K(C)|, M) \arrow[r, "\cong", "\varphi'_{K(C)}"'] & {|F^G(K(C), M)|} .
\end{tikzcd} \tag{4.7}$$

To see this, take $\gamma^G_\beta(l) \in F^G(|D|, M)$. If we chase $\gamma^G_\beta(l)$ along the top of the diagram and then down, we have that

$$|t^G_{K(p)}|\varphi'_{K(D)}\varphi^G_{D*}(\gamma^G_\beta(l)) = \left[\sum_{\iota\in\mathfrak{I}} \gamma^G_{\sigma_\iota} M^*(\widehat{K(p)}_{\sigma_\iota})(l), t\right], \tag{4.8}$$

where $\beta^{-1}(0,1] = \{w_0 < \cdots < w_m\}$, $t = (\beta(w_0), \ldots, \beta(w_m))$, and $\{[\sigma_\iota] \mid \iota \in \mathcal{I}\} = K(p)^{-1}(w_0, \ldots, w_m)/G_{(w_0,\ldots,w_m)}$.

If we now chase the same element down and then along the bottom of the diagram, we have that

$$\varphi'_{K(C)}\varphi^G_{C*}t^G_{|p|}(\gamma^G_\beta(l)) = \sum_{\kappa\in\mathcal{K}} \left[\gamma^G_{\alpha_\kappa^{-1}(0,1]} M^*(\widehat{|p|_{\alpha_\kappa}})(l), s_\kappa\right], \tag{4.9}$$

where $\{[\alpha_\kappa] \mid \kappa \in \mathcal{K}\} = |p|^{-1}(\beta)/G_\beta$, $\alpha_\kappa^{-1}(0,1] = (v_0^\kappa < \cdots < v_m^\kappa)$, and $s_\kappa = (\alpha_\kappa(v_0^\kappa), \ldots, \alpha_\kappa(v_m^\kappa))$.

First we define a bijection

$$f : |p|^{-1}(\beta) \longrightarrow p^{-1}(\beta^{-1}(0,1])$$

for each $\beta \in |D|$ by $f(\alpha) = \alpha^{-1}(0,1]$. To see that f is well defined, recall that $|p|(\alpha)(w) = \sum_{v\in p^{-1}(w)} \alpha(v)$. Hence

$$\begin{aligned} \beta^{-1}(0,1] &= \left\{ w \in V_D \mid \sum_{v\in p^{-1}(w)} \alpha(v) > 0 \right\} \\ &= \left\{ w \in V_D \mid \alpha(v) > 0 \text{ for some } v \in p^{-1}(w) \right\} \\ &= p(\alpha^{-1}(0,1]). \end{aligned}$$

To see that f is surjective, take $\sigma \in p^{-1}(\beta^{-1}(0,1])$ and let $\beta^{-1}(0,1] = \{w_0 < \cdots < w_m\}$. Then by Property 1 in Definition 4.1, we have that $\sigma = \{v_0 < \cdots < v_m\}$, where $p(v_i) = w_i$, $i = 0, \ldots, m$. Define

$$\alpha : V_C \longrightarrow I \quad \text{by} \quad \alpha(v) = \begin{cases} \beta p(v) & \text{if } v = v_i,\ i = 0, \ldots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f(\alpha) = \sigma$.

On the other hand, by 4.5, f is also injective.

Furthermore, there is also an obvious bijection

$$f' : p^{-1}(\beta^{-1}(0,1]) \longrightarrow K(p)^{-1}(w_0, \ldots, w_m)$$

given for $\sigma = \{v_0 < \cdots < v_m\}$ by $f'(\sigma) = (v_0, \ldots, v_m)$. Thus we have a bijection $|p|^{-1}(\beta) \longrightarrow K(p)^{-1}(w_0, \ldots, w_m)$.

Recall the G-equivariant homeomorphism $|D| \longrightarrow |K(D)|$ that sends an element $\beta \in |D|$ to the element $[w_0, \ldots, w_m; \beta(w_0), \ldots, \beta(w_m)]$. Thus the isotropy groups G_β and $G_{[w_0,\ldots,w_m,t]}$ coincide, where $t = (\beta(w_0), \ldots, \beta(w_m))$ is as above. Since the representative $(w_0, \ldots, w_m, t)$ is nondegenerate, by [2, 2.4], the groups

$G_{[w_0,\dots,w_m,t]}$ and $G_{(w_0,\dots,w_m)}$ also coincide. Hence $G_\beta = G_{(w_0,\dots,w_m)}$ and we have a bijection

$$|p|^{-1}(\beta)/G_\beta \longrightarrow K(p)^{-1}(w_0,\dots,w_m)/G_{(w_0,\dots,w_m)}\,.$$

So we have that $\mathcal{K} = \mathcal{I}$.

Finally, since $\beta(w_i) = |p|(\alpha_\iota)(w_i) = \alpha_\iota(v_i^\iota)$, $i = 0,\dots,m$, for any $\alpha_\iota \in |p|^{-1}(\beta)$, where $\alpha_\iota^{-1}(0,1] = \{v_0^\iota < \cdots < v_m^\iota\}$, we have $s_\iota = (\alpha_\iota(v_0^\iota),\dots,\alpha^\iota(v_m^\iota)) = (\beta(w_0),\dots,\beta(w_m)) = t \in \Delta^n$. Hence (4.8) and (4.9) are equal and diagram (4.7) commutes. So the transfer $t_{|p|}$ is continuous. ■

Remark 4.10 In the previous result, notice that since $|p| : |C| \longrightarrow |D|$ is a ramified covering G-map in the category of ρ-spaces, if the Mackey functor is homological, then by [6, Thm. 4.12] it need not be special in order for $t_{|p|}^G$ to be continuous.

To finish this section we define the transfer in Bredon-Illman homology by just applying the homotopy-group functors π_q to the transfer between topological groups. Using Propositions 4.6 and 1.6, we obtain the following.

Theorem 4.11 *Let $p : C \longrightarrow D$ be a special n-fold G-equivariant ramified covering map of simplicial complexes, and let M be a Mackey functor for G. Then there exists a transfer*

$$\tau_{|p|} : \widetilde{H}_*^G(|D|;M) \longrightarrow \widetilde{H}_*^G(|C|;M)$$

with the following properties: Pullback, Normalization, Additivity, Quasiadditivity, Functoriality, Invariance under change of coefficients, and if M is homological, then the composite

$$|p|_* \circ \tau_{|p|} : \widetilde{H}^G(|D|;M) \longrightarrow \widetilde{H}^G(|D|;M)$$

is multiplication by n.

Proof: Assume that $p : C \longrightarrow D$ is a special n-fold G-equivariant ramified covering map of simplicial complexes and that $f : D' \longrightarrow D$ is a G-equivariant pointed simplicial map. The pullback property follows from the fact that there are canonical homeomorphisms

$$|D'| \times_{|D|} |C| \approx |K(D')| \times_{|K(D)|} |K(C)| \approx |K(D') \times_{K(D)} K(C)|\,.$$

The first one follows from the homeomorphism mentioned in the proof of 4.6. To see the second one, notice that there is a natural homeomorphism $|Q' \times K| \approx |Q'| \times |K|$ for arbitrary simplicial sets Q' and K (see [10]), which restricts to a homeomorphism $|Q' \times_Q K| \approx |Q'| \times_{|Q|} |K|$ for any maps $K \xrightarrow{p} Q \xleftarrow{f} Q'$. Furthermore, under these homeomorphisms, the pullback diagram

$$\begin{tikzcd} |D'| \times_{|D|} |C| \arrow[r, "\widetilde{|f|}"] \arrow[d, "|p|'"'] & |C| \arrow[d, "|p|"] \\ |D'| \arrow[r, "|f|"'] & |D| \end{tikzcd}$$

corresponds to the diagram

$$\begin{tikzcd} |K(D') \times_{K(D)} K(C)| \arrow[r, "\widetilde{K(f)}"] \arrow[d, "|K(p)'|"'] & |K(C)| \arrow[d, "|K(p)|"] \\ |K(D')| \arrow[r, "|K(f)|"'] & |K(D)| \,. \end{tikzcd}$$

Therefore, since the pullback property of the transfer holds in the category of simplicial sets, it holds also in this case.

In order to prove the additivity property, assume first that for each $\alpha = 1, 2, \ldots, k$, $p_\alpha : C_\alpha \longrightarrow D$ is an n_α-fold G-equivariant ramified covering map of simplicial complexes. One can take the wedge sum $C = C_1 \vee C_2 \vee \cdots \vee C_k$. If the set of vertices of each C_α is partially ordered, so that every simplex is totally ordered, then the partial orders define a partial order in the set of vertices of C such that it induces the given order in the set of vertices of each C_α and each simplex in C, which is a simplex in some C_α, is totally ordered. Then one has a homeomorphism of topological spaces

$$|C_1 \vee C_2 \vee \cdots \vee C_k| \approx |C_1| \vee |C_2| \vee \cdots \vee |C_k| \,.$$

By [5, Thm. 4.2], each $|p_\alpha| : |C_\alpha| \longrightarrow |D|$ is a (topological) n_α-fold ramified covering G-map. Hence, from [6, 3.2(a)], $\pi : |C_1| \vee |C_2| \vee \cdots \vee |C_k| \longrightarrow |D|$, given by $\pi|_{|C_\alpha|} = |p_\alpha|$, is an $(n_1 + n_2 + \cdots + n_k)$-fold ramified covering G-map. By the additivity property of the transfer for G-functions with multiplicity [6, 2.19], the desired additivity property follows, namely, for all $\xi \in H_*^G(|D|; M)$,

$$\tau_\pi(\xi) = i_{1*}\tau_{|p_1|}(\xi) + i_{2*}\tau_{|p_2|}(\xi) + \cdots + i_{k*}\tau_{|p_k|}(\xi) \in H_*^G(|C_1| \vee |C_2| \vee \cdots \vee |C_k|; M) \,,$$

where i_α is the inclusion. Notice that using [5, Thm. 3.1], one can easily show that $p : K(C_1) \vee K(C_2) \vee \cdots \vee K(C_k) \longrightarrow K(D)$ given by $p|_{K(C_\alpha)} = K(p_\alpha)$ is a G-equivariant ramified covering map of simplicial sets, and it has the property that its realization corresponds to π. Thus the transfer of π corresponds to the realization of the transfer of p defined on simplicial sets.

The functoriality follows from the fact that if $p : C \longrightarrow D$ and $q : D \longrightarrow E$ are G-equivariant ramified covering maps of simplicial complexes, then by [1, 4.20] the composite $|q| \circ |p|$ is a (topological) ramified covering G-map and the corresponding property of G-functions with multiplicity [6, 2.21]. Notice that the composite $q \circ p$ is a G-equivariant ramified covering map of simplicial complexes such that $|q \circ p| = |q| \circ |p|$ and $t^G_{|q|\circ|p|} = t^G_{|q\circ p|}$. ■

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