CORRECTIONS TO "EQUIVARIANT HOMOTOPICAL HOMOLOGY WITH COEFFICIENTS IN A MACKEY FUNCTOR"

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In our paper [1], the proof of Theorem 4.1, as well as those of Propositions 3.12 and 3.14 are not correct. This occurs because Proposition 3.3 is not true, namely, for a simplicial G-set K, the groups $F^G(|K|, M)$ and $|F^G(K, M)|$, which are isomorphic as groups, are not in general isomorphic as topological groups. One obtains a correct proof of these results simply by changing the definition of the topological group $F^G(X, M)$, given in 3.4 of [1]. In fact, the new definition was used in the construction of the original one.

Definition 0.1. Let X be a pointed G-space and M a Mackey functor for G. If $\mathcal{S}(X)$ denotes the singular simplicial G-set of X, then define

$$F^G(X, M) = |F^G(\mathcal{S}(X), M)|.$$

Moreover, if $f : X \longrightarrow Y$ is a pointed G-map, then we define $f_*^G : F^G(X, M) \longrightarrow F^G(Y, M)$ by $f_*^G = |\mathcal{S}(f)_*^G|$.

Remark 0.2. One may similarly redefine $F(X, M) = |F(\mathcal{S}(X), M)|$, and the map $\beta_X = |\beta_{\mathcal{S}(X)}| : F(X, M) \twoheadrightarrow F^G(X, M)$ is an identification, as stated in 3.7 of [1]. Our new definition also replaces 3.1, 3.2, and 3.6.

If K is a simplicial set, such that $K_m = C$ for all m and $f^K = \mathrm{id}$ for all f in Δ, then $|K|$ is a discrete space homeomorphic to C. Therefore, if X has the discrete topology, then $F^G(X, M) = |F^G(\mathcal{S}(X), M)| = F^G(X^\delta, M)$, where we denote by X^δ the underlying G-set of the G-space X, hence the new definition is consistent with the definition for the discrete case.

For any space X, $F^G(X, M)$ is now not only of the homotopy type of a CW-complex, as stated in Proposition 4.8 of [1], but it is in fact a regular CW-complex.

Then, with the new definition of the topological group $F^G(X, M)$, the proof of Theorem 4.1 is correct, since in the diagram the arrows $(\rho_*^G)_*$ and ψ_* on the bottom can be replaced by the identity.

On the other hand, the proof of Proposition 3.12 is as follows.

First recall that if $p : A \longrightarrow C$ is an n-to-1 G-function, then its transfer is given by

$$t_p^G(\gamma_x^G(l)) = \sum_{[a] \in p^{-1}(x)/G_x} \gamma_a^G(M^*(\widehat{p}_a)(l))\,. \tag{0.3}$$

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Consider now the pullback diagram

$$\begin{array}{ccc} A' & \xrightarrow{\alpha'} & A \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ D & \xrightarrow[\alpha]{} & C\,. \end{array}$$

Take the restriction of α' from the fiber $(p')^{-1}(y)$ to the fiber $p^{-1}(\alpha(y))$. This function induces a surjective function

$$q : (p')^{-1}(y)/G_y \longrightarrow p^{-1}(\alpha(y))/G_{\alpha(y)}\,,$$

given by $q([a']) = [\alpha'(a')]$. If $[a_0] \in p^{-1}(\alpha(y))/G_{\alpha(y)}$, then one easily shows that there is a bijection

$$\overline{\delta} : G_y\backslash G_{\alpha(y)}/G_{a_0} \longrightarrow q^{-1}([a_0])\,, \tag{0.4}$$

given by $\overline{\delta}({}_{G_y}[g]_{G_{a_0}}) = [a']$, where $p'(a') = y$ and $\alpha'(a') = ga_0$.

One has the following.

Lemma 0.5. *Let $H, H' \subset K \subset G$ be subgroups of G and consider the fibered product*

$$G/H \times_{G/K} G/H' = \{([g]_H, [g']_{H'}) \mid g, g' \in G \text{ and } g^{-1}g' \in K\}\,.$$

Take the set of double cosets $H\backslash K/H' = \{{}_H[g_\kappa]_{H'} \mid \kappa = 1, \ldots, k\}$, where $g_1, \ldots, g_k \in K$ are fixed representatives. If $H''_\kappa = H \cap g_\kappa H' g_\kappa^{-1}$, then there is an isomorphism of G-sets

$$\sqcup_{\kappa=1}^k G/H''_\kappa \xrightarrow{\cong} G/H \times_{G/K} G/H'\,.$$

Proof. The isomorphism is given by $\varphi[g]_{H''_\kappa} = ([g]_H, [gg_\kappa]_{H'})$. □

And we also have the following result.

Lemma 0.6. *Let $H, H' \subset K \subset G$ be subgroups of G and let M be a Mackey functor. Take $w \in M(G/H \times_{G/K} G/H')$; then*

$$w = \sum_{\kappa=1}^k M_*(\varphi_\kappa)M^*(\varphi_\kappa)(w)\,,$$

where $\varphi_\kappa = \varphi \circ i_\kappa$ and $i_\kappa : G/H''_\kappa \hookrightarrow \sqcup_{\kappa=1}^k G/H''_\kappa$ is the inclusion.

Proof. By the additivity property of the Mackey functor, we have an isomorphism

$$\oplus_{\kappa=1}^k M(G/H''_\kappa) \longrightarrow M(\sqcup_{\kappa=1}^k G/H''_\kappa)$$

induced by the family $M_*(i_\kappa)$, whose inverse is induced by the family $M^*(i_\kappa)$. Using the previous lemma, we obtain the result. □

Let now $p : E \longrightarrow X$ be an n-fold covering G-map. Then for any r the G-function $\mathcal{S}_r(p) : \mathcal{S}_r(E) \longrightarrow \mathcal{S}_r(X)$ has fibers with n elements. We need the next.

Proposition 0.7. *The transfers*

$$t^G_{\mathcal{S}_q(p)} : F^G(\mathcal{S}_n(X)^+, M) \longrightarrow F^G(\mathcal{S}_q(E)^+, M)$$

determine a homomorphism of simplicial abelian groups

$$t^G_{\mathcal{S}(p)} : F^G(\mathcal{S}(X)^+, M) \longrightarrow F^G(\mathcal{S}(E)^+, M)\,.$$

Proof. Let $f : \overline{r} \longrightarrow \overline{s}$ be a morphism in Δ. In order to see that $t^G_{\mathcal{S}(p)}$ is a homomorphism of simplicial groups, we have to prove that the following diagram commutes:

$$\begin{array}{ccc} F^G(\mathcal{S}_s(X)^+, M) & \xrightarrow{(f^{\mathcal{S}(X)^+})^G_*} & F^G(\mathcal{S}_r(X)^+, M) \\ \Big\downarrow{\scriptstyle t^G_{\mathcal{S}_s(p)}} & & \Big\downarrow{\scriptstyle t^G_{\mathcal{S}_r(p)}} \\ F^G(\mathcal{S}_s(E)^+, M) & \xrightarrow[(f^{\mathcal{S}(E)^+})^G_*]{} & F^G(\mathcal{S}_r(E)^+, M) \end{array} .$$

To see this, take a generator $\gamma^G_\sigma(l) \in F^G(\mathcal{S}_s(X)^+, M)$, where $\sigma \in \mathcal{S}_s(X)$ and $l \in M(G/G_\sigma)$. Since p is a covering map, the following is a pullback diagram:

$$\begin{array}{ccc} \mathcal{S}_s(E) & \xrightarrow{f^{\mathcal{S}(E)}} & \mathcal{S}_r(E) \\ \Big\downarrow{\scriptstyle \mathcal{S}_s(p)} & & \Big\downarrow{\scriptstyle \mathcal{S}_r(p)} \\ \mathcal{S}_s(X) & \xrightarrow[f^{\mathcal{S}(X)}]{} & \mathcal{S}_r(X) \,. \end{array}$$

Consider the surjective function

$$q : \mathcal{S}_s(p)^{-1}(\sigma)/G_\sigma \longrightarrow \mathcal{S}_r(p)^{-1}(\sigma \circ f_\#)/G_{\sigma\circ f_\#}$$

induced by the restriction of $f^{\mathcal{S}(E)}$ to the fibers. Then, by the formula (0.3), we have

$$t^G_{\mathcal{S}_r(p)}(f^{\mathcal{S}(X)^+})^G_*(\gamma^G_\sigma(l)) = \sum_{[\tau]\in\mathcal{S}_r(p)^{-1}(\sigma\circ f_\#)/G_{\sigma\circ f_\#}} \gamma^G_\tau M^*(\widehat{\mathcal{S}_r(p)}_\tau)M_*(\widehat{f^{\mathcal{S}(X)^+}}_\sigma)(l) \,.$$

On the other hand, we have

$$(f^{\mathcal{S}(E)^+})^G_* t^G_{\mathcal{S}_s(p)}(\gamma^G_\sigma(l)) = \sum_{[\tau']\in\mathcal{S}_s(p)^{-1}(\sigma)/G_\sigma} \gamma^G_{\tau'\circ f_\#} M_*(\widehat{f^{\mathcal{S}(E)^+}}_{\tau'})M^*(\widehat{\mathcal{S}_s(p)}_{\tau'})(l) \,.$$

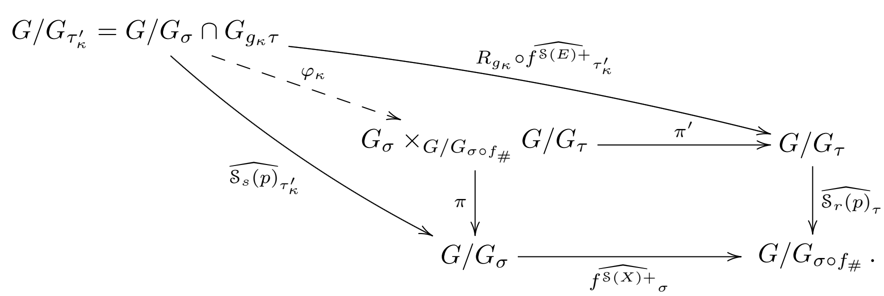
By (0.4), we can write $\mathcal{S}_s(p)^{-1}(\sigma)/G_\sigma = \sqcup_{[\tau]} q^{-1}([\tau])$, where $[\tau] \in \mathcal{S}_r(p)^{-1}(\sigma \circ f_\#)/G_{\sigma\circ f_\#}$, and each $q^{-1}([\tau]) = \{[\tau'_\kappa] \mid \kappa = 1, \ldots, k\}$, where $\mathcal{S}_s(p)(\tau'_\kappa) = \sigma$, $\tau'_\kappa \circ f_\# = f^{\mathcal{S}(E)}(\tau'_\kappa) = g_\kappa\tau$, and the group-elements g_κ are such that

$$\{{}_{G_\sigma}[g_\kappa]_{G_\tau}\}_{\kappa=1}^k = G_\sigma\backslash G_{\sigma\circ f_\#}/G_\tau$$

(notice that the set $\{g_\kappa\}_{\kappa=1}^k$ depends on each representative τ). Since γ^G is equivariant, we have

$$\gamma^G_{g_\kappa\tau} M_*(\widehat{f^{\mathcal{S}(E)^+}}_{\tau'_\kappa})M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l) = \gamma^G_\tau M_*(R_{g_\kappa} \circ \widehat{f^{\mathcal{S}(E)^+}}_{\tau'_\kappa})M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l) \,.$$

Consider the following pullback diagram



Hence, $M^*(\widehat{\mathcal{S}_r(p)}_\tau) \circ M_*(\widehat{f^{\mathcal{S}(X)+}}_\sigma) = M_*(\pi') \circ M^*(\pi)$. Using Lemma 0.6, we can write

$$M^*(\pi)(l) = \sum_{\kappa=1}^{k} M_*(\varphi_\kappa)M^*(\varphi_\kappa)M^*(\pi)(l) = \sum_{\kappa=1}^{k} M_*(\varphi_\kappa)M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l)\,.$$

Composing with $M_*(\pi')$ on the left, we obtain

$$\begin{aligned} M_*(\pi')M^*(\pi)(l) &= \sum_{\kappa=1}^{k} M_*(\pi')M_*(\varphi_\kappa)M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l) \\ &= \sum_{\kappa=1}^{k} M_*(R_{g_\kappa} \circ \widehat{f^{\mathcal{S}(E)+}}_{\tau'_\kappa})M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l)\,. \end{aligned}$$

Therefore, by the pullback property of M,

$$M^*(\widehat{\mathcal{S}_r(p)}_\tau)M_*(\widehat{f^{\mathcal{S}(X)+}}_\sigma)(l) = \sum_{\kappa=1}^{k} M_*(R_{g_\kappa} \circ \widehat{f^{\mathcal{S}(E)+}}_{\tau'_\kappa})M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l)\,.$$

Hence $\gamma_\tau^G M^*(\widehat{\mathcal{S}_r(p)}_\tau)M_*(\widehat{f^{\mathcal{S}(X)+}}_\sigma)(l) = \sum_{\kappa=1}^{k} \gamma_{g_\kappa\tau}^G M_*(\widehat{f^{\mathcal{S}(E)+}}_{\tau'_\kappa})M^*(\widehat{\mathcal{S}_s(p)}_{\tau'_\kappa})(l)$ for each $[\tau]$ and thus the result follows. □

Thus we can give the following.

Definition 0.7. Let $p : E \longrightarrow X$ be an n-fold covering G-map. Define the transfer $t_p^G : F^G(X^+, M) \longrightarrow F^G(E^+, M)$ by

$$t_p^G = |t_{\mathcal{S}(p)}^G|\,,$$

thus the transfer is continuous. (Notice that for any space X, one has $\mathcal{S}_n(X^+) = \mathcal{S}_n(X)^+$.)

With this, we recover Proposition 3.12 in [1]:

Proposition 0.8. *The transfer* $t_p^G : F^G(X^+, M) \longrightarrow F^G(E^+, M)$ *is a continuous homomorphism.* □

Remark 0.9. Observe that the proof of Proposition 0.7 is the correct one for Proposition 3.10 in [1]. On the other hand, the first part of 3.10 is not true in general and hence part (a) of 3.12 is not true either. However, these results were not used in that paper.

Proposition 3.14 in [1] also requires a new proof. We need the following.

Lemma 0.10. *Let K and Q be simplicial pointed G-sets and be $\alpha_0, \alpha_1 : K \longrightarrow Q$ be morphisms. If α_0 and α_1 are G-homotopic, then $\alpha_{0*}^G, \alpha_{1*}^G : F^G(K, M) \longrightarrow F^G(Q, M)$ are homotopic homomorphisms.*

Proof. Let $\mathcal{H} : K \times \Delta[1] \longrightarrow Q$ be a G-homotopy between α_0 and α_1, since $\mathcal{H}$ is G-equivariant (where $\Delta[1]$ has the trival action), it induces a homomorphism

$$\mathcal{H}_*^G : F^G(K \times \Delta[1], M) \longrightarrow F^G(Q, M)\,.$$

Let $\iota^G : F^G(K, M) \times \Delta[1] \longrightarrow F^G(K \times \Delta[1], M)$ be given by

$$\iota_n^G(u, a)(\sigma, b) = \begin{cases} u(\sigma) & \text{if } b = a\,, \\ 0 & \text{if } b \neq a\,, \end{cases}$$

where $(u,a) \in F^G(K_n,M) \times \Delta[1]_n$ and $(\sigma,b) \in K_n \times \Delta[1]_n$. Since u is a G-invariant element, it follows that $\iota_n^G(u,a)$ is also G-invariant. We have that $\iota_n^G(u+u',a) = \iota_n^G(u,a) + \iota_n^G(u',a)$. Therefore $\iota_n^G(\sum_\sigma \gamma_\sigma^G(l_\sigma),a) = \sum_\sigma \gamma_{(\sigma,a)}^G(l_\sigma)$. One can easily see that ι^G is a morphism of simplicial pointed sets. The composite $\mathcal{H}_*^G \circ \iota^G$ is a homotopy between α_{0*}^G and α_{1*}^G. □

Proposition 0.11. *If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps, then $f_{0*}^G, f_{1*}^G : F^G(X,M) \longrightarrow F^G(Y,M)$ are homotopic homomorphisms.*

Proof. For convenience, we shall take the standard 1-simplex Δ^1 instead of the unit interval I. Thus let $H : X \times \Delta^1 \longrightarrow Y$ be a pointed G-homotopy from f_0 to f_1. Consider the morphism of simplicial G-sets $R : \mathcal{S}(X) \times \Delta[1] \longrightarrow \mathcal{S}(Y)$ given as follows. If $s \in \Delta^n$, then define $R_n : \mathcal{S}_n(X) \times \Delta[1]_n \longrightarrow \mathcal{S}_n(Y)$ by

$$R_n(\sigma,a)(s) = H(\sigma(s), a_\#(s))\,,$$

where $a_\# : \Delta^n \longrightarrow \Delta^1$ is the affine map determined by a. Then R is a G-equivariant homotopy between $\mathcal{S}(f_0)$ and $\mathcal{S}(f_1)$. Thus, by the previous lemma, there is a homotopy T between the morphisms $\mathcal{S}(f_0)_*^G$ and $\mathcal{S}(f_1)_*^G$. Then

$$H' : |F^G(\mathcal{S}(X),M)| \times |\Delta[1]| \xleftarrow{\approx} |F^G(\mathcal{S}(X),M) \times \Delta[1]| \xrightarrow{|T|} |F^G(\mathcal{S}(Y),M)|$$

is a homotopy between $f_{0*}^G = |\mathcal{S}(f_0)_*^G|$ and $f_{1*}^G = |\mathcal{S}(f_1)_*^G|$. □

All the results concerning the topological groups $F^G(X,M)$ hold with this new definition. Finally, with this definition, one needs to make some changes in the notation of the statements of 4.9, 5.4, 5.7, and 5.8 of [1]. They should read as follows.

Corollary 4.9. *The isomorphism $H_q(F^G(\mathcal{S}(X),M)) \longrightarrow \pi_q(F^G(X,M))$ sends a homology class $[u]$ represented by a special chain $u = \sum_\alpha \gamma_{\sigma_\alpha}^G(u(\sigma_\alpha))$ to the homotopy class $[\overline{u}]$ given by $\overline{u}(t) = \sum_\alpha [\gamma_{\sigma_\alpha}^G(u(\sigma_\alpha)), t]$.*

Proposition 5.4. *Let M be a Mackey functor and let $p : E \longrightarrow X$ be an n-fold G-equivariant covering map. Then*

$$p_*^G t_p^G([\gamma_\sigma^G(l),t]) = \sum_{\kappa \in K} [\gamma_\sigma^G(M_*(\widehat{\mathcal{S}_r(p)}_{\widetilde{\sigma}_\kappa}) M^*(\widehat{\mathcal{S}_r(p)}_{\widetilde{\sigma}_\kappa})(l)), t]\,,$$

where $\mathcal{S}_r(p)^{-1}(\sigma)/G_\sigma = \{[\widetilde{\sigma}_\kappa] \mid \kappa \in K\}$.

Theorem 5.7. *Let M be a Mackey functor. Then the formula*

$$p_*^G t_p^G([\gamma_\sigma^G(l),t]) = \sum_{\kappa \in K} [G_\sigma : G_{\widetilde{\sigma}_\kappa}][\gamma_\sigma^G(l),t]$$

holds for any n-fold G-equivariant covering map $p : E \longrightarrow X$ if and only if M is homological.

Theorem 5.8. *Let M be a Mackey functor for G with values in k-vector spaces, such that the filed k has characteristic zero or prime to $|G|$. Then the formula*

$$p_*^G t_p^G([\gamma_\sigma^G(l),t]) = \sum_{\kappa \in K} [G_\sigma : G_{\widetilde{\sigma}_\kappa}][\gamma_\sigma^G(l),t]$$

holds for any n-fold G-equivariant covering map $p : E \longrightarrow X$ if and only if M is the fixed point Mackey functor M_L of some kG-module L.

REMARK 0.12. When M is a *homological* Mackey functor, it is possible to use the original abelian group $F^G(X, M)$ as defined in [1], but with a different topology given as follows.

Let us take the surjective homomorphism

$$\pi_X^G : |F^G(\mathcal{S}(X), M)| \twoheadrightarrow F(X^\delta, M)^G$$

defined by

$$\pi_X^G\left(\left[\sum_\sigma \gamma_\sigma^G(l_\sigma),\ t\right]\right) = \sum_\sigma \gamma_{\sigma(t)}^G M_*(p_{\sigma,t})(l_\sigma)\,,$$

where $p_{\sigma,t} : G/G_\sigma \longrightarrow G/G_{\sigma(t)}$ is the quotient map. Then we give $F(X^\delta, M)^G$ the identification topology. We denote the resulting topological group by $\mathbb{F}^G(X, M)$. In this case one can prove (see [2]) that $\mathbb{F}^G(X, M)$ has all the properties of $F^G(X, M)$ given in [1], including the transfer for G-equivariant covering maps, as defined in that paper.

We apologize for the mistake.

References

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