



Periodic solutions of symmetric autonomous Newtonian systems

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Abstract

In this paper we show the existence and bifurcation of T -periodic solutions of a special form for an autonomous Newtonian system with symmetry. If the phase-space \mathbb{R}^{2n} is equipped with the structure of an orthogonal representation $(\mathbb{W}, \rho_{\mathbb{W}})$ and the potential $V : \mathbb{W} \rightarrow \mathbb{R}$ is invariant, then for every such a solution the set of indices of nonvanishing Fourier coefficients is finite and depends on \mathbb{W} only. If the potential V depends on the squares of complex coordinates, then for every such a solution T is the minimal period.

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1. Introduction

In this paper we show that a symmetry with respect to the circle group \mathbb{S}^1 of a Newtonian system given by a potential field provides us with detailed information on the form of its periodic solutions. This problem is of variational type and it has a natural \mathbb{S}^1 -symmetry given by the symmetry of the domain of the functions that give the periodic solutions, namely by the shift of the arguments. We call this action ‘internal’; the functional of the problem is invariant with respect to this action. On the other hand, the Euclidean phase-space of the solutions can be equipped with the structure of an orthogonal representation of \mathbb{S}^1 . We call this action ‘external.’ In order for the functional to be invariant, we have to impose as additional condition on the potential, the fact that it is also invariant with respect to the external action. Using a similar idea to what was done in [5], we introduce in the function space of the problem a new action, which combines the internal and the external actions. Then the functional is invariant with respect to this action, and due to the Palais symmetry principle, the critical points of its restriction to the fixed point subspace of the action are the solutions to our problem. In the case that we discuss, we notice that the fixed point subspace is finite-dimensional and has an orthogonal representation structure given either by the external, or by the internal action. Consequently, the solutions correspond to the critical points of an invariant function on a finite-dimensional orthogonal representation, whose structure is determined by the external action. This allows us to prove more refined versions of previous results on the bifurcation and existence of periodic solutions, for instance, the minimal periods and the description and finiteness of the set of nonvanishing Fourier coefficients.

In the first part of Section 3, we study the bifurcation of T -periodic solutions of a parameterized Newtonian problem in \mathbb{R}^{2n} with a potential $V : \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R}$. We suppose that \mathbb{R}^{2n} is equipped with the structure of an orthogonal representation \mathbb{W} of \mathbb{S}^1 , with $\mathbb{W}^{\mathbb{S}^1} = \{0\}$. Assuming the potential $V : \mathbb{W} \times \mathbb{R} \rightarrow \mathbb{R}$ to be \mathbb{S}^1 -invariant, we obtain information about the minimal periods of bifurcating periodic solutions (Theorem 3.1.1). We do this by adapting a bifurcation theorem of the third author and col. [10]. Perhaps the most interesting part is the fact that the isotypical coordinates of bifurcating periodic solutions are of the pure-harmonic form, i.e. they have a nonvanishing Fourier coefficient for only one index. We also consider the same problem without the assumption that $\mathbb{W}^{\mathbb{S}^1} = \{0\}$ obtaining a similar result (Theorem 3.1.2).

Next we study Newtonian systems given by a potential $V : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ looking for their periodic solutions of any period. The problem can be reformulated as a bifurcation problem of T -periodic solutions, with fixed T , of systems with a real parameter. This allows us to use our previous results (3.1.1)–(3.1.2) to get analogous information about the form of bifurcating periodic solutions in this case (Theorems 3.1.3 and 3.1.4).

In the second part of Section 3 we study the Newtonian problem in $\mathbb{R}^{2n} (\cong \mathbb{C}^n)$ with a special potential of a very natural form, which we call toroidal. This means that the potential depends on the squares of the norms of the complex variables. This guarantees that the potential is invariant with respect to any orthogonal action of \mathbb{S}^1 that preserves the complex structure. Consequently, we can always impose on \mathbb{R}^{2n} a structure of a free, or a one-orbit-type representation of \mathbb{S}^1 , and study the Newtonian problem with a toroidal potential, which is \mathbb{S}^1 -invariant. Our approach based on [10] provides an unbounded continuum of periodic solutions, consisting of pure-harmonics of the same period (3.2.1)–(3.2.2).

In Section 4 we discuss an asymptotically linear Newtonian problem in $\mathbb{R}^{2n} = \mathbb{W}$ with an invariant potential $V : (\mathbb{W}, \rho_{\mathbb{W}}) \rightarrow \mathbb{R}$. Adapting the Amann–Zehnder equivariant index and using our method once more, we get an estimate of the number of periodic solutions of this problem. Additionally, our method leads to an effective finite-dimensional reduction, namely, to

a G -subspace which is isomorphic to $(\mathbb{W}, \rho_{\mathbb{W}})$. This implies that the solutions found by this method are of a special form, as previously. Each of their isotypical coordinates is a pure-harmonic. Also, the assumption that the potential V is toroidal yields that the minimal period of the solutions to this problem is equal to T .

2. Preliminaries

In this section we shall discuss an orthogonal linear action ρ of the circle group $\mathbb{S}^1 = \{e^{is} \in \mathbb{C} \mid s \in \mathbb{R}\}$ in a separable Hilbert space \mathbb{H} , where $\text{Aut}(\mathbb{H})$ is equipped with the norm-of-operators topology. This action is defined as the composite of two commuting orthogonal actions ρ_I and ρ_E . This is inspired by the example of the Hilbert periodic-function space $\mathbb{H} = \mathbb{H}_T^1(\mathbb{S}^1, \mathbb{W}) = \mathbb{H}_T^1(\mathbb{S}^1, \mathbb{R}) \otimes \mathbb{W}$, where $(\mathbb{W}, \rho_{\mathbb{W}})$ is a representation of \mathbb{S}^1 , with an internal action given by $\rho_I(e^{is})u(t) = u(t + \frac{Ts}{2\pi})$ (see Section 2.3), and with an external action given by $\rho_E(e^{is})u(t) = \rho_{\mathbb{W}}(e^{is})u(t)$.

2.1. Representation structure induced by two commuting representations

Denote by $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ a separable Hilbert space. Assume that $\rho_I, \rho_E : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H})$ are continuous homomorphisms and that $(\mathbb{H}, \rho_I), (\mathbb{H}, \rho_E)$ are real orthogonal \mathbb{S}^1 -representations satisfying the following condition

$$\rho_I(e^{is_1}) \circ \rho_E(e^{is_2}) = \rho_E(e^{is_2}) \circ \rho_I(e^{is_1}) \quad \text{for every } e^{is_1}, e^{is_2} \in \mathbb{S}^1. \tag{2.1.1}$$

Define a continuous map $\rho : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H})$ by the formula $\rho(g) = \rho_I(g) \circ \rho_E(g)$.

Lemma 2.1.1. *Under the above assumptions $\rho : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H})$ is a homomorphism and (\mathbb{H}, ρ) is a real orthogonal \mathbb{S}^1 -representation.*

Proof. Notice that for every $e^{is_1}, e^{is_2} \in \mathbb{S}^1$:

$$\begin{aligned} \rho(e^{is_1}e^{is_2}) &= \rho_I(e^{is_1}e^{is_2}) \circ \rho_E(e^{is_1}e^{is_2}) = \rho_I(e^{is_1}) \circ \rho_I(e^{is_2}) \circ \rho_E(e^{is_1}) \circ \rho_E(e^{is_2}) \\ &= \rho_I(e^{is_1}) \circ \rho_E(e^{is_1}) \circ \rho_I(e^{is_2}) \circ \rho_E(e^{is_2}) = \rho(e^{is_1}) \circ \rho(e^{is_2}). \end{aligned}$$

From the above it follows that (\mathbb{H}, ρ) is a real orthogonal \mathbb{S}^1 -representation. \square

Denote by $\Psi(\mathbb{S}^1)$ the set of closed subgroups of \mathbb{S}^1 and fix $K \in \Psi(\mathbb{S}^1)$. Define

$$\begin{aligned} \mathbb{H}^{K, \rho_L} &= \{u \in \mathbb{H} \mid \rho_L(g)u = u \ \forall g \in K\}, \quad \text{for } L = I, E, \\ \mathbb{H}^{K, \rho} &= \{u \in \mathbb{H} \mid \rho(g)u = u \ \forall g \in K\} = \{u \in \mathbb{H} \mid \rho_E(g)u = \rho_I(g)^{-1}u \ \forall g \in K\}. \end{aligned} \tag{2.1.2}$$

Lemma 2.1.2. *Assume that $\mathbb{H}^{\mathbb{S}^1, \rho_I} = \{0\}$ and that $K \subset \ker \rho_I$. Then*

$$\mathbb{H}^{\mathbb{S}^1, \rho} \subset \mathbb{H}^{K, \rho_E} \quad \text{and} \quad \mathbb{H}^{\mathbb{S}^1, \rho} \cap \mathbb{H}^{\mathbb{S}^1, \rho_E} = \{0\}.$$

Proof. Let us remind that

$$\mathbb{H}^{\mathbb{S}^1, \rho} = \{u \in \mathbb{H} \mid \rho(g)u = u \ \forall g \in \mathbb{S}^1\} = \{u \in \mathbb{H} \mid (\rho_I(g) \circ \rho_E(g))u = u \ \forall g \in \mathbb{S}^1\}.$$

Notice that for every $g \in K$ and $u \in \mathbb{H}^{\mathbb{S}^1, \rho}$ we have

$$u = \rho(g)u = (\rho_I(g) \circ \rho_E(g))u = \rho_E(g)u.$$

Therefore we obtain $\mathbb{H}^{\mathbb{S}^1, \rho} \subset \mathbb{H}^{K, \rho_E}$. Suppose, contrary to our second claim, that there is $u \in \mathbb{H}^{\mathbb{S}^1, \rho} \cap \mathbb{H}^{\mathbb{S}^1, \rho_E} \setminus \{0\}$. Thus for every $g \in \mathbb{S}^1$ we obtain $u = \rho(g)u = (\rho_I(g) \circ \rho_E(g))u = \rho_I(g)u$, which contradicts the assumption that (\mathbb{H}, ρ_I) is a fixed point free \mathbb{S}^1 -representation. \square

Lemma 2.1.3. Fix $K \in \Psi(\mathbb{S}^1)$. Then $\mathbb{H}^{K, \rho}$ is a (\mathbb{S}^1, ρ_L) -invariant subspace, and consequently $(\mathbb{H}^{K, \rho}, \rho_L)$ is a real orthogonal \mathbb{S}^1 -representation for $L = I, E$. Moreover, if $K = \mathbb{S}^1$, then $(\mathbb{H}^{\mathbb{S}^1, \rho}, \rho_I) \cong (\mathbb{H}^{\mathbb{S}^1, \rho}, \rho_E)$. Consequently, for all $u \in \mathbb{H}$, the isotropy subgroups of both actions $(\mathbb{S}^1)_u^{\rho_I}$ and $(\mathbb{S}^1)_u^{\rho_E}$ are equal.

Proof. For the first part, it is enough to show that $\rho_L(g)u \in \mathbb{H}^{K, \rho}$ for every $g \in \mathbb{S}^1$, $u \in \mathbb{H}^{K, \rho}$, and $L = I, E$. Fix $k \in K$, $g \in \mathbb{S}^1$, and $u \in \mathbb{H}^{K, \rho}$. Taking into account that \mathbb{S}^1 is a commutative group and (2.1.1) we obtain the following

$$\begin{aligned} \rho(k)(\rho_I(g)u) &= (\rho_I(k) \circ \rho_E(k))(\rho_I(g)u) = (\rho_I(k) \circ \rho_E(k) \circ \rho_I(g))u \\ &= (\rho_I(g) \circ \rho_I(k) \circ \rho_E(k))u = \rho_I(g)(\rho(k)u) = \rho_I(g)u. \end{aligned}$$

The other case is similar.

The second part is a consequence of the following fact. For all $g \in \mathbb{S}^1$ and all $u \in \mathbb{H}$, $\rho(g)u = u \Leftrightarrow \rho_E(g)\rho_I(g)u = u \Leftrightarrow \rho_E(g)u = \rho_I(g)^{-1}u$. Thus, for all $g \in \mathbb{S}^1$, $\rho_E(g) = \rho_I(g)^{-1}$. But $\rho_I(g)$ is real orthogonal, so that $\rho_I(g)^{-1} = \rho_I(g)^* = \rho_I(g)$. \square

Fix $K \in \Psi(\mathbb{S}^1)$, $u \in \mathbb{H}^{K, \rho}$ and define $(\mathbb{S}^1)_u^{\rho_L} = \{g \in \mathbb{S}^1 \mid \rho_L(g)u = u\}$ the isotropy subgroup of $u \in \mathbb{H}^{K, \rho}$, for $L = I, E$.

2.2. Tensor product representations

In this paragraph we recall some facts concerning the tensor product of \mathbb{S}^1 -representations. They will play a role in the proofs of the main results of this paper.

Fix $m \in \mathbb{N}$ and denote by $\mathbb{R}[1, m] = (\mathbb{R}^2, \rho_m)$ a two-dimensional \mathbb{S}^1 -representation with homomorphism $\rho_m : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{R}^2)$ defined as follows $\rho_m(e^{is}) = \begin{bmatrix} \cos ms & -\sin ms \\ \sin ms & \cos ms \end{bmatrix}$. For $k \in \mathbb{N}$ we define $\mathbb{R}[k, m] = \bigoplus_{i=1}^k \mathbb{R}[1, m]$. Moreover, by $\mathbb{R}[1, 0]$ we denote trivial 1-dimensional \mathbb{S}^1 -representation with $\rho_0(e^{is}) = 1$. For $k \in \mathbb{N}$ we define $\mathbb{R}[k, 0] = \bigoplus_{i=1}^k \mathbb{R}[1, 0]$.

First of all, recall that every finite-dimensional \mathbb{S}^1 -representation \mathbb{W} can be written as $\bigoplus_{i=0}^p \mathbb{R}[k_i, m_i]$, where $k_i, m_i \in \mathbb{N}$, $1 \leq i \leq p$, $k_0 \in \mathbb{N} \cup \{0\}$, $0 = m_0 < m_1 < m_2 < \dots < m_p$, and these numbers are unique.

Lemma 2.2.1. Fix $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. Then

$$\left((\mathbb{R}[1, n] \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_n \otimes \rho_m}, \rho_n \otimes \text{Id}_{\mathbb{R}[1, m]} \right) \cong \left((\mathbb{R}[1, n] \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_n \otimes \rho_m}, \text{Id}_{\mathbb{R}[1, n]} \otimes \rho_m \right)$$

and both are isomorphic to $\mathbb{R}[1, n]$ if $n = m$, or to $\{0\}$ if $n \neq m$, where the action is given by Lemma 2.1.3.

Proof. The fact that if $n \neq m$, then the representation above is the zero-representation, is well known, but we include its proof for the convenience of the reader. Assume $n, m \in \mathbb{N}$ and notice that

$$\sum_{i, j=1}^2 x_{ij} e_i \otimes f_j \in (\mathbb{R}[1, n] \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_n \otimes \rho_m}$$

if and only if

$$\sum_{i, j=1}^2 x_{ij} \rho_n(e^{is}) e_i \otimes f_j = \sum_{i, j=1}^2 x_{ij} e_i \otimes \rho_m(e^{-is}) f_j, \tag{2.2.1}$$

for every $s \in \mathbb{R}$. Notice that (2.2.1) is equivalent to

$$\left(\rho_n(e^{is}) \otimes \text{Id}_{\mathbb{R}[1, m]} - \text{Id}_{\mathbb{R}[1, n]} \otimes \rho_m(e^{-is}) \right) \left(\sum_{i, j=1}^2 x_{ij} (e_i \otimes f_j) \right) = 0,$$

for every $s \in \mathbb{R}$. It is easy to check that a linear map

$$\mathbb{R}[1, n] \otimes \mathbb{R}[1, m] \ni \sum_{i, j=1}^2 x_{ij} (e_i \otimes f_j)$$

$$\mapsto \left(\rho_n(e^{is}) \otimes \text{Id}_{\mathbb{R}[1, m]} - \left(\text{Id}_{\mathbb{R}[1, n]} \otimes \rho_m(e^{-is}) \right) \right) \left(\sum_{i, j=1}^2 x_{ij} (e_i \otimes f_j) \right) \in \mathbb{R}[1, n] \otimes \mathbb{R}[1, m]$$

is given by

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} \rightarrow A_{nm}(s) \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix},$$

where

$$A_{nm}(s) = \begin{bmatrix} \cos ns - \cos ms & -\sin ms & -\sin ns & 0 \\ \sin ms & \cos ns - \cos ms & 0 & -\sin ns \\ \sin ns & 0 & \cos ns - \cos ms & -\sin ms \\ 0 & \sin ns & \sin ms & \cos ns - \cos ms \end{bmatrix}.$$

One can easily show that

$$\det A_{nm}(s) = 4(\cos ns - \cos ms)^2 = 16 \sin\left(\frac{(m+n)s}{2}\right) \sin\left(\frac{(m-n)s}{2}\right).$$

Therefore, if $n \neq m$, then $(\mathbb{R}[1, n] \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_n \otimes \rho_m} = \{0\}$.

Now, if $n = m$, then

$$A_{nn}(s) = \begin{bmatrix} 0 & -\sin ns & -\sin ns & 0 \\ \sin ns & 0 & 0 & -\sin ns \\ \sin ns & 0 & 0 & -\sin ns \\ 0 & \sin ns & \sin ns & 0 \end{bmatrix}$$

and $\det A_{nn}(s) = 0$. From the above it is easy to see that

$$\begin{aligned} & (\mathbb{R}[1, n] \otimes \mathbb{R}[1, n])^{\mathbb{S}^1, \rho_n \otimes \rho_n} \\ &= \{x(e_1 \otimes f_1 - e_2 \otimes f_2) + y(e_1 \otimes f_2 + e_2 \otimes f_1) \in \mathbb{R}[1, n] \otimes \mathbb{R}[1, n] \mid x, y \in \mathbb{R}\}. \end{aligned} \quad (2.2.2)$$

What is left to show is that one has isomorphisms

$$\begin{aligned} & ((\mathbb{R}[1, n] \otimes \mathbb{R}[1, n])^{\mathbb{S}^1, \rho_n \otimes \rho_n}, \rho_n \otimes \text{Id}_{\mathbb{R}[1, n]}) \cong \mathbb{R}[1, n], \\ & ((\mathbb{R}[1, n] \otimes \mathbb{R}[1, n])^{\mathbb{S}^1, \rho_n \otimes \rho_n}, \text{Id}_{\mathbb{R}[1, n]} \otimes \rho_n) \cong \mathbb{R}[1, n]. \end{aligned}$$

It is easy to verify that the map

$$\mathbb{S}^1 \times (\mathbb{R}[1, n] \otimes \mathbb{R}[1, n])^{\mathbb{S}^1, \rho_n \otimes \rho_n} \rightarrow (\mathbb{R}[1, n] \otimes \mathbb{R}[1, n])^{\mathbb{S}^1, \rho_n \otimes \rho_n}$$

defined by $(e^{is}, v) \mapsto (\text{Id}_{\mathbb{R}[1, n]} \otimes \rho_n(e^{is}))(v)$ can be represented as $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \rho_n(e^{is})\begin{bmatrix} x \\ y \end{bmatrix}$. We have just shown that $((\mathbb{R}[1, n] \otimes \mathbb{R}[1, n])^{\mathbb{S}^1, \rho_n \otimes \rho_n}, \text{Id}_{\mathbb{R}[1, n]} \otimes \rho_n) \cong \mathbb{R}[1, n]$. To obtain the isomorphism for $\rho_n \otimes \text{Id}_{\mathbb{R}[1, n]}$, it is enough to apply Lemma 2.1.3 to $\rho = (\rho_n \otimes \text{Id}) \circ (\text{Id} \otimes \rho_n)$. Assume that $n \in \mathbb{N}, m = 0$, and notice that

$$\sum_{i=1}^2 x_i e_i \otimes f \in (\mathbb{R}[1, n] \otimes \mathbb{R}[1, 0])^{\mathbb{S}^1, \rho_n \otimes \rho_0}$$

if and only if

$$(\rho_n(e^{is}) \otimes \text{Id}_{\mathbb{R}[1, 0]}) \left(\sum_{i=1}^2 x_i e_i \otimes f \right) = \sum_{i=1}^2 x_i e_i \otimes f, \quad (2.2.3)$$

for every $s \in \mathbb{R}$. Observe that (2.2.3) is equivalent to

$$(\rho_n(e^{is}) - \text{Id}_{\mathbb{R}[1,n]}) \otimes \text{Id}_{\mathbb{R}[1,0]} \left(\sum_{i=1}^2 x_i e_i \right) \otimes f = 0,$$

for every $s \in \mathbb{R}$. Therefore $(\mathbb{R}[1, n] \otimes \mathbb{R}[1, 0])^{\mathbb{S}^1, \rho_n \otimes \rho_0} = \{0\}$. \square

Remark 2.2.1. Assume that

$$(\mathbb{W}, \rho_{\mathbb{W}}) = \bigoplus_{k=1}^p \mathbb{R}[p_k, n_k], \quad n_k, p_k \in \mathbb{N}, \quad k = 1, \dots, p,$$

$$(\mathbb{U}, \rho_{\mathbb{U}}) = \bigoplus_{j=1}^q \mathbb{R}[q_j, m_j] \oplus \mathbb{R}[t, 0], \quad m_j, q_j \in \mathbb{N}, \quad j = 1, \dots, q.$$

Define the homomorphism $\rho : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{W} \otimes \mathbb{U})$ as $\rho = \rho_{\mathbb{W}} \otimes \rho_{\mathbb{U}}$.

Then, we have

$$((\mathbb{W} \otimes \mathbb{U})^{\mathbb{S}^1, \rho}, \rho_{\mathbb{W}} \otimes \text{Id}_{\mathbb{U}}) = \bigoplus_{k=1}^p \bigoplus_{j=1}^q \bigoplus_{r=1}^{p_i} \bigoplus_{s=1}^{q_j} ((\mathbb{R}[1, n_k] \otimes \mathbb{R}[1, m_j])^{\mathbb{S}^1, \rho_{n_k} \otimes \rho_{m_j}}, \rho_{\mathbb{W}} \otimes \text{Id}_{\mathbb{U}}),$$

$$((\mathbb{W} \otimes \mathbb{U})^{\mathbb{S}^1, \rho}, \text{Id}_{\mathbb{W}} \otimes \rho_{\mathbb{U}}) = \bigoplus_{k=1}^p \bigoplus_{j=1}^q \bigoplus_{r=1}^{p_i} \bigoplus_{s=1}^{q_j} ((\mathbb{R}[1, n_k] \otimes \mathbb{R}[1, m_j])^{\mathbb{S}^1, \rho_{n_k} \otimes \rho_{m_j}}, \text{Id}_{\mathbb{W}} \otimes \rho_{\mathbb{U}}).$$

Corollary 2.2.1. If $(\mathbb{W}, \rho_{\mathbb{W}}) = \mathbb{R}[p, m]$, $(\mathbb{U}, \rho_{\mathbb{U}}) = \mathbb{R}[q, m] \oplus \bigoplus_{j=1}^r \mathbb{R}[q_j, m_j]$, $0 \leq m_1 < m_2 < \dots < m_r$, $m \in \mathbb{N}$, $m_j \neq m$ for $j = 1, \dots, r$, then

$$((\mathbb{W} \otimes \mathbb{U})^{\mathbb{S}^1, \rho}, \rho_{\mathbb{W}} \otimes \text{Id}_{\mathbb{U}}) \cong \mathbb{R}[pq, m], \quad ((\mathbb{W} \otimes \mathbb{U})^{\mathbb{S}^1, \rho}, \text{Id}_{\mathbb{W}} \otimes \rho_{\mathbb{U}}) \cong \mathbb{R}[pq, m].$$

2.3. Functional space

We begin this section with a definition of an appropriate Hilbert space. Fix $T > 0$ and define

$$\mathbb{H}_T^1 = \{u : [0, T] \rightarrow \mathbb{R}^n \mid u \text{ is abs. cont., } u(0) = u(T), \dot{u} \in L^2([0, T], \mathbb{R}^n)\}.$$

It is known that \mathbb{H}_T^1 is a separable Hilbert space with a scalar product given by the formula $\langle u, v \rangle_{\mathbb{H}_T^1} = \int_0^T (\dot{u}(t), \dot{v}(t)) + (u(t), v(t)) dt$, where (\cdot, \cdot) and $\|\cdot\|$ are the usual scalar product and norm in \mathbb{R}^n , respectively.

A pair (\mathbb{H}_T^1, ρ_T) is an orthogonal \mathbb{S}^1 -representation with a homomorphism $\rho_T : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H}_T^1)$ given by $\rho_T(e^{is})(u(t)) = u(t + \frac{Ts}{2\pi})$.

Consider \mathbb{R}^n as an \mathbb{S}^1 -representation

$$(\mathbb{W}, \rho_{\mathbb{W}}) = (\mathbb{R}^n, \rho_{\mathbb{W}}) = \mathbb{R}[k_0, m_0] \oplus \mathbb{R}[k_1, m_1] \oplus \dots \oplus \mathbb{R}[k_p, m_p],$$

where $k_i, m_i \in \mathbb{N}$, $1 \leq i \leq p$, $k_0 \in \mathbb{N} \cup \{0\}$, $0 = m_0 < m_1 < \dots < m_p$.

A pair (\mathbb{H}_T^1, ρ_E) is an orthogonal \mathbb{S}^1 -representation with a homomorphism $\rho_E: \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H}_T^1)$ given by $\rho_E(e^{is})(u(t)) = \rho_{\mathbb{W}}(e^{is})u(t)$.

Remark 2.3.1. Define a map $\rho: \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H}_T^1)$ by $\rho(e^{is}) = \rho_I(e^{is}) \circ \rho_E(e^{is})$. It is easy to verify that homomorphisms $\rho_I, \rho_E: \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H}_T^1)$ satisfy (2.1.1). Therefore directly from Lemma 2.1.1 it follows that (\mathbb{H}_T^1, ρ) is an orthogonal \mathbb{S}^1 -representation.

Lemma 2.3.1. Under the above conditions: $((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \cong \mathbb{W} \cong ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_E)$.

Proof. Notice that $\mathbb{H}_T^1 = \overline{\bigoplus_{m=0}^{\infty} \mathbb{R}[n, m]} = \overline{\bigoplus_{m=0}^{\infty} \mathbb{W} \otimes \mathbb{R}[1, m]}$, where

$$\mathbb{R}[n, m] = \{a \cdot \cos(mt) + b \sin(mt) \mid a, b \in \mathbb{W}\}, \quad \mathbb{R}[1, m] = \text{span}_{\mathbb{R}}\{\cos(mt), \sin(mt)\}.$$

Since $\rho(e^{is})(\mathbb{R}[n, m]) \subset \mathbb{R}[n, m]$ for every $s \in \mathbb{R}$,

$$(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} = \overline{\bigoplus_{m=0}^{\infty} \mathbb{R}[n, m]^{\mathbb{S}^1, \rho}} = \overline{\bigoplus_{m=0}^{\infty} (\mathbb{W} \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_{\mathbb{W}} \otimes \rho_m}}. \tag{2.3.1}$$

Notice that

$$\begin{aligned} (\mathbb{W} \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_{\mathbb{W}} \otimes \rho_m} &= \left(\bigoplus_{i=0}^p (\mathbb{R}[k_i, m_i] \otimes \mathbb{R}[1, m]) \right)^{\mathbb{S}^1, \rho_{\mathbb{W}} \otimes \rho_m} \\ &= \bigoplus_{i=0}^p (\mathbb{R}[k_i, m_i] \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, (\bigoplus_{j=1}^{k_i} \rho_{m_j}) \otimes \rho_m}. \end{aligned}$$

Fix any $m \in \mathbb{N}$ such that $m > m_p$. Then, from Lemma 2.2.1, it follows that

$$(\mathbb{W} \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_{\mathbb{W}} \otimes \rho_m} = \{0\}. \tag{2.3.2}$$

Finally, combining (2.3.1), (2.3.2) with Lemma 2.2.1 we obtain

$$\begin{aligned} (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} &= \overline{\bigoplus_{m=0}^{\infty} (\mathbb{W} \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, \rho_{\mathbb{W}} \otimes \rho_m}} = \overline{\bigoplus_{m=0}^{m_p} \bigoplus_{i=0}^p (\mathbb{R}[k_i, m_i] \otimes \mathbb{R}[1, m])^{\mathbb{S}^1, (\bigoplus_{j=1}^{k_i} \rho_{m_j}) \otimes \rho_m}} \\ &= \bigoplus_{i=0}^p (\mathbb{R}[k_i, m_i] \otimes \mathbb{R}[1, m_i])^{\mathbb{S}^1, (\bigoplus_{j=1}^{k_i} \rho_{m_j}) \otimes \rho_{m_i}}. \quad \square \end{aligned}$$

As a direct consequence of Corollary 2.2.1 and Lemma 2.3.1 we obtain the following corollary.

Corollary 2.3.1. Fix $m \in \mathbb{N}$. If $(\mathbb{W}, \rho_{\mathbb{W}}) = \mathbb{R}[n, m]$ then $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \cong \mathbb{R}[n, m]$. Moreover, taking into account (2.2.2) we obtain that $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ consists of elements of the form

$$\begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{bmatrix} \cos(2\pi mt/T) + \begin{bmatrix} ry_1 \\ -x_1 \\ \vdots \\ y_n \\ -x_n \end{bmatrix} \sin(2\pi mt/T),$$

where $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$.

Lemma 2.3.2. If $u \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ then

$$(\mathbb{S}^1)_u^{\rho_I} = (\mathbb{S}^1)_u^{\rho_E} \in \{ \mathbb{Z}_{\gcd(m_{i_1}, \dots, m_{i_s})} \mid \{m_{i_1}, \dots, m_{i_s}\} \subset \{m_1, \dots, m_p\} \} \cup \{ \mathbb{S}^1 \}.$$

Moreover, if $k_0 = 0$ and $u \neq 0$ then $(\mathbb{S}^1)_u^{\rho_I} \neq \mathbb{S}^1$.

Proof. By the last part of Theorem 2.1.3, we obtain $(\mathbb{S}^1)_u^{\rho_I} = (\mathbb{S}^1)_u^{\rho_E}$. From Lemma 2.3.1 we obtain $((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \cong \mathbb{W} = \mathbb{R}[k_0, m_0] \oplus \mathbb{R}[k_1, m_1] \oplus \dots \oplus \mathbb{R}[k_p, m_p]$. \square

Lemma 2.3.3. If $k_0 = 0$ and there is $m \in \mathbb{N}$ such that $\mathbb{Z}_m \subset \ker \rho_{\mathbb{W}}$, then $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \setminus \{0\}$ consists of periodic functions with minimal period belonging to $\{ \frac{T}{m_i} \mid i \in \mathbb{N} \}$.

Proof. Since $\mathbb{W}^{\mathbb{S}^1, \rho_{\mathbb{W}}} = \{0\}$, $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho_E} = \{0\}$. That is why $(\mathbb{S}^1)_u^{\rho_E} \neq \mathbb{S}^1$ for every $u \in \mathbb{H}_T^1$. Fix $u \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$. By the last part of Theorem 2.1.3, we obtain $(\mathbb{S}^1)_u^{\rho_I} \neq \mathbb{S}^1$. Therefore, $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ consist of nonconstant T -periodic functions. Since $\mathbb{Z}_m \subset \ker \rho_{\mathbb{W}}$, $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \subset (\mathbb{H}_T^1)^{\mathbb{Z}_m, \rho_I}$. \square

2.4. Global bifurcations of critical \mathbb{S}^1 -orbits

In this paragraph, for the convenience of the reader, we recall the degree for \mathbb{S}^1 -equivariant gradient maps and some of its properties given in [12]. This degree will be denoted by $\nabla_{\mathbb{S}^1}$ -deg.

Put $U(\mathbb{S}^1) = \mathbb{Z} \oplus (\bigoplus_{k=1}^{\infty} \mathbb{Z})$ and define actions

$$\begin{aligned} +, \star : U(\mathbb{S}^1) \times U(\mathbb{S}^1) &\rightarrow U(\mathbb{S}^1), \\ \cdot : \mathbb{Z} \times U(\mathbb{S}^1) &\rightarrow U(\mathbb{S}^1) \end{aligned}$$

as follows:

$$\begin{aligned} \alpha + \beta &= (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \dots, \alpha_k + \beta_k, \dots), \\ \alpha \star \beta &= (\alpha_0 \cdot \beta_0, \alpha_0 \cdot \beta_1 + \beta_0 \cdot \alpha_1, \dots, \alpha_0 \cdot \beta_k + \beta_0 \cdot \alpha_k, \dots), \\ \gamma \cdot \alpha &= (\gamma \cdot \alpha_0, \gamma \cdot \alpha_1, \dots, \gamma \cdot \alpha_k, \dots), \end{aligned}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k, \dots), \beta = (\beta_0, \beta_1, \dots, \beta_k, \dots) \in U(\mathbb{S}^1), \gamma \in \mathbb{Z}$. It is easy to check that $(U(\mathbb{S}^1), +, \star)$ is a commutative ring with unit $\mathbb{I} = (1, 0, \dots) \in U(\mathbb{S}^1)$.

Let $(\mathbb{W}, \rho_{\mathbb{W}})$ be a real, finite-dimensional and orthogonal \mathbb{S}^1 -representation. If $w \in \mathbb{W}$ then the subgroup $\mathbb{S}^1_w = \{g \in \mathbb{S}^1 \mid g \cdot w = w\}$ is said to be the isotropy group of $w \in \mathbb{W}$. Let $\Omega \subset \mathbb{W}$ be an open, bounded and \mathbb{S}^1 -invariant subset and let $H \subset \mathbb{S}^1$ be closed subgroup.

Recall that

- $\Omega^H = \{w \in \Omega \mid H \subset \mathbb{S}^1_w\} = \{w \in \Omega \mid gw = w \forall g \in H\}$,
- $\Omega_H = \{w \in \Omega \mid H = \mathbb{S}^1_w\}$.

Fix $k \in \mathbb{N}$ and set $C^k_{\mathbb{S}^1}((\mathbb{W}, \rho_{\mathbb{W}}), \mathbb{R}) = \{f \in C^k(\mathbb{W}, \mathbb{R}) \mid f \text{ is } \mathbb{S}^1\text{-invariant}\}$. Take $f \in C^1_{\mathbb{S}^1}((\mathbb{W}, \rho_{\mathbb{W}}), \mathbb{R})$. Since $(\mathbb{W}, \rho_{\mathbb{W}})$ is an orthogonal \mathbb{S}^1 -representation, gradient $\nabla f : \mathbb{W} \rightarrow \mathbb{W}$ is an \mathbb{S}^1 -equivariant C^0 -map. If $H \subset \mathbb{S}^1$ is a closed subgroup then \mathbb{W}^H is a finite-dimensional \mathbb{S}^1 -representation and $(\nabla f)^H = \nabla(f|_{\mathbb{W}^H}) : \mathbb{W}^H \rightarrow \mathbb{W}^H$ is well-defined \mathbb{S}^1 -equivariant gradient map. Choose an open, bounded and \mathbb{S}^1 -invariant subset $\Omega \subset \mathbb{W}$ such that $(\nabla f)^{-1}(0) \cap \partial\Omega = \emptyset$. Under this assumption, a degree for \mathbb{S}^1 -equivariant gradient maps $\nabla_{\mathbb{S}^1}\text{-deg}(\nabla f, \Omega) \in U(\mathbb{S}^1)$ with coordinates

$$\nabla_{\mathbb{S}^1}\text{-deg}(\nabla f, \Omega) = (\nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{S}^1}(\nabla f, \Omega), \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_1}(\nabla f, \Omega), \dots, \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_m}(\nabla f, \Omega), \dots)$$

was defined in [12], where also its general properties are proved.

For $\gamma > 0$ and $w_0 \in \mathbb{W}^{\mathbb{S}^1}$ we put $B_\gamma(\mathbb{W}, w_0) = \{w \in \mathbb{W} \mid |w - w_0| < \gamma\}$.

To apply successfully any degree theory we need computational formulas for this invariant. Below we show how to compute degree for \mathbb{S}^1 -equivariant gradient maps of a linear, self-adjoint, \mathbb{S}^1 -equivariant isomorphism.

Let $\mathbb{W} \cong \mathbb{R}[k_0, 0] \oplus \mathbb{R}[k_1, m_1] \oplus \dots \oplus \mathbb{R}[k_p, m_p]$.

Lemma 2.4.1. (See [12].) *If $L : \mathbb{W} \rightarrow \mathbb{W}$ is a self-adjoint, \mathbb{S}^1 -equivariant, linear isomorphism and $\gamma > 0$ then*

- (1) $L = \text{diag}(L_0, L_1, \dots, L_r)$,
- (2) $\nabla_{\mathbb{S}^1}\text{-deg}_H(L, B_\gamma(V, 0)) = \begin{cases} (-1)^{m^-(L_0)}, & \text{for } H = \mathbb{S}^1, \\ (-1)^{m^-(L_0)} \cdot \frac{m^-(L_i)}{2}, & \text{for } H = \mathbb{Z}_{m_i}, \\ 0, & \text{for } H \notin \{\mathbb{S}^1, \mathbb{Z}_{m_1}, \dots, \mathbb{Z}_{m_p}\}, \end{cases}$
- (3) *if, in particular, $L = -\text{Id}$, then*

$$\nabla_{\mathbb{S}^1}\text{-deg}_H(-\text{Id}, B_\gamma(V, 0)) = \begin{cases} (-1)^{k_0}, & \text{for } H = \mathbb{S}^1, \\ (-1)^{k_0} \cdot k_i, & \text{for } H = \mathbb{Z}_{m_i}, \\ 0, & \text{for } H \notin \{\mathbb{S}^1, \mathbb{Z}_{m_1}, \dots, \mathbb{Z}_{m_p}\}. \end{cases}$$

Lemma 2.4.2. *If $V \in C^2_{\mathbb{S}^1}((\mathbb{W}, \rho_{\mathbb{W}}), \mathbb{R})$ satisfy the following conditions:*

- (1) $\nabla V(0) = 0$,
- (2) $\ker \nabla^2 V(0) \subset \mathbb{R}[k_0, 0]$,
- (3) $0 \in \mathbb{W}$ *is isolated in* $(\nabla V)^{-1}(0)$,

then for sufficiently small positive α and $\mathbb{W} = \mathbb{W}_0 \oplus \mathbb{W}_1 = \mathbb{R}[k_0, 0] \oplus \mathbb{R}[k_0, 0]^\perp$ we have

$$\begin{aligned} & \nabla_{\mathbb{S}^1}\text{-deg}(\nabla V, B_\alpha(\mathbb{W})) \\ &= \text{deg}_{\mathbb{B}}(\nabla V^{\mathbb{S}^1}, B_\alpha(\mathbb{W}_0), 0) \cdot \nabla_{\mathbb{S}^1}\text{-deg}(\nabla^2 V(0)|_{\mathbb{W}_1}, B_\alpha(\mathbb{W}_1)). \end{aligned}$$

In particular, if $m \notin \{m_1, \dots, m_p\}$ then $\nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_m}(\nabla V, B_\alpha(\mathbb{W})) = 0$.

Let $V \in C_{\mathbb{S}^1}^2((\mathbb{W}, \rho_{\mathbb{W}}) \times \mathbb{R}, \mathbb{R})$ and $\lambda_0 \in \mathbb{R}$ satisfy the following conditions:

- (f1) $\nabla_v V(0, \lambda) = 0$, for all $\lambda \in \mathbb{R}$,
- (f2) $(0, \lambda)$ is isolated in $(\nabla_v V(\cdot, \lambda))^{-1}(0)$, for every $\lambda \in \mathbb{R}$,
- (f3) $\det(\nabla_v^2 V(0, \lambda_0 \pm \varepsilon)|_{\mathbb{R}[k_0, 0]^\perp}) \neq 0$, for sufficiently small positive ε ,
- (f4) $\nabla_{\mathbb{S}^1}\text{-deg}(\nabla_v V(0, \lambda_0 + \varepsilon), B_\alpha(\mathbb{W})) \neq \nabla_{\mathbb{S}^1}\text{-deg}(\nabla_v V(0, \lambda_0 - \varepsilon), B_\alpha(\mathbb{W}))$.

Denote by $\mathcal{C}(\lambda_0) \subset \mathbb{W} \times \mathbb{R}$ connected component of

$$\text{cl}(\{(v, \lambda) \in (\mathbb{W} \setminus \{0\}) \times \mathbb{R} \mid \nabla_v V(v, \lambda) = 0\})$$

such that $(0, \lambda_0) \in \mathcal{C}(\lambda_0)$.

Using the degree for \mathbb{S}^1 -equivariant gradient maps one can prove the following global bifurcation theorems.

Theorem 2.4.1. *Let $V \in C_{\mathbb{S}^1}^2((\mathbb{W}, \rho_{\mathbb{W}}) \times \mathbb{R}, \mathbb{R})$ satisfy conditions (f1)–(f4). Then the continuum $\mathcal{C}(\lambda_0)$ is unbounded or*

$$\mathcal{C}(\lambda_0) \cap (\{0\} \times \{\lambda \in \mathbb{R} \setminus \{\lambda_0\} \mid \det(\nabla_v^2 V(0, \lambda)|_{\mathbb{R}[k_0, 0]^\perp}) = 0\}) \neq \emptyset.$$

Theorem 2.4.2. *Let the assumptions of Theorem 2.4.1 be fulfilled. If moreover $\{\lambda \in \mathbb{R} \mid \det(\nabla_v^2 V(0, \lambda)|_{\mathbb{R}[k_0, 0]^\perp}) = 0\}$ does not have finite accumulation points then the continuum $\mathcal{C}(\lambda_0)$ is unbounded or*

$$\mathcal{C}(\lambda_0) \cap (\{0\} \times \{\lambda \in \mathbb{R} \mid \det(\nabla_v^2 V(0, \lambda)|_{\mathbb{R}[k_0, 0]^\perp}) = 0\}) = \{(0, \lambda_0), \dots, (0, \lambda_p)\}$$

and

$$\sum_{i=0}^p \nabla_{\mathbb{S}^1}\text{-deg}(\nabla_v f(0, \lambda_i + \varepsilon), B_\alpha(\mathbb{W})) - \nabla_{\mathbb{S}^1}\text{-deg}(\nabla_v f(0, \lambda_i - \varepsilon), B_\alpha(\mathbb{W})) = \emptyset.$$

3. Bifurcation of periodic solutions

Let $(\mathbb{W}, \rho_{\mathbb{W}})$ be an orthogonal \mathbb{S}^1 -representation. In this section we study bifurcations of nonstationary T -periodic solutions of the following system

$$\begin{cases} \ddot{u}(t) = -V'_u(u(t), \lambda), \\ u(0) = u(T), \\ \dot{u}(0) = \dot{u}(T), \end{cases} \tag{3.1}$$

where $V \in C_{\mathbb{S}^1}^2((\mathbb{W}, \rho_{\mathbb{W}}) \times \mathbb{R}, \mathbb{R})$ is such that $V'_x(0, \lambda) = 0$ for every $\lambda \in \mathbb{R}$.

Define homomorphisms $\rho, \rho_I, \rho_E : \mathbb{S}^1 \rightarrow \text{Aut}(\mathbb{H}_T^1)$ as follows

$$\begin{aligned} \rho_I(e^{is})(u(t)) &= u(t + (Ts/2\pi)), & \rho_E(e^{is})(u(t)) &= \rho_{\mathbb{W}}(e^{is})u(t), \\ \rho(e^{is}) &= \rho_I(e^{is}) \circ \rho_E(e^{is}). \end{aligned} \tag{3.2}$$

It is easy to check that the homomorphisms ρ_I, ρ_E satisfy condition (2.1.1). By Remark 2.3.1 (\mathbb{H}_T^1, ρ) is an orthogonal \mathbb{S}^1 -representation. Solutions of (3.1) are in one-to-one correspondence with critical points of an \mathbb{S}^1 -invariant C^2 -functional $\Phi \in C_{\mathbb{S}^1}^2((\mathbb{H}_T^1, \rho) \times \mathbb{R}, \mathbb{R})$ defined as follows

$$\Phi(u, \lambda) = \int_0^T \frac{1}{2} \|\dot{u}(t)\|^2 - V(u(t), \lambda) dt. \tag{3.3}$$

Let $\tilde{\Phi} : (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R} \rightarrow \mathbb{R}$ be the restriction of Φ to $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$.

Remark 3.1. Since $\nabla_u \Phi : (\mathbb{H}_T^1, \rho) \times \mathbb{R} \rightarrow (\mathbb{H}_T^1, \rho)$ is \mathbb{S}^1 -equivariant,

$$(\nabla_u \Phi)^{\mathbb{S}^1} : (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R} \rightarrow (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}.$$

Moreover,

- (1) $\nabla_u \Phi(u, \lambda) = (\nabla_{\tilde{u}} \tilde{\Phi}(\tilde{u}, \lambda), 0)$ for $u = (\tilde{u}, 0) \in \mathbb{H}_T^1 = (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \oplus ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho})^\perp$,
- (2) $\tilde{\Phi} : ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \times \mathbb{R} \rightarrow \mathbb{R}$ is \mathbb{S}^1 -invariant.

Let $u = (\tilde{u}, 0) \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \oplus ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho})^\perp$. By the above, the study of solutions of equation $\nabla_u \Phi(u, \lambda) = 0$ is equivalent to the study of solutions of equation $\nabla_{\tilde{u}} \tilde{\Phi}(\tilde{u}, \lambda) = 0$. This is the so-called Palais symmetry principle.

From now on for simplicity of notations we put $A(\lambda) = V_x''(0, \lambda)$. By assumption we have $V(x) = \frac{1}{2} \langle A(\lambda)x, x \rangle + \eta(x, \lambda)$, where $\eta'_x(0, \lambda) = 0, \eta''_x(0, \lambda) = 0$. Therefore, we obtain $V'_x(x, \lambda) = A(\lambda)x + \eta'_x(x, \lambda)$. Repeating the reasoning from [10] we obtain functional (3.3) in the following form

$$\Phi(u, \lambda) = \frac{1}{2} \langle u - L_{A(\lambda)}(u), u \rangle_{\mathbb{H}_T^1} - N(u, \lambda), \tag{3.4}$$

where $L_{A(\lambda)} : (\mathbb{H}_T^1, \rho) \rightarrow (\mathbb{H}_T^1, \rho)$ is a linear, self-adjoint, \mathbb{S}^1 -equivariant, and compact operator defined by the formula $\langle L_{A(\lambda)}(u), v \rangle_{\mathbb{H}_T^1} = \int_0^T (u(t) + A(\lambda)u(t), v(t)) dt$, for every $\lambda \in \mathbb{R}$, and $N : (\mathbb{H}_T^1, \rho) \times \mathbb{R} \rightarrow \mathbb{R}$ is an \mathbb{S}^1 -invariant potential defined by $N(u, \lambda) = \int_0^T \eta(u(t), \lambda) dt$.

Finally, notice that $\nabla_u \Phi(u, \lambda) = u - L_{A(\lambda)}u - \nabla_u N(u, \lambda)$, where $\nabla_u N : (\mathbb{H}_T^1, \rho) \times \mathbb{R} \rightarrow (\mathbb{H}_T^1, \rho)$ is \mathbb{S}^1 -equivariant, compact, and such that $\nabla_u N(0, \lambda) = 0, \nabla_u^2 N(0, \lambda) = 0$. Fix $\lambda_0 \in \mathbb{R}$ and denote by $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ a connected component of

$$\text{cl}(\{(u, \lambda) \in ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \setminus \{0\}) \times \mathbb{R} : (\nabla_u \Phi)^{\mathbb{S}^1}(u, \lambda) = 0\})$$

such that $(0, \lambda_0) \in \mathcal{C}(\lambda_0)$.

3.1. \mathbb{S}^1 -invariant potential

Let $(\mathbb{W}, \rho_{\mathbb{W}}) = \mathbb{R}[k_0, m_0] \oplus \mathbb{R}[k_1, m_1] \oplus \dots \oplus \mathbb{R}[k_p, m_p]$, where $k_i, m_i \in \mathbb{N}$, $1 \leq i \leq p$, $k_0 \in \mathbb{N} \cup \{0\}$, $0 = m_0 < m_1 < \dots < m_p$.

Put $n = k_0 + 2(k_1 + \dots + k_p)$ and assume that $V \in C_{\mathbb{S}^1}^2((\mathbb{W}, \rho_{\mathbb{W}}) \times \mathbb{R}, \mathbb{R})$ satisfy the following conditions:

- (a1) $V'_x(0, \lambda) = 0$, for every $\lambda \in \mathbb{R}$,
- (a2) $0 \in \mathbb{R}^n$ is isolated in $(V'_x(\cdot, \lambda))^{-1}(0)$, for every $\lambda \in \mathbb{R}$, and
- (a3) $\text{deg}_{\mathbb{B}}(V'_x(\cdot, 0), B_\alpha^n, 0) \neq 0 \in \mathbb{Z}$, for sufficiently small positive α .

Since $A(\lambda) = V''_x(0, \lambda)$ is \mathbb{S}^1 -invariant, $A(\lambda) = \text{diag}(A_0(\lambda), A_1(\lambda), \dots, A_p(\lambda))$. Fix $i_0 \in \{0, 1, \dots, p\}$, $\lambda_0 \in \mathbb{R}$ and define

$$\mathcal{J}(\lambda_0, i_0) = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} m^- \left(A_{i_0}(\lambda_0 + \varepsilon) - \frac{4m_{i_0}^2 \pi^2}{T^2} \text{Id} \right) - m^- \left(A_{i_0}(\lambda_0 - \varepsilon) - \frac{4m_{i_0}^2 \pi^2}{T^2} \text{Id} \right),$$

where $m^-(L)$ is the Morse index of a symmetric matrix L .

Theorem 3.1.1. *Suppose that $k_0 = 0$ and that condition (a1) is satisfied. If there are $\lambda_0 \in \mathbb{R}$ and $i_0 \in \{1, \dots, p\}$ such that*

- (i) $\det(A_{i_0}(\lambda_0) - \frac{4m_{i_0}^2 \pi^2}{T^2} \text{Id}) = 0$,
- (ii) $\prod_{i=1}^p \det(A_i(\lambda_0 \pm \varepsilon) - \frac{4m_i^2 \pi^2}{T^2} \text{Id}) \neq 0$, for every sufficiently small positive ε , and
- (iii) $\mathcal{J}(\lambda_0, i_0) \neq 0$,

then the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R}$ is unbounded or

$$\mathcal{C}(\lambda_0) \cap \{0\} \times \left(\left\{ \lambda \in \mathbb{R} \setminus \{\lambda_0\} : \prod_{i=1}^p \det \left(A_i(\lambda) - \frac{4m_i^2 \pi^2}{T^2} \text{Id} \right) = 0 \right\} \right) \neq \emptyset.$$

Moreover,

- (a) for $(u_1(t), \dots, u_p(t)) \in \mathcal{C}(\lambda_0)$ we have

$$u_i(t) = a_i \cos(2\pi m_i t / T) + J a_i \sin(2\pi m_i t / T),$$

where $a_i \in \mathbb{R}[k_i, m_i]$ and

$$J = \text{diag} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right), \quad 1 \leq i \leq p,$$

- (b) for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\{0\} \times \mathbb{R})$, there are $m_{i_1}, \dots, m_{i_r} \in \{m_1, \dots, m_p\}$ such that the minimal period of $\tilde{u}(t)$ equals $\frac{T}{\gcd(m_{i_1}, \dots, m_{i_r})}$,
- (c) if $\det(A_i(\lambda_0) - \frac{4m_i^2\pi^2}{T^2}\text{Id}) \neq 0$, for $i \neq i_0$, then for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\{0\} \times \mathbb{R})$ sufficiently close to $(0, \lambda_0)$, the minimal period of $u(t)$ equals $\frac{T}{m_{i_0}}$, and
- (d) if $\mathcal{C}(\lambda_0)$ is bounded and $\text{card}\{i \in \{1, \dots, p\} \mid \det(A_i(\lambda) - \frac{4m_i^2\pi^2}{T^2}\text{Id}) = 0\} \leq 1$, for every $\lambda \in \mathbb{R}$, then $\mathcal{C}(\lambda_0)$ contains solutions with at least two different minimal periods.

Proof. Fix $\bar{\lambda} \in \mathbb{R}$. First of all notice that by Corollary 5.1.1 of [10] and Remark 3.2.1 the following conditions are equivalent:

- (1) operator $\text{Id} - L_{A(\bar{\lambda})} : \mathbb{H}_T^1 \rightarrow \mathbb{H}_T^1$ is an isomorphism, and
- (2) $\det(A(\bar{\lambda}) - \frac{4m^2\pi^2}{T^2}\text{Id}) \neq 0$, for every $m \in \mathbb{N} \cup \{0\}$.

It is known that if $(0, \bar{\lambda}) \in \mathbb{H}_T^1 \times \mathbb{R}$ is a bifurcation point of solutions of the equation $\nabla_u \Phi(u, \lambda) = 0$, then $\nabla_u^2 \Phi(0, \bar{\lambda}) = \text{Id} - L_{A(\bar{\lambda})}$ is not an isomorphism. By assumption (i) we obtain

$$\det\left(A(\lambda_0) - \frac{4m_{i_0}^2\pi^2}{T^2}\text{Id}\right) = \prod_{i=1}^p \det\left(A_i(\lambda_0) - \frac{4m_i^2\pi^2}{T^2}\text{Id}\right) = 0.$$

Hence, the necessary condition for the existence of bifurcation of solutions of $\nabla_u \Phi(u, \lambda) = 0$ at $(0, \lambda_0)$ is fulfilled, i.e. $\nabla_u^2 \Phi(0, \lambda_0) = \text{Id} - L_{A(\lambda_0)}$ is not an isomorphism.

By Lemma 2.3.1 we obtain

$$\left((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I\right) = \left(\bigoplus_{i=1}^p (\mathbb{R}[k_i, m_i] \otimes \mathbb{R}[1, m_i])^{\mathbb{S}^1, (\bigoplus_{j=1}^{k_i} \rho_{m_j}) \otimes \rho_{m_i}}, \rho_I\right) \cong (\mathbb{W}, \rho_I).$$

From Remark 3.1, it follows that in order to complete the proof, it is enough to study critical orbits of a functional $\tilde{\Phi} : ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \times \mathbb{R} \rightarrow \mathbb{R}$.

By Lemma 5.1.1 of [10],

$$\nabla_u^2 \tilde{\Phi}(0, \lambda)(u_1, \dots, u_p) = (\Lambda(m_1, \lambda)u_1, \dots, \Lambda(m_p, \lambda)u_p),$$

where $\Lambda(m_i, \lambda) = (\frac{4m_i^2\pi^2}{4m_i^2\pi^2 + T^2}\text{Id} - \frac{T^2}{4m_i^2\pi^2 + T^2}A_i(\lambda))$. From assumption (i) it follows that $\nabla_u^2 \tilde{\Phi}(0, \lambda_0)$ is not an isomorphism. By assumption (ii) $\nabla_u^2 \tilde{\Phi}(0, \lambda_0 \pm \varepsilon)$ is an isomorphism for every sufficiently small positive ε . Taking into account assumption (iii) and Lemma 2.4.1 we obtain

$$\begin{aligned} & \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_{i_0}}}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho})) - \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_{i_0}}}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho})) \\ & = \mathcal{J}(\lambda_0, i_0) \neq 0. \end{aligned}$$

Applying Theorem 2.4.1, we obtain the continuum $\mathcal{C}(\lambda_0)$ satisfying the conclusion of this theorem.

(a) By Corollary 2.3.1 for $(u_1(t), \dots, u_p(t)) \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ we have

$$u_i(t) = a_i \cos(2\pi m_i t/T) + J a_i \sin(2\pi m_i t/T),$$

where $a_i \in \mathbb{R}[k_i, m_i]$ and $J = \text{diag}([\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}], \dots, [\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}])$, $1 \leq i \leq p$.

(b) Since $((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \cong (\bigoplus_{i=1}^p \mathbb{R}[k_i, m_i], \rho_I)$, the conclusion follows from Lemma 2.3.2.

(c) In this situation there is $k'_{i_0} \in \mathbb{N}$ such that $(\ker \nabla_u^2 \tilde{\Phi}(0, \lambda_0), \rho_I) \cong (\mathbb{R}[k'_{i_0}, m_{i_0}], \rho_I)$. Therefore, from $(0, \lambda_0)$ only critical orbits with minimal period $\frac{T}{m_{i_0}}$ (isotropy group $\mathbb{Z}_{m_{i_0}}$) can bifurcate.

(d) Since $\mathcal{C}(\lambda_0)$ is bounded there exist $\lambda'_0 \in \mathbb{R} \setminus \{\lambda_0\}$ and exactly one $i'_0 \in \{1, \dots, p\} \setminus \{i_0\}$ and such that

- $\det(A_{i'_0}(\lambda'_0) - (4m_{i'_0}^2 \pi^2 / T^2) \text{Id}) = 0,$
- $(0, \lambda'_0) \in \mathcal{C}(\lambda_0).$

Since $(\ker \nabla_u^2 \tilde{\Phi}(0, \lambda_0), \rho_I) \cong (\mathbb{R}[k'_{i_0}, m_{i_0}], \rho_I), (\ker \nabla_u^2 \tilde{\Phi}(0, \lambda'_0), \rho_I) \cong (\mathbb{R}[k''_{i_0}, m'_{i_0}], \rho_I)$ and $m_{i_0} \neq m'_{i_0}$, the rest of the proof is a direct consequence of (b). \square

For $\lambda_0 \in \mathbb{R}$ define $\mathcal{J}(\lambda_0) = (\mathcal{J}(\lambda_0, 1), \dots, \mathcal{J}(\lambda_0, p)).$

Corollary 3.1.1. *Under the assumptions of Theorem 3.1.1, and if moreover*

$$\left\{ \lambda \in \mathbb{R} \mid \prod_{i=1}^p \det \left(A_i(\lambda) - \frac{4m_i^2 \pi^2}{T^2} \text{Id} \right) = 0 \right\}$$

does not possess finite accumulation points, then the conclusion of Theorem 3.1.1 holds true. Moreover, if $\mathcal{C}(\lambda_0)$ is bounded, then

$$\mathcal{C}(\lambda_0) \cap \{0\} \times \left\{ \lambda \in \mathbb{R} \mid \prod_{i=1}^p \det \left(A_i(\lambda) - \frac{4m_i^2 \pi^2}{T^2} \text{Id} \right) = 0 \right\} = \{(0, \lambda_0), (0, \lambda_1), \dots, (0, \lambda_s)\}$$

and

$$\sum_{j=0}^s \mathcal{J}(\lambda_j) = \emptyset. \tag{3.1.1}$$

Proof. First of all notice that from Lemma 2.4.1 it follows that for every $i \in \{1, \dots, p\}$

$$\begin{aligned} \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_i}}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) - \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_i}}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) \\ = \mathcal{J}(\lambda_0, i). \end{aligned}$$

Additionally by Lemma 2.4.1 we obtain that for every $\tilde{m} \notin \{m_1, \dots, m_p\}$

$$\nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{\tilde{m}}}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) = \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{\tilde{m}}}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) = 0.$$

The rest of the proof is the same as in the proof of Theorem 3.1.1, but instead of Theorem 2.4.1 we apply Theorem 2.4.2. \square

Theorem 3.1.2. *Suppose that*

- (1) $k_0 > 0$,
- (2) *assumptions (a1)–(a3) are satisfied, and*
- (3) *assumptions (i)–(iii) of Theorem 3.1.1 are satisfied.*

Then the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R}$ is unbounded or

$$\mathcal{C}(\lambda_0) \cap \{0\} \times \left(\left\{ \lambda \in \mathbb{R} \setminus \{\lambda_0\} \mid \prod_{i=1}^p \det \left(A_i(\lambda) - \frac{4m_i^2 \pi^2}{T^2} \text{Id} \right) = 0 \right\} \right) \neq \emptyset.$$

Moreover,

- (a) *for $(u_0(t), u_1(t), \dots, u_p(t)) \in \mathcal{C}(\lambda_0)$ we have*

$$u_0(t) = a_0, \quad u_i(t) = a_i \cos(2\pi m_i t/T) + J a_i \sin(2\pi m_i t/T),$$

where $a_0 \in \mathbb{R}[k_0, 0]$, $a_i \in \mathbb{R}[k_i, m_i]$ and $J = \text{diag}([\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}], \dots, [\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}])$, $1 \leq i \leq p$,

- (b) *for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\mathbb{R}[k_0, 0] \times \mathbb{R})$ there are $m_{i_1}, \dots, m_{i_r} \in \{m_1, \dots, m_p\}$ such that the minimal period of $u(t)$ equals $\frac{T}{\text{gcd}(m_{i_1}, \dots, m_{i_r})}$,*
- (c) *if $\det(A_i(\lambda_0) - \frac{4m_i^2 \pi^2}{T^2} \text{Id}) \neq 0$, for $i \neq i_0$, then for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\{0\} \times \mathbb{R})$ sufficiently close to $(0, \lambda_0)$, the minimal period of $u(t)$ equals $\frac{T}{m_{i_0}}$, and*
- (d) *if $\mathcal{C}(\lambda_0)$ is bounded and $\text{card}\{i \in \{1, \dots, p\} \mid \det(A_i(\lambda) - \frac{4m_i^2 \pi^2}{T^2} \text{Id}) = 0\} \leq 1$, for every $\lambda \in \mathbb{R}$, then $\mathcal{C}(\lambda_0)$ contains solutions with at least two different minimal periods.*

Proof. Repeating the argument of the proof of Theorem 3.1.1 one can show that the necessary condition for the existence of a bifurcation of the solutions of $\nabla_u \Phi(u, \lambda) = 0$ at $(0, \lambda_0)$ is fulfilled at $(0, \lambda_0)$, i.e. $\nabla_u^2 \Phi(0, \lambda_0) = \text{Id} - L_{A(\lambda_0)}$ is not an isomorphism.

By Lemma 2.3.1, we obtain

$$((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) = \left(\mathbb{R}[k_0, 0] \oplus \bigoplus_{i=1}^p (\mathbb{R}[k_i, m_i] \otimes \mathbb{R}[1, m_i])^{\mathbb{S}^1, (\bigoplus_{j=1}^{k_i} \rho_{m_j}) \otimes \rho_{m_i}}, \rho_I \right) \cong (\mathbb{W}, \rho_I).$$

From Remark 3.1, it follows that in order to complete the proof it is enough to study the critical orbits of the functional $\tilde{\Phi} : ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \times \mathbb{R} \rightarrow \mathbb{R}$.

From Lemma 5.1.1 of [10],

$$\nabla_u^2 \tilde{\Phi}(0, \lambda)(u_0, u_1, \dots, u_p) = (\Lambda(m_0, \lambda)u_0, \Lambda(m_1, \lambda)u_1, \dots, \Lambda(m_p, \lambda)u_p),$$

where $\Lambda(m_i, \lambda) = (\frac{4m_i^2\pi^2}{4m_i^2\pi^2+T^2}\text{Id} - \frac{T^2}{4m_i^2\pi^2+T^2}A_i(\lambda))$. From assumption (i) it follows that $\nabla_{\tilde{u}}^2\tilde{\Phi}(0, \lambda_0)$ is not an isomorphism. By assumption (ii), $\nabla_{\tilde{u}}^2\tilde{\Phi}(0, \lambda_0 \pm \varepsilon)|_{\mathbb{R}[k_0, 0]^\perp}$ is an isomorphism for every sufficiently small positive ε . Since V' is \mathbb{S}^1 -equivariant,

$$\text{deg}_{\mathbb{B}}(V'_x(\cdot, 0), B_\alpha(\mathbb{V}), 0) = \text{deg}_{\mathbb{B}}(V'_x(\cdot, 0)^{\mathbb{S}^1}, B_\alpha(\mathbb{V}^{\mathbb{S}^1}), 0).$$

Combining assumptions (a3) and (iii) with Lemmas 2.4.1 and 2.4.2, we obtain

$$\begin{aligned} &\nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_{i_0}}}(\nabla_{\tilde{u}}\tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) - \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_{i_0}}}(\nabla_{\tilde{u}}\tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) \\ &= \text{deg}_{\mathbb{B}}(V'_x(\cdot, 0), B_\alpha^n, 0) \cdot \mathcal{J}(\lambda_0, i_0) \neq 0. \end{aligned}$$

Applying Theorem 2.4.1 we obtain that the continuum $\mathcal{C}(\lambda_0)$ satisfies the conclusion of this theorem.

(a) Consequence of Corollary 2.3.1.

(b) Since $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \cong (\bigoplus_{i=0}^p \mathbb{R}[k_i, m_i], \rho_I)$, the conclusion follows from Lemma 2.3.2.

(c) In this situation there is $k'_0 \in \mathbb{N} \cup \{0\}, k'_{i_0} \in \mathbb{N}$, such that $(\ker \nabla_{\tilde{u}}^2\tilde{\Phi}(0, \lambda_0), \rho_I) \cong (\mathbb{R}[k'_0, 0] \oplus \mathbb{R}[k'_{i_0}, m_{i_0}], \rho_I)$. Since there is no bifurcation of stationary solutions from $\{0\} \times \mathbb{R}^n$, the only minimal period of bifurcating critical orbits is $\frac{T}{m_{i_0}}$.

(d) The proof is the same as the proof of (d) in Theorem 3.1.1. \square

Corollary 3.1.2. *Under the assumptions of Theorem 3.1.2, and if moreover*

$$\left\{ \lambda \in \mathbb{R} \mid \prod_{i=1}^p \det\left(A_i(\lambda) - \frac{4m_i^2\pi^2}{T^2}\text{Id}\right) = 0 \right\}$$

does not possess finite accumulation points, then the conclusion of Theorem 3.1.2 holds true. Moreover, if $\mathcal{C}(\lambda_0)$ is bounded, then

$$\mathcal{C}(\lambda_0) \cap \{0\} \times \left\{ \lambda \in \mathbb{R} \mid \prod_{i=1}^p \det\left(A_i(\lambda) - \frac{4m_i^2\pi^2}{T^2}\text{Id}\right) = 0 \right\} = \{(0, \lambda_0), (0, \lambda_1), \dots, (0, \lambda_s)\}$$

and

$$\sum_{j=0}^s \mathcal{J}(\lambda_j) = \Theta. \tag{3.1.2}$$

Proof. First of all notice that from Lemmas 2.4.1 and 2.4.2 it follows that for every $i \in \{1, \dots, p\}$

$$\begin{aligned} &\nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_i}}(\nabla_{\tilde{u}}\tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) - \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{m_i}}(\nabla_{\tilde{u}}\tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) \\ &= \text{deg}_{\mathbb{B}}(V'_x(\cdot, 0), B_\alpha^n, 0) \cdot \mathcal{J}(\lambda_0, i). \end{aligned}$$

Additionally, by Lemmas 2.4.1 and 2.4.2, we obtain that for every $\tilde{m} \notin \{m_1, \dots, m_p\}$

$$\nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{\tilde{m}}}(\nabla_{\tilde{u}}\tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) = \nabla_{\mathbb{S}^1}\text{-deg}_{\mathbb{Z}_{\tilde{m}}}(\nabla_{\tilde{u}}\tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}) = 0.$$

Since $\text{deg}_B(V'_x(\cdot, 0), B_\alpha^n, 0) \neq 0$, the rest of the proof of this theorem is in fact the same as in the proof of Theorem 3.1.1, but instead of Theorem 2.4.1 we apply Theorem 2.4.2. \square

Remark 3.1.1. Notice that in Theorems 3.1.1 and 3.1.2 it can happen that $\det V''_x(0, \lambda) = 0$ for every $\lambda \in \mathbb{R}$.

At the end of this section we study bifurcations of nonstationary periodic solutions of system $\ddot{u}(t) = -V'(u(t))$ with arbitrary period. We assume that $V \in C^2_{\mathbb{S}^1}((\mathbb{W}, \rho_{\mathbb{W}}), \mathbb{R})$ satisfy the following conditions:

- (a1') $V'(0) = 0$,
- (a2') $0 \in \mathbb{R}^n$ is isolated in $(V')^{-1}(0)$,
- (a3') $\text{deg}_B(V', B_\alpha^n, 0) \neq 0 \in \mathbb{Z}$, for sufficiently small positive α .

From now on we study bifurcations of nonstationary T -periodic solutions of the following system

$$\begin{cases} \ddot{u}(t) = -\lambda^2 V'(u(t)), \\ u(0) = u(T), \\ \dot{u}(0) = \dot{u}(T). \end{cases} \tag{3.1.3}$$

Since $V' \in C^1_{\mathbb{S}^1}((\mathbb{W}, \rho_{\mathbb{W}}), (\mathbb{W}, \rho_{\mathbb{W}}))$, $A = V''(0) = \text{diag}(A_1, \dots, A_p)$.

Define $\mathcal{B}_i = \bigcup_{\alpha \in \sigma_+(A_i)} \{ \frac{2m_i\pi}{T\sqrt{\alpha}} \}$ and $\mathcal{B} = \bigcup_{i=0}^p \mathcal{B}_i$.

Theorem 3.1.3. *Suppose that $k_0 = 0$ and that condition (a1') is satisfied. Then for every $\lambda_0 \in \mathcal{B}$, the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R}$ is unbounded. Moreover,*

- (a) for $(u_1(t), \dots, u_p(t)) \in \mathcal{C}(\lambda_0)$ we have

$$u_i(t) = a_i \cos(2\pi m_i t/T) + J a_i \sin(2\pi m_i t/T),$$

where $a_i \in \mathbb{R}[k_i, m_i]$ and $J = \text{diag}([\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}], \dots, [\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}]), 1 \leq i \leq p$,

- (b) for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\{0\} \times \mathbb{R})$ there are $m_{i_1}, \dots, m_{i_r} \in \{m_1, \dots, m_p\}$ such that the minimal period of $u(t)$ equals $\frac{T}{\text{gcd}(m_{i_1}, \dots, m_{i_r})}$, and
- (c) if $\lambda_0 \in \mathcal{B}_{i_0}$ and $\lambda_0 \notin \mathcal{B}_i$ for $i \neq i_0$, then for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\{0\} \times \mathbb{R})$ sufficiently close to $(0, \lambda_0)$, the minimal period of $u(t)$ equals $\frac{T}{m_{i_0}}$.

Proof. To prove this theorem we shall apply Theorem 3.1.1 and Corollary 3.1.1 with $A(\lambda)$ replaced by $\lambda^2 A$. First of all notice that, if $(0, \lambda_0) \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ is a bifurcation point of solutions of the equation $\nabla_{\tilde{u}}\tilde{\Phi}(u, \lambda) = 0$, then $\lambda_0 \in \mathcal{B}$. Hence, there is $i_0 \in \{1, \dots, p\}$ such that $\lambda_0 \in \mathcal{B}_{i_0}$. Moreover, it is easy to see that all nontrivial coordinates of $\mathcal{J}(\lambda_0)$ have the same sign and that

$J(\lambda_0, i_0) \neq 0$. Thus, we immediately obtain that formula (3.1.1) cannot be satisfied. The rest of the proof is a consequence of Theorem 3.1.1. \square

The proof of the following theorem is the same as the proof of Theorem 3.1.3, replacing Theorem 3.1.1 and Corollary 3.1.1 by Theorem 3.1.2 and Corollary 3.1.2.

Theorem 3.1.4. *Suppose that $k_0 > 0$, and that conditions (a1')–(a3') are satisfied. Then for every $\lambda_0 \in \mathcal{B}$ the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^{1,p}} \times \mathbb{R}$ is unbounded. Moreover,*

(a) *for $(u_0(t), u_1(t), \dots, u_p(t)) \in \mathcal{C}(\lambda_0)$, we have*

$$u_0(t) = a_0, \quad u_i(t) = a_i \cos(2\pi m_i t/T) + J a_i \sin(2\pi m_i t/T),$$

where $a_0 \in \mathbb{R}[k_0, 0]$, $a_i \in \mathbb{R}[k_i, m_i]$ and $J = \text{diag}([\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}], \dots, [\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}])$, $1 \leq i \leq p$,

(b) *for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\mathbb{R}[k_0, 0] \times \mathbb{R})$, there are $m_{i_1}, \dots, m_{i_r} \in \{m_1, \dots, m_p\}$ such that the minimal period of $u(t)$ equals $\frac{T}{\text{gcd}(m_{i_1}, \dots, m_{i_r})}$, and*

(c) *if $\lambda_0 \in \mathcal{B}_{i_0}$ and $\lambda_0 \notin \mathcal{B}_i$ for $i \neq i_0$, then for every $(u(t), \lambda) \in \mathcal{C}(\lambda_0) \setminus (\{0\} \times \mathbb{R})$ sufficiently close to $(0, \lambda_0)$, the minimal period of $u(t)$ equals $\frac{T}{m_{i_0}}$.*

3.2. Toroidal potential

A potential $V \in C^2(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$ is called *toroidal*, if it satisfies the following condition:

(b1) There is $Q \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ such that

$$V(x_1, x_2, \dots, x_{2n-1}, x_{2n}, \lambda) = \frac{1}{2} Q(x_1^2 + x_2^2, \dots, x_{2n-1}^2 + x_{2n}^2, \lambda).$$

Remark 3.2.1. Notice that by assumption (a1) in Section 3.1, we obtain

$$\begin{aligned} V'_{x_{2i-1}}(x, \lambda) &= Q'_{y_i}(x_1^2 + x_2^2, \dots, x_{2n-1}^2 + x_{2n}^2, \lambda)x_{2i-1}, \\ V'_{x_{2i}}(x, \lambda) &= Q'_{y_i}(x_1^2 + x_2^2, \dots, x_{2n-1}^2 + x_{2n}^2, \lambda)x_{2i}, \\ V''_{x_{2i-1}x_{2i-1}}(0, \lambda) &= Q''_{y_i}(0, \lambda), \\ V''_{x_{2i}x_{2i}}(0, \lambda) &= Q''_{y_i}(0, \lambda), \quad i = 1, \dots, n, \\ V''_{x_k x_l}(0, \lambda) &= 0, \quad \text{for } k \neq l. \end{aligned}$$

Taking Remark 3.2.1 into account we may rewrite system (3.1) in the following form

$$\begin{cases} \ddot{u}_{2i-1}(t) = -Q'_{y_i}(u_1^2(t) + u_2^2(t), \dots, u_{2n-1}^2(t) + u_{2n}^2(t), \lambda)u_{2i-1}(t), \\ \ddot{u}_{2i}(t) = -Q'_{y_i}(u_1^2(t) + u_2^2(t), \dots, u_{2n-1}^2(t) + u_{2n}^2(t), \lambda)u_{2i}(t), \quad i = 1, \dots, n, \\ u(0) = u(T), \\ \dot{u}(0) = \dot{u}(T). \end{cases} \quad (3.2.1)$$

Moreover, the linearization of (3.2.1) at a stationary solution $(0, \lambda)$ is the following

$$\begin{cases} \ddot{u}_{2i-1}(t) = -Q'_{y_i}(0, \lambda)u_{2i-1}(t), \\ \ddot{u}_{2i}(t) = -Q'_{y_i}(0, \lambda)u_{2i}(t), & i = 1, \dots, n, \\ u(0) = u(T), \\ \dot{u}(0) = \dot{u}(T). \end{cases} \tag{3.2.2}$$

Fix $\lambda_0 \in \mathbb{R}$, $m \in \mathbb{N}$ and define

$$\mathcal{D}(m) = \left\{ \lambda \in \mathbb{R} \mid \prod_{i=1}^n \left(Q'_{y_i}(0, \lambda) - \frac{4m^2\pi^2}{T^2} \right) = 0 \right\},$$

$$\mathcal{B}_-(\lambda_0) = \text{card}\{i \in \{1, \dots, n\} \mid Q'_{y_i}(0, \lambda_0) = 0 \text{ and } Q'_{y_i}(0, \lambda) \text{ is decreasing at } \lambda_0\},$$

$$\mathcal{B}_+(\lambda_0) = \text{card}\{i \in \{1, \dots, n\} \mid Q'_{y_i}(0, \lambda_0) = 0 \text{ and } Q'_{y_i}(0, \lambda) \text{ is increasing at } \lambda_0\}.$$

Theorem 3.2.1. *Let condition (b1) be satisfied. If, besides, there are $\lambda_0 \in \mathbb{R}$ and $m \in \mathbb{N}$ such that*

- (i) $\lambda_0 \in \mathcal{D}(m)$,
- (ii) λ_0 is isolated in $\mathcal{D}(m)$, and
- (iii) $\mathcal{B}_+(\lambda_0) \neq \mathcal{B}_-(\lambda_0)$,

then the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^{1,\rho}} \times \mathbb{R}$ is unbounded or $\mathcal{C}(\lambda_0) \cap \{0\} \times (\mathcal{D}(m) \setminus \{\lambda_0\}) \neq \emptyset$, where ρ is given by (3.2) with $(\mathbb{W}, \rho_{\mathbb{W}}) = \mathbb{R}[n, m]$. Moreover, all elements of $\mathcal{C}(\lambda_0)$ are of the form

$$u(t) = a \cos(2\pi mt/T) + Ja \sin(2\pi mt/T), \tag{3.2.3}$$

where $a \in \mathbb{R}[n, m]$ and $J = \text{diag}([\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}], \dots, [\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}])$.

Proof. Fix $\bar{\lambda} \in \mathbb{R}$. First of all notice that by Corollary 5.1.1 of [10] and Remark 3.2.1 the following conditions are equivalent:

- (1) operator $\text{Id} - L_{A(\bar{\lambda})} : \mathbb{H}_T^1 \rightarrow \mathbb{H}_T^1$ is an isomorphism, and
- (2) $W'_{y_i}(0, \bar{\lambda}) \neq \frac{4m^2\pi^2}{T^2}$, for every $m \in \mathbb{N} \cup \{0\}$ and $1 \leq i \leq n$.

It is known that if $(0, \bar{\lambda}) \in \mathbb{H}_T^1 \times \mathbb{R}$ is a bifurcation point of the solutions of the equation $\nabla_u \Phi(u, \lambda) = 0$, then $\nabla_u^2 \Phi(0, \bar{\lambda}) = \text{Id} - L_{A(\bar{\lambda})}$ is not, the necessary condition for the existence of a bifurcation point of the solutions of the equation $\nabla_u \Phi(u, \lambda) = 0$ is fulfilled, i.e. $\nabla_u^2 \Phi(0, \lambda_0) = \text{Id} - L_{A(\lambda_0)}$ is not an isomorphism.

Define $(\mathbb{W}, \rho_{\mathbb{W}}) = \mathbb{R}[n, m]$. By Lemma 2.3.1, we obtain

$$((\mathbb{H}_T^1)^{\mathbb{S}^{1,\rho}}, \rho_I) = ((\mathbb{W} \otimes \mathbb{R}[1, m])^{\rho_{\mathbb{W}} \otimes \rho_m}, \rho_I) \cong \mathbb{R}[n, m].$$

By Corollary 2.3.1, $(\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ consists of elements of the form

$$u(t) = a \cos(2\pi mt/T) + J a \sin(2\pi mt/T),$$

where $a \in \mathbb{R}^{2n}$ and $J = \text{diag}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$.

From Remark 3.1 it follows that in order to complete the proof it is enough to study the critical orbits of the potential $\tilde{\Phi} : ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \times \mathbb{R} \rightarrow \mathbb{R}$. From Lemma 5.1.1 of [10] it follows that $\nabla_u^2 \tilde{\Phi}(0, \lambda)(u) = \Lambda(m, \lambda)u$, where $\Lambda(m, \lambda) = (\frac{4m^2\pi^2}{4m^2\pi^2+T^2} \text{Id} - \frac{T^2}{4m^2\pi^2+T^2} A(\lambda))$. From assumption (i) it follows that $\nabla_u^2 \tilde{\Phi}(0, \lambda_0)$ is not an isomorphism. By assumption (ii), $\nabla_u^2 \tilde{\Phi}(0, \lambda_0 \pm \varepsilon)$ is an isomorphism for a sufficiently small positive ε . By assumption (iii),

$$\nabla_{\mathbb{S}^1}\text{-deg}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 - \varepsilon), B_\alpha((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho})) \neq \nabla_{\mathbb{S}^1}\text{-deg}(\nabla_u^2 \tilde{\Phi}(0, \lambda_0 + \varepsilon), B_\alpha((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho})).$$

The rest of the proof is a direct consequence of Theorem 2.4.1. \square

Theorem 3.2.2. *Under the assumptions of Theorem 3.2.1 and if, moreover, $\mathcal{D}(m)$ does not possess finite accumulation points, then the conclusion of Theorem 3.2.1 holds true. Moreover, if $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R}$ is bounded, then*

$$\mathcal{C}(\lambda_0) \cap \{0\} \times \mathcal{D}(m) = \{(0, \lambda_0), \dots, (0, \lambda_s)\}$$

and

$$\sum_{i=0}^s \mathcal{B}_+(\lambda_i) = \sum_{i=0}^s \mathcal{B}_-(\lambda_i). \tag{3.2.4}$$

Proof. The proof of this theorem is the same as the proof of Theorem 3.2.1, but instead of Theorem 2.4.1, we have to apply Theorem 2.4.2. \square

The following corollary is a direct consequence of Theorem 3.1.2. Under the assumptions of this corollary formula (3.2.4) can never be satisfied.

Corollary 3.2.1. *Let the assumptions of Theorem 3.2.2 be fulfilled. If, moreover, $\mathcal{B}_+(\lambda'_0) = 0$ ($\mathcal{B}_-(\lambda'_0) = 0$) for every $\lambda'_0 \in \mathcal{D}(m)$, then for every $\lambda_0 \in \mathcal{D}(m)$ such that $\mathcal{B}_-(\lambda_0) > 0$ ($\mathcal{B}_+(\lambda_0) > 0$), the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R}$ is unbounded. Moreover, all elements of $\mathcal{C}(\lambda_0)$ have the form (3.2.3).*

To finish this section we shall discuss a special case of the results proved above. Namely, we assume additionally

$$(b2) \quad V(x_1, x_2, \dots, x_{2n-1}, x_{2n}, \lambda) = \frac{\lambda^2}{2} Q(x_1^2 + x_2^2, \dots, x_{2n-1}^2 + x_{2n}^2).$$

In other words, we shall discuss bifurcations of nonstationary periodic solutions of the system $\ddot{u}(t) = -V'(u(t))$ with an arbitrary period.

Define $\mathbb{I} = \{i \in \{1, \dots, n\} : Q'_{y_i}(0) > 0\}$. By condition (a3) we obtain

$$\mathcal{D}_+(1) = \left\{ \lambda \in (0, +\infty) \mid \prod_{i=1}^n \left(Q'_{y_i}(0, \lambda) - \frac{4\pi^2}{T^2} \right) = 0 \right\} = \left\{ \frac{2\pi}{T\sqrt{Q'_{y_i}(0)}} \mid i \in \mathbb{I} \right\}.$$

Moreover, it is easy to verify that for every $\lambda_0 \in \mathcal{D}_+(1)$, we have $\mathcal{B}_+(\lambda_0) > 0$ and $\mathcal{B}_-(\lambda_0) = 0$.

Corollary 3.2.2. *Let conditions (b1)–(b2) be satisfied. Then, for every $\lambda_0 \in \mathcal{D}_+(1)$, the continuum $\mathcal{C}(\lambda_0) \subset (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \times \mathbb{R}$ is unbounded. Moreover, all elements of $\mathcal{C}(\lambda_0)$ have the form (3.2.3) with $m = 1$.*

4. Existence of periodic solutions

In this section we shall discuss the asymptotically linear Newtonian system (3.1) without parameter. To derive a least number of its nontrivial (nonstationary) periodic orbits, we shall use the approach of Amann and Zehnder [1]. We give a slight generalization of their theorem by allowing degeneracy of the Hessians at 0 and ∞ . We also assume, as before, that the phase-space has a structure of an orthogonal representation $(\mathbb{W}, \rho_{\mathbb{W}})$ of \mathbb{S}^1 and that the potential is invariant. Imposing the combined action on the function space once more, and using the Palais symmetry principle (Remark 3.1), we can restrict the functional of the problem to the fixed point set of this representation, which is isomorphic to $(\mathbb{W}, \rho_{\mathbb{W}})$ (Lemma 2.1.3). Due to our assumption that the potential V is $\rho_{\mathbb{W}}$ -invariant, the corresponding finite-dimensional reduction is effective. Consequently, the Amann–Zehnder index of such a problem is computable in terms of $V : \mathbb{W} \rightarrow \mathbb{R}$ and its second derivatives at 0 and at ∞ .

Let $(\mathbb{W}, \rho_{\mathbb{W}})$ be an orthogonal representation of $G = \mathbb{S}^1$ of the form

$$(\mathbb{W}, \rho_{\mathbb{W}}) = \bigoplus_{i=0}^p \mathbb{R}[k_i, m_i], \quad m_i, k_i \in \mathbb{N}, \quad i = 1, \dots, p, \quad m_0 = 0, \tag{4.1}$$

i.e. $\mathbb{W}^G = \mathbb{R}[k_0, 0] \cong \mathbb{R}^{k_0}$. Note that $d = \dim_{\mathbb{R}} \mathbb{W} = 2n + k_0$, where $n = \sum_1^p k_i$. Put $\mathbb{W}^i = \mathbb{R}[k_i, m_i]$. Then the above decomposition into a direct sum of subrepresentations has the form $\mathbb{W} = \bigoplus_{i=0}^p \mathbb{W}^i$, where $\mathbb{W}^0 = \mathbb{W}^G$. Let next $\varphi : \mathbb{W} \rightarrow \mathbb{R}$ be a G -invariant function of class C^2 .

We shall assume that

- (i) $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\{0\}$ is an isolated zero of the gradient of φ .

The above assumption yields that locally near 0 the function φ is of the form

$$\varphi(z) = \frac{1}{2} \langle A_0 z, z \rangle + o(|z|^2),$$

where $A_0 = \varphi''(0)$ is a self-adjoint linear operator with respect to a G -invariant scalar product.

Note that A_0 is G -equivariant, and consequently $\frac{1}{2} \langle A_0 z, z \rangle$ is a G -invariant quadratic form.

(ii) φ is asymptotically linear, i.e. there exists a self-adjoint linear operator $A_\infty : \mathbb{W} \rightarrow \mathbb{W}$ such that

$$\varphi'(z) = A_\infty z + o(|z|) \quad \text{as } |z| \rightarrow \infty. \tag{4.2}$$

Since the scalar product and the norm induced by it are G -invariant and the map $\varphi' : \mathbb{W} \rightarrow \mathbb{W}$ is G -equivariant, we have for every $g \in G$

$$A_\infty g z - g A_\infty z = o(|z|).$$

Fixing $z = z_0$, $g = g_0$ and then taking $z = z_0 t$, $t \in [1, \infty)$, we show that A_∞ is G -equivariant.

For the following we need some notation. Let \mathbb{W}_0^+ , \mathbb{W}_0^- , and \mathbb{W}_0^0 be the positive part, the negative part, and the kernel of A_0 , respectively. Analogously, let \mathbb{W}_∞^+ , \mathbb{W}_∞^- and \mathbb{W}_∞^0 be the corresponding parts of A_∞ . Note that all the above are linear G -invariant subspaces (subrepresentations) of \mathbb{W} . Denote by d_0^+ , d_0^- , and d_0^0 , and correspondingly by d_∞^+ , d_∞^- , and d_∞^0 , their real dimensions.

We shall also assume the following:

(iii) For the fixed point subspace either $\mathbb{W}^G \subset \mathbb{W}_\infty^+$ or $\mathbb{W}^G \subset \mathbb{W}_\infty^-$.

To formulate the main theorem of this section we need a notion of index which is a slight adaptation of the Amann–Zehnder index [1]. First note that $d_0^+ + d_0^- + d_0^0 = d = d_\infty^+ + d_\infty^- + d_\infty^0$. We define a nonnegative integer $i(\varphi)$ of an asymptotically linear G -invariant function $\varphi : \mathbb{W} \rightarrow \mathbb{R}$ of class C^2 , that satisfies (i) and (ii) by the formula

$$i(\varphi; 0, \infty) := \begin{cases} \frac{1}{2} \max\{0, d_0^- - (d_\infty^- + d_\infty^0)\} & \text{if } \mathbb{W}^G \subset \mathbb{W}_\infty^+, \\ \frac{1}{2} \max\{0, d_0^+ - (d_\infty^+ + d_\infty^0)\} & \text{if } \mathbb{W}^G \subset \mathbb{W}_\infty^-. \end{cases} \tag{4.3}$$

Remark 4.1. Note that if condition (iii) holds, then d_0^+ , d_0^- , and $d_0^- - (d_\infty^- + d_\infty^0)$, $d_0^+ - (d_\infty^+ + d_\infty^0)$ are even numbers, since they are either dimensions or differences of dimensions of orthogonal representations of \mathbb{S}^1 with trivial fixed point subspace. Observe that (iii) holds if $\mathbb{W}^G = \{0\}$. Notice also that if $d_0^0 = d_\infty^0 = 0$, then

$$d_0^+ - d_\infty^+ = [d - d_0^-] - [d - d_\infty^-] = d_\infty^- - d_0^- = -(d_0^- - d_\infty^-).$$

Thus, if one of these integers is nonzero, then at least one of them is positive. Consequently, if A_0 and A_∞ are nondegenerate, then $i(\varphi; 0, \infty) = \frac{1}{2}|d_0^- - d_\infty^-|$, provided condition (iii) holds, e.g. if $\mathbb{W}^G = \{0\}$.

Proposition 4.1. *Let $\varphi : \mathbb{W} \rightarrow \mathbb{R}$ be a G -invariant function of class C^2 . Suppose also that 0 is the only critical point of φ in \mathbb{W}^G , i.e. such that $\varphi'(z) = 0$ and $z \in \mathbb{W}^G$ implies $z = 0$. Then, under the assumptions (i)–(iii), the function φ has at least $i(\varphi; 0, \infty)$ different nonstationary critical orbits.*

Remark 4.2. By the Palais symmetry principle it is enough to check that 0 is the only critical point of $\varphi|_{\mathbb{W}^G}$. In Remark 4.3 below, we shall formulate another condition which can replace the one discussed.

The proof of Proposition 4.1 follows from the proof of Theorem 4 in [1]. The main step consists in estimating the number of nontrivial critical orbits of an \mathbb{S}^1 -invariant function of class C^2 [1, Lemma 10].

To state the next lemma we have to recall the notion of an equivariant minimax invariant.

Definition 4.1. Let Z be a G -space and \mathcal{F} the family of all invariant subsets of Z . We say that a function $\gamma : \mathcal{F} \rightarrow \mathbb{N} \cup \{\infty\}$ is an *equivariant minimax invariant* relative to the fixed point set, if it has the following properties:

- ($\gamma 0$) If $X = \emptyset$, then $\gamma(X) = 0$, and if $X \neq \emptyset$ then $\gamma(X) \geq 1$.
- ($\gamma 1$) If $X, Y \in \mathcal{F}$, then $\gamma(X \cup Y) \leq \gamma(X) + \gamma(Y)$.
- ($\gamma 2$) If $X, Y \in \mathcal{F}$ and $f : X \rightarrow Y$ is a continuous equivariant map, then $\gamma(X) \leq \gamma(Y)$. Equality holds if f is a homeomorphism.
- ($\gamma 3$) If $X \in \mathcal{F}$ is compact, then there is a compact neighborhood $N(X) \in \mathcal{F}$ such that $\gamma(X) = \gamma(N(X))$.
- ($\gamma 4$) If $X \cap Z^G \neq \emptyset$, then $\gamma(X) = \infty$.
- ($\gamma 5$) If $1 < \gamma(X) < \infty$, then the orbit space X/G is an infinite set.
- ($\gamma 6$) (Normalization) If $E \subset \mathbb{W}$ is an invariant linear subspace of an orthogonal representation with $\dim_{\mathbb{R}} E = k$ that satisfies $E \cap \mathbb{W}^G = \{0\}$, then $\gamma(E \cap S_R) = \frac{1}{2}k$ for every $R > 0$, where S_R is the sphere of radius R in \mathbb{W} .

Proposition 4.2. For the group $G = \mathbb{S}^1$ there exists an equivariant minimax invariant relative to the fixed point set.

Proof. As a matter of fact, for compact Lie groups there are many constructions of equivariant minimax invariants that satisfy ($\gamma 0$)–($\gamma 5$). A G -cohomological index was introduced for $G = \mathbb{S}^1$ by Fadell and Rabinowitz in [9], and used by Amann and Zehnder. Then it was generalized to every G by Fadell and Husseini [8]. A G -genus was first considered for $G = \mathbb{Z}_2$ by Krasnoselski, and independently by Yang. It was generalized to any compact Lie group by Bartsch [2] and Clapp and Puppe [7]. A G -capacity was introduced by Clapp [6]. An \mathbb{S}^1 -equivariant geometrical index was defined by Benci [4], and a G -index for any orthogonal compact Lie group action by Marzantowicz [11]. For the group $G = \mathbb{S}^1$ all these constructions have the normalization property ($\gamma 6$) due to a version of the Borsuk–Ulam theorem (see [3] for more information). \square

To pick up critical orbits by a standard minimax procedure, we define for each $n \in \mathbb{N}$ the family of subsets $\Gamma_n = \{X \in \mathcal{F} \mid n \leq \gamma(X) < \infty\}$. Next we define for a function $\varphi : Z \rightarrow \mathbb{R}$ the minimax levels $c_n(\varphi) = \inf_{X \in \Gamma_n} \sup_{x \in X} (\varphi(x))$, provided that $\Gamma_n \neq \emptyset$. It is obvious that $c_1(\varphi) \leq c_2(\varphi) \leq \dots$. The basic properties of the critical levels of any equivariant minimax invariant are given in the following lemma (see [3] for references).

Lemma 4.1. Let Z be a smooth G -manifold and $\gamma : \mathcal{F} \rightarrow [0, \infty]$ an equivariant minimax invariant relative to the fixed point set (that satisfies conditions ($\gamma 0$)–($\gamma 5$)). Assume moreover that $\varphi \in C^2(Z, \mathbb{R})$ is invariant and satisfies the Palais–Smale condition. Suppose also that for some integers n and k one has

$$-\infty < c = c_n(\varphi) = c_{n+1}(\varphi) = \dots = c_{n+k}(\varphi) < 0.$$

Then, if $\gamma(K_c) < \infty$, we have $\gamma(K_c) \geq k + 1$, where $K_c = K_c(\varphi) = \{z \in Z \mid \varphi(z) = c \text{ and } \varphi'(z) = 0\}$ is the critical set of φ at the level c .

We are now in position to formulate the following lemma [1, Lemma 10].

Lemma 4.2. *Let $\varphi \in C^2(\mathbb{W}, \mathbb{R})$ such that $\varphi(0) = 0$ be invariant under the orthogonal action ρ of \mathbb{S}^1 and satisfy the Palais–Smale condition and let γ be an equivariant minimax invariant relative to the fixed point set. Assume moreover the following:*

- (1) *There is an invariant subspace $\mathbb{W}' \subset \mathbb{W}$, with $\mathbb{W}' \cap \mathbb{W}^G = \{0\}$, such that $\varphi(z) < 0$ for all $z \in \mathbb{W}'$ satisfying $|z| = \rho$, some fixed $\rho > 0$.*
- (2) *There is an invariant subspace $\mathbb{W}'' \subset \mathbb{W}$ with $\mathbb{W}^G \subset \mathbb{W}''$, such that φ is bounded below on \mathbb{W}'' .*
- (3) *$d' + d'' > d = \dim_{\mathbb{R}} \mathbb{W}$, where $d' = \dim_{\mathbb{R}} \mathbb{W}'$ and $d'' = \dim_{\mathbb{R}} \mathbb{W}''$.*

Then we have that $-\infty < c_k(\varphi) < 0$ for all $k \in \mathbb{N}$ such that $\frac{d-d''}{2} < k \leq \frac{d'}{2}$, and φ has at least $\frac{1}{2}(d' + d'' - d)$ different nonstationary critical orbits, provided $K_{c_k} \cap \mathbb{W}^G = \emptyset$.

Proof. A proof of the above lemma can be found in [1]. For the convenience of the reader, we include it here.

For $a \in \mathbb{R}$ let φ_a be the set $\varphi^{-1}(\infty, a] \in \mathbb{W}'$, and let $S_\rho(\mathbb{W}')$ be the sphere of radius ρ in \mathbb{W}' . By assumption (1), there is a positive $\sigma > 0$ such that $\varphi(\zeta) \leq -\sigma < 0$ for $\zeta \in S_\rho(\mathbb{W}') = S_\rho(\mathbb{W}) \cap \mathbb{W}'$. Hence, $S_\rho(\mathbb{W}') \subset \varphi_{-\sigma}$ and by $(\gamma 2)$, $\gamma(S_\rho(\mathbb{W}')) \leq \gamma(\varphi_{-\sigma})$. Since $\mathbb{W}^G \cap \mathbb{W}' = \{0\}$, we have $\gamma(S_\rho(\mathbb{W}')) = d'/2$ by $(\gamma 6)$ and therefore, $c_\tau(\varphi) \leq -\sigma < 0$ and so $c_k(\varphi) < 0$ and so $c_k < 0$ for $k \leq d'/2$. In view of assumption (2) we can choose $\tau < 0$ with $\varphi_\tau \cap \mathbb{W}'' = \emptyset$. If $\pi : \mathbb{W} \rightarrow (\mathbb{W}'')^\perp$ denotes the orthogonal projection, then $\pi : \varphi_\tau \rightarrow (\mathbb{W}'')^\perp \setminus \{0\}$ is continuous and equivariant, hence by $(\gamma 2)$ and $(\gamma 6)$ we find $\gamma(\varphi_\tau) \leq \frac{1}{2} \dim(\mathbb{W}'')^\perp = \frac{1}{2}(\dim \mathbb{W} - \dim \mathbb{W}'')$, since $\mathbb{W}^G \cap (\mathbb{W}'')^\perp = \{0\}$. Let $j > \frac{1}{2}(\dim \mathbb{W} - \dim \mathbb{W}'')$ and assume that there is $X \in \mathcal{F}$ such that $j \leq \gamma(X) < \infty$, then $\sup_{x \in X} \varphi(x) > \tau$. If, in addition, $j \leq d'/2$, then there is indeed such a set by the assumptions (1), (3), and $(\gamma 6)$. Consequently, $c_j(\varphi) \geq \tau > -\infty$ for $\frac{1}{2}(d - d'') < j \leq \frac{d'}{2}$ and the statement follows from Lemma 4.1, in the case that the levels $c_k(\varphi)$ are different and $K_{c_k} \cap \mathbb{W}^G = \emptyset$. If, however, two or more of these levels coincide, say they are equal to c , then by $(\gamma 5)$ the orbit space K_c/G is an infinite set. This finishes the proof of the lemma. \square

Remark 4.3. To avoid problems with the fixed point set, one can pose another condition on φ , besides that $\varphi'(z) = 0$ and $z \in \mathbb{W}^G$ imply $z = 0$. It is enough to know that $K_{c_k} \cap \mathbb{W}^G = \emptyset$ for every $k \in \mathbb{N}$. In particular, we can assume that $\varphi|_{\mathbb{W}^G} \geq 0$ if $\mathbb{W}^G \subset \mathbb{W}_\infty^+$, respectively $\varphi|_{\mathbb{W}^G} \leq 0$ if $\mathbb{W}^G \subset \mathbb{W}_\infty^-$. If $\mathbb{W}^G = \{0\}$, then $K_c \cap \mathbb{W}^G = \emptyset$ for every $c \neq 0$ and this assumption is satisfied.

Proof of Proposition 4.1. We shall see that the assumptions of Lemma 4.2 hold. Suppose first that $\mathbb{W}^G \subset \mathbb{W}_\infty^+$ and $d_0^- - (d_\infty^- + d_\infty^0) > 0$ which is equivalent to $d_0^- + d_\infty^+ > d$.

Let us take $\mathbb{W}' = \mathbb{W}_0^-$ and $\mathbb{W}'' = \mathbb{W}_\infty^+$. We have to check conditions (1)–(3).

- (1) Since $\varphi(0) = 0$, $\varphi'(0) = 0$ and φ is C^2 , we have

$$\varphi(z) = \varphi''(0)(z, z) + o(|z|^2) = \frac{1}{2}(A_0 z, z) + o(|z|^2)$$

for small $|z|$. If $z \in \mathbb{W}_0^-$ then $\frac{1}{2}\langle A_0z, z \rangle \leq -\delta|z|^2$, where δ is the largest (smallest up to the module) negative eigenvalue of $A_0|_{\mathbb{W}_0^-}$. Take $\rho > 0$ such that $o(|z|) < \frac{1}{2}\delta|z|^2$ if $|z| \leq \rho$. Then for $|z| = \rho$ we have $\varphi(z) < -\frac{1}{2}\delta|z|^2 = -\frac{1}{2}\rho^2$ for all $z \in S_\rho(\mathbb{W}_0^-)$.

(2) First observe that for every $z \in \mathbb{W}_\infty^+$, $\frac{1}{2}\langle A_\infty z, z \rangle \geq \frac{\delta}{2}|z|^2$, where δ is the smallest eigenvalue of $A_\infty|_{\mathbb{W}_\infty^+}$.

On the other hand, equality (4.2) implies that for every $\tilde{\delta}$ there exists $c > 0$ such that $|\varphi'(z) - A_\infty z| \leq \tilde{\delta}|z| + c$ for each $z \in \mathbb{W}$. Indeed, let $R > 0$ be such that $|\varphi'(z) - A_\infty z| \leq \tilde{\delta}|z|$ for $|z| \geq R$. Then we have $|\varphi'(z) - A_\infty z| \leq \max\{c = \sup_{|z| \leq R} |\varphi'(z) - A_\infty z|, \tilde{\delta}|z|\}$, which gives the required inequality.

Moreover, by the mean value theorem, we have

$$\left| \varphi(z) - \frac{1}{2}\langle A_\infty z, z \rangle \right| \leq \int_0^T |(\varphi'(tz) - A_\infty tz, z)| dt \leq \int_0^T [\tilde{\delta}t|z|^2 + c|z|] dt \leq T^2\tilde{\delta}|z|^2 + Tc|z|$$

for every $z \in \mathbb{W}_\infty^+$.

Consequently, for $z \in \mathbb{W}_\infty^+$,

$$\begin{aligned} \varphi(z) &= \frac{1}{2}\langle A_\infty z, z \rangle + \varphi(z) - \frac{1}{2}\langle A_\infty z, z \rangle \\ &\geq (\delta/2)|z|^2 - (T^2\tilde{\delta}|z|^2 + Tc|z|) = (\delta/2 - \tilde{\delta}T^2)|z|^2 - cT|z|. \end{aligned}$$

If $\tilde{\delta}$ is such that $\delta/2 - \tilde{\delta}T^2 > 0$ then the coefficient at $|z|^2$ is positive, which shows that φ is bounded from below on \mathbb{W}_∞^+ .

Applying Lemma 4.2 to the function φ we get the first case of the statement of Proposition 4.1.

If $\mathbb{W}^G \subset \mathbb{W}_\infty^-$ then we can consider the function $\tilde{\varphi} = -\varphi$. Note that the critical points (orbits) of $\tilde{\varphi}$ are the same as of φ . Moreover, we have $\tilde{d}_0^0 = d_0^0$, $\tilde{d}_\infty^0 = d_\infty^0$, $\tilde{d}_0^\pm = d_0^\mp$, and $\tilde{d}_\infty^\pm = d_\infty^\mp$. Applying the first part of the proof to $\tilde{\varphi}$ we get the second case of the statement of Proposition 4.1. \square

4.1. Computations in terms of the potential

In this subsection we derive the Amann–Zehnder index (4.3) of the restriction $\tilde{\Phi}$ of the functional (3.3) to the space $(\mathbb{H}_T^1)^{G,\rho} \cong \mathbb{W}$, i.e. the function $\varphi : \mathbb{W} \rightarrow \mathbb{R}$ is equal to $\tilde{\Phi}$. To do it we use formulas for $\nabla\Phi$ and $\nabla\tilde{\Phi}$ given in Section 3. Note that here the potential V and consequently Φ does not depend on the parameter λ .

Let $V : \mathbb{W} \rightarrow \mathbb{R}$ be an \mathbb{S}^1 -invariant potential of class C^2 asymptotically linear at infinity. Let $A(0) = D^2V(0)$, i.e. near $z = 0$ the potential is given as $V(z) = \langle A(0)z, z \rangle + \eta(z)$, $\eta(z) = o(|z|^2)$. Analogously, we denote the linear part of V at infinity by $A(\infty)$, i.e.

$$\nabla V(z) = A(\infty)z + o(|z|) \quad \text{as } |z| \rightarrow \infty.$$

We can write V as $V(z) = \frac{1}{2}\langle A(\infty), z \rangle + \zeta(z)$, where $\frac{|\zeta(z)|}{|z|^2}$ is bounded by the inequality in the last part of the proof of Proposition 4.1.

By the Palais symmetry principle (cf. Remark 3.1), we have

- (1) $\nabla_u \Phi(u, \lambda) = (\nabla_{\tilde{u}} \tilde{\Phi}(\tilde{u}, \lambda), 0)$ for $u = (\tilde{u}, 0) \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}$ and
- (2) $\tilde{\Phi} : ((\mathbb{H}_T^1)^{\mathbb{S}^1, \rho}, \rho_I) \times \mathbb{R} \rightarrow \mathbb{R}$ is \mathbb{S}^1 -invariant.

Notice that the study of solutions of the equation $\nabla_u \Phi(u, \lambda) = 0$ is equivalent to the study of solutions of the equation $\nabla_{\tilde{u}} \tilde{\Phi}(\tilde{u}, \lambda) = 0$. To emphasize that our variable $u = (\tilde{u}, 0) \in (\mathbb{H}_T^1)^{\mathbb{S}^1, \rho} \cong \mathbb{W}$ lies in a finite-dimensional space we shall denote it by $z = (z_0, \dots, z_p) \in \mathbb{W} = \bigoplus_{i=0}^p \mathbb{W}^i$.

Consequently, we have to check the assumptions of Proposition 4.1 for $\varphi(z) = \tilde{\Phi}(\tilde{u})$. Let us recall that the function φ has the following properties already listed in Section 3.

- $\varphi : \mathbb{W} \rightarrow \mathbb{R}$ is \mathbb{S}^1 -invariant and of class C^2 ,
- $\varphi(0) = \varphi'(0) = 0$,
- $A_0 = \varphi''(0)(z_0, \dots, z_p) = (\Lambda(m_0)z_0, \dots, \Lambda(m_p)z_p)$, where

$$A_0(m_i) = \left(\frac{4m_i^2\pi^2}{4m_i^2\pi^2 + T^2} \text{Id} - \frac{T^2}{4m_i^2\pi^2 + T^2} A_i(0) \right),$$

and $A_i(0) = A(0)|_{\mathbb{W}^i}$.

Moreover, since V is asymptotically linear, so is φ' and the self-adjoint operator A_∞ of the derivative of φ' at infinity is equal to

- $A_\infty = (\Lambda(m_0)z_0, \dots, \Lambda(m_p)z_p)$, where

$$A_\infty(m_i) = \left(\frac{4m_i^2\pi^2}{4m_i^2\pi^2 + T^2} \text{Id} - \frac{T^2}{4m_i^2\pi^2 + T^2} A_i(\infty) \right),$$

and $A_i(\infty) = A(\infty)|_{\mathbb{W}^i}$.

The above formulas allow us to derive the subrepresentations $\mathbb{W}_0^+ = \bigoplus_{i=0}^p \mathbb{W}_0^{i+}$, $\mathbb{W}_0^- = \bigoplus_{i=0}^p \mathbb{W}_0^{i-}$, $\mathbb{W}_0^0 = \bigoplus_{i=0}^p \mathbb{W}_0^{i0}$, and the corresponding spaces $\mathbb{W}_\infty^\pm, \mathbb{W}_\infty^0$.

For every $0 \leq i \leq p$ let \mathbb{W}_0^{i+} be the linear subspace corresponding to the eigenvalues of $A(0)$ which are smaller than $\frac{4m_i^2\pi^2}{T^2}$, and let d_0^{i+} be its dimension; analogously, let \mathbb{W}_0^{i-} be the linear subspace corresponding to all eigenvalues of $A(0)$ which are greater than $\frac{4m_i^2\pi^2}{T^2}$, and let d_0^{i-} be its dimension. Finally, let \mathbb{W}_0^{i0} be the linear subspace corresponding to the eigenvalues of $A(0)$ which are equal to $\frac{4m_i^2\pi^2}{T^2}$ and let d_0^{i0} be its dimension.

Similarly, we define the linear subspaces $\mathbb{W}_\infty^{i+}, \mathbb{W}_\infty^{i-}$, and \mathbb{W}_∞^{i0} , and their dimensions $d_\infty^{i+}, d_\infty^{i-}$, and d_∞^{i0} , respectively. Note that all the above subspaces are invariant, i.e. they are subrepresentations because they are defined as eigenspaces of restrictions of self-adjoint equivariant operators to invariant subspaces.

Remark 4.1.1. Note that the derivative of $\varphi|_{\mathbb{W}^G} = \tilde{\Phi}|_{\mathbb{W}^G}$ is equal to $-\nabla V(z), z \in \mathbb{W}^G$.

By the considerations above, we obtain the following.

Proposition 4.1.1. *Let $V : \mathbb{W} \rightarrow \mathbb{R}$ be an invariant potential of class C^2 , asymptotically linear at infinity, satisfying assumptions (i) and (ii), and let $\varphi = \tilde{\Phi} : \mathbb{W} \rightarrow \mathbb{R}$ be a C^2 , invariant function induced by functional (3.3). Then the positive, negative, and zero subspaces of φ at 0, respectively, at ∞ , are equal to*

$$\mathbb{W}_0^+ = \bigoplus_{i=0}^p \mathbb{W}_0^{i+}, \quad \mathbb{W}_0^- = \bigoplus_{i=0}^p \mathbb{W}_0^{i-}, \quad \mathbb{W}_0^0 = \bigoplus_{i=0}^p \mathbb{W}_0^{i0},$$

respectively,

$$\mathbb{W}_\infty^+ = \bigoplus_{i=0}^p \mathbb{W}_\infty^{i+}, \quad \mathbb{W}_\infty^- = \bigoplus_{i=0}^p \mathbb{W}_\infty^{i-}, \quad \mathbb{W}_\infty^0 = \bigoplus_{i=0}^p \mathbb{W}_\infty^{i0},$$

where all the summands are as defined above. Consequently, their dimensions are equal to

$$d_0^+ = \sum_{i=0}^p d_0^{i+}, \quad d_0^- = \sum_{i=0}^p d_0^{i-}, \quad d_0^0 = \sum_{i=0}^p d_0^{i0},$$

respectively,

$$d_\infty^+ = \sum_{i=0}^p d_\infty^{i+}, \quad d_\infty^- = \sum_{i=0}^p d_\infty^{i-}, \quad d_\infty^0 = \sum_{i=0}^p d_\infty^{i0}.$$

Moreover, assumption (iii) $\mathbb{W}^G \subset \mathbb{W}_\infty^+$, or correspondingly $\mathbb{W}^G \subset \mathbb{W}_\infty^-$, is equivalent to $\mathbb{W}^G = \mathbb{W}^0 \subset \mathbb{W}_\infty^+$, i.e. $A(\infty)$ is positive definite on \mathbb{W}^G , respectively, $A(\infty)$ is negative definite on \mathbb{W}^G , provided that the subspace \mathbb{W}^G is nonzero.

By the previous proposition and Proposition 4.1, we get the following.

Theorem 4.1.1. *Suppose that Eq. (3.1) is given by an \mathbb{S}^1 -invariant potential V of class C^2 which is asymptotically linear at infinity. Assume next that if $\mathbb{W}^G \neq \{0\}$ then its derivative at infinity $A(\infty)$ is either positive definite or negative definite on \mathbb{W}^G . Assume next that either 0 is the only critical point of $V|_{\mathbb{W}^G}$ or*

- (a) $V(z) \geq 0$ for $z \in \mathbb{W}^G$, respectively,
- (b) $V(z) \leq 0$ for $z \in \mathbb{W}^G$.

Then in case (a) there exist at least $i(\varphi; 0, \infty) = \frac{1}{2} \max\{0, d_0^- - (d_\infty^- + d_\infty^0)\}$ nonstationary periodic solutions of (3.1), and in case (b) there exist at least $\frac{1}{2} \max\{0, d_0^+ - (d_\infty^+ + d_\infty^0)\}$ nonstationary periodic solutions of (3.1).

If $\mathbb{W}_0^0 = \mathbb{W}_\infty^0 = \{0\}$, then there exist at least $i(\varphi; 0, \infty) = \frac{1}{2} |d_0^- - d_\infty^-|$ nonstationary periodic solutions of (3.1).

Moreover, in case (a) every such solution has the form $u(t) = (u_1(t), \dots, u_p(t))$, where $u_i(t) \in \mathbb{W}^i$, and $u_i(t) \neq 0$ if $\mathbb{W}_0^{i-} \neq \{0\}$. Consequently, for every i such that $\mathbb{W}_0^{i-} \neq \{0\}$ one

has $u_i(t) = a_i \cos\left(\frac{2\pi m_i t}{T}\right) + J a_i \sin\left(\frac{2\pi m_i t}{T}\right)$, and thus the minimal period of u is $\frac{T}{\gcd(m_i)}$, where i is as above.

In case (b) we have an analogous statement with \mathbb{W}_0^{i-} replaced by \mathbb{W}_0^{i+} .

Corollary 4.1.1. *Assume that in Eq. (3.1) the potential $V : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is toroidal and that all other analytical assumptions of the previous theorem are satisfied. Then \mathbb{R}^{2n} can be equipped with a free orthogonal action of \mathbb{S}^1 , for which V is invariant. Consequently, each of the solutions given by the previous theorem has the form*

$$u(t) = a \cos\left(\frac{2\pi t}{T}\right) + J a \sin\left(\frac{2\pi t}{T}\right),$$

where $a \in \mathbb{R}^{2n}$.

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