APORTACIONES MATEMÁTICAS
Comunicaciones **42** (2011) 49–78

A BASIC INTRODUCTION TO 2-DIMENSIONAL TOPOLOGICAL FIELD THEORIES

ANA GONZALEZ, ERNESTO LUPERCIO, CARLOS SEGOVIA AND MIGUEL XICOTENCATL

ABSTRACT. In this expository paper we introduce the basic notions related to 2-dimensional topological field theories and Frobenius algebras providing a sketch of the proof of the famous folk theorem that relates them. The paper is very elementary and self-contained. This paper is dedicated to *Escuela Superior de Física y Matemáticas del IPN*, México on its 50th birthday.

1. INTRODUCTION

The purpose of this note is to give a brief introduction to the definition of a topological quantum field theory (TFT) in geometry and topology. The subject has a long and very interesting history in physics before it entered the mathematician's language, where it was incepted primarily though the influence of E. Witten [20]. It was he who proved that the concept was very fruitful to study a host of mathematical phenomena in geometry and topology, specifically giving remarkable applications to knot theory.

Let us start by describing briefly what is usually meant by a quantum field theory in physics. We start by a *space-time* M which is a given smooth manifold of dimension $n+1$. We are also given for every manifold M (with boundary) a space of fields $\mathcal{F}(M)$. For every $x \in M$ we have (complex valued) local observables of the form $O_x\colon \mathcal{F}(M) \longrightarrow \mathbb{C}$, so that $O_x(\phi) \in \mathbb{C}$ for every field $\phi \in \mathcal{F}(M)$. The notation $O_x(\phi)$ is meant to signify that its value depends on ϕ_x, the germ of ϕ around x. The most important part of the structure is a probability measure μ on $\mathcal{F}(M)$ called the *Feynman measure*. All the physics of a quantum system is then contained in the expectation values $\langle O_x \rangle$, and the correlation values $\langle O^{(1)}_{x_1} O^{(2)}_{x_2} O^{(3)}_{x_3} \cdots O^{(k)}_{x_k} \rangle$.

In a great majority of examples we have that

$$\mu = e^{-iS(\phi)}\mathcal{D}\phi,$$

2000 *Mathematics Subject Classification.* 81T45.

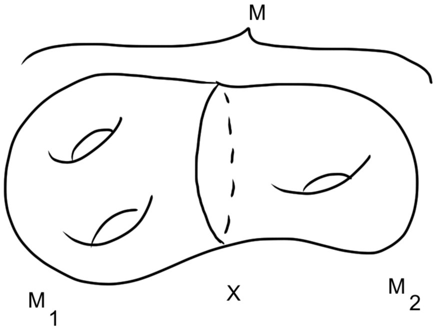
Key words and phrases. Topological Field Theories, Frobenius Algebras.

where the *action* $S\colon \mathcal{F}(M) \to \mathbb{R}$ if of the form

$$S(\phi) = \int_M \mathcal{L}(\phi, D\phi) dx,$$

where $\mathcal{L}\colon TM \to \mathbb{R}$ is called the *Lagrangian* of the theory.

Following Atiyah [20] and Segal [18], [19] we will extract an algebraic gadget out of this picture. To do this notice that whenever we cut up a manifold M into two submanifolds M_1 and M_2 with common boundary X as in the picture:



We can use the fact that $S(\phi) = S(\phi_1) + S(\phi_2)$ where ϕ_i is the restriction of ϕ to M_i, and roughly write:

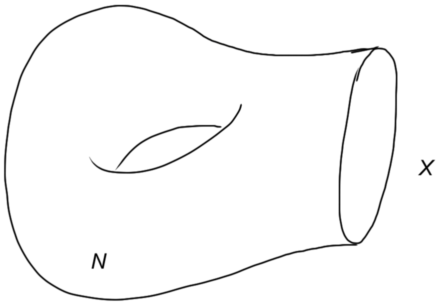
$$Z_M = \int_{\mathcal{F}(M)} e^{-iS(\phi)} \mathcal{D}\phi = \int_{\mathcal{F}(X)} Z_1(\psi) Z_2(\psi) \mathcal{D}\psi,$$

where

$$Z_i(\psi) = \int_{\phi_i \in \mathcal{F}(M_i),\ \phi_i|_X = \psi} e^{-iS(\phi_i)} \mathcal{D}\phi_i.$$

Let us denote by $H_X := \mathrm{Maps}(\mathcal{F}(X), \mathbb{C})$. Clearly H_X has the structure of a vector space, and we have that since $Z_i\colon \mathcal{F}(X) \to \mathbb{C}$, then $Z_i = Z_{M_i} \in H_X$ for a $n+1$ dimensional manifold M_i with boundary X. In other words, whenever a $n+1$ dimensional manifold N has as its boundary a n dimensional manifold X we set:

$$Z_N(\psi) = \int_{\phi_i \in \mathcal{F}(N),\ \phi_i|_X = \psi} e^{-iS(\phi_i)} \mathcal{D}\phi_i.$$

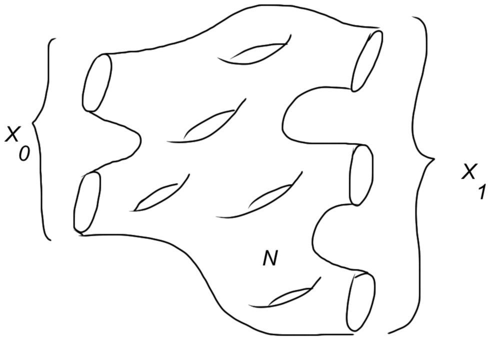


Obtaining in this manner a vector

$$Z_N \in H_X.$$

In this way a quantum field theory (of dimension $n+1$) provides us with an assignment $X \mapsto H_X$ of a vector space for every n-dimensional manifold, and a vector $N \mapsto Z_N$ whenever a $n+1$ dimensional manifold has boundary $\partial N = X$.

We can do a little better. Suppose now that we think of the manifold as having an *initial* boundary $\partial_0 N = X_0$ and a *final* boundary $\partial_1 = X_1$:



Let $H_{X_i} := \mathrm{Maps}(\mathcal{F}(X_i), \mathbb{C})$. Then we can write a linear operator of the form:

$$Z_N \colon H_{X_0} \longrightarrow H_{X_1},$$

by the formula:

$$(Z_N(\Psi))(\psi_1) = \int_{\mathcal{F}(X_0)} K(\psi_1, \psi_0)\Psi(\psi_0)\mathcal{D}\psi_0,$$

where the kernel K is given by

$$K(\phi_1, \phi_2) = \int_{\phi \in \mathcal{F}(N),\ \phi|_{X_i} = \psi_i} e^{-iS(\phi)} \mathcal{D}\phi.$$

We should also note that (formally at least) since $H_X := \mathrm{Maps}(\mathcal{F}(X), \mathbb{C})$, then we have that

$$\begin{aligned} H_{X_1 \coprod X_2} &:= \mathrm{Maps}(\mathcal{F}(X_1 \coprod X_2), \mathbb{C}) \\ &= \mathrm{Maps}(\mathcal{F}(X_1), \mathbb{C}) \times \mathrm{Maps}(\mathcal{F}(X_2), \mathbb{C}) = H_{X_1} \times H_{X_2}. \end{aligned}$$

If as in the picture above X_0 (resp. X_1) can be written as the disjoint union of its connected components $X_{01} \coprod X_{02}$ (resp. $X_{11} \coprod X_{12} \coprod X_{13}$,) then the map

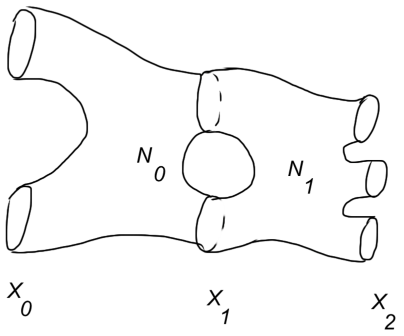
$$Z_N \colon H_{X_{01}} \times H_{X_{02}} \longrightarrow H_{X_{11}} \times H_{X_{12}} \times H_{X_{13}},$$

is actually a map

$$Z_N \colon H_{X_{01}} \otimes H_{X_{02}} \longrightarrow H_{X_{11}} \otimes H_{X_{12}} \otimes H_{X_{13}},$$

for the required multilinearity conditions are easy to verify.

Also easy to verify is that whenever we glue two cobordisms $N = N_0 \cup N_1$ as depicted below:



we have that

$$Z_N = Z_{N_1} \circ Z_{N_0}.$$

What is quite surprising at first is that for ma ny examples, roughly speaking, the assignments

$$X \mapsto H_X, \ \ N \mapsto Z_N,$$

for all X and for all N, contain *all* the information of the field theory, namely we can recover all correlations from those mappings. For topological field theories and

2-dimensional conformal field theories this is the case. This is great news for mathematicians since the purported measure on the space of fields $\mathcal{F}(M)$ often does not exists. Nevertheless the assignments do exist and provide a mathematical definition for the field theories in question.

When the assignment $N \mapsto Z_N$ depends on the metric of N we refer to the theory as an *Euclidean field theory*, when it depends only on the conformal structure we call it a *conformal field theory*, and when it only depends on the topology of N we call it a *topological field theory*. In the last case the correlations will be independent of the metric.

2. TOPOLOGICAL FIELD THEORIES IN DIMENSION 1+1.

Michael Atiyah in [2] and [3] defined *nD-Topological Field Theory* (nD-TFT) Z^A, using the following data:

1. A vector space $\mathrm{Z}^A(\Sigma)$ associated to each $(n-1)$-dimensional closed manifold Σ.
2. A vector $\mathrm{Z}^A(M) \in \mathrm{Z}^A(\partial M)$ associated to each oriented n-dimensional manifold M with boundary ∂M.
3. An isomorphism $\mathrm{Z}(f) : \mathrm{Z}(\Sigma_1) \to \mathrm{Z}(\Sigma_2)$, where $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving diffeomorphism.

This data is subject to the following axioms:

(i) Z^A is *functorial* with respect to orientation-preserving diffeomorphisms of Σ and M.

(ii) Z^A is *involutory*, i.e. $\mathrm{Z}^A(\Sigma^*) = \mathrm{Z}^A(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $\mathrm{Z}^A(\Sigma)^*$ is the dual vector space of $\mathrm{Z}^A(\Sigma)$.

(iii) Z^A is *multiplicative*

$$\mathrm{Z}^A(\Sigma_1 \sqcup \Sigma_2) = \mathrm{Z}^A(\Sigma_1) \otimes \mathrm{Z}^A(\Sigma_2).$$

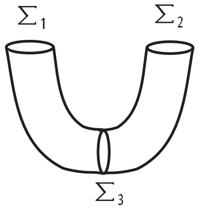
(iv) $\mathrm{Z}^A(\emptyset) = \Bbbk$, where $\emptyset$ is interpreted as the empty $(n-1)$-dimensional closed manifold.

(v) $\mathrm{Z}^A(\emptyset) = 1$, where $\emptyset$ is interpreted as the empty n-dimensional manifold.

(vi) If $f : \Sigma_1 \to \Sigma_2$ is an orientation-preserving diffeomorphism, then $\mathrm{Z}(f) : \mathrm{Z}(\Sigma_1) \to \mathrm{Z}(\Sigma_2)$ is an isomorphism.

These axioms are meant to be understood as follows. The functoriality axiom means first that an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma'$ induces an isomorphism $\mathrm{Z}^A(f) : \mathrm{Z}^A(\Sigma) \to \mathrm{Z}^A(\Sigma')$ and that $\mathrm{Z}^A(gf) = \mathrm{Z}^A(g)\,\mathrm{Z}^A(f)$ for $g : \Sigma' \to \Sigma''$. Also if f extends to an orientation-preserving diffeomorphism $M \to M'$, with $\partial M = \Sigma$ and $\partial M' = \Sigma'$, then $\mathrm{Z}^A(f)$ takes the element $\mathrm{Z}^A(M)$ to $\mathrm{Z}^A(M')$. The multiplicative axiom is clear. Moreover if $\partial M_1 = \Sigma_1 \sqcup \Sigma_3^*$, $\partial M_2 = \Sigma_3 \sqcup \Sigma_2$

and $M = M_1 \sqcup_{\Sigma_3} M_2$ is the manifold obtained by gluing together the common Σ_3-component:



Then we require:

$$\mathrm{Z}^A(M) = \langle \mathrm{Z}^A(M_1), \mathrm{Z}^A(M_2) \rangle$$

where $\langle\,,\rangle$ denotes the natural pairing from the duality map,

$$\mathrm{Z}^A(\Sigma_1) \otimes \mathrm{Z}^A(\Sigma_3)^* \otimes \mathrm{Z}^A(\Sigma_3) \otimes \mathrm{Z}^A(\Sigma_2) \to \mathrm{Z}^A(\Sigma_1) \otimes \mathrm{Z}^A(\Sigma_2)$$

defined by $a \otimes \varphi \otimes b \otimes c \longmapsto \varphi(b) a \otimes c$. This is a very powerful axiom which implies that $Z^A(M)$ can be computed (in many different ways) by "cutting M in half" along Σ_3.

3. CATEGORICAL DEFINITION OF A TQFT.

The first step is to define an appropiate category of cobordisms that permits us to give a functorial definition of a nD-TFT.

Definition 3.1. Let Σ_0 and Σ_1 two compact, connected, oriented $(n-1)$-manifolds, we say that they are *cobordant* if there is a n-manifold M, with boundary $\Sigma_1^* \sqcup \Sigma_2$, in this case we say that M is a *n-cobordism* of Σ_1 to Σ_2.

If we fix a positive integer n, we can construct a category $\widetilde{nCob}$ where the objects are the closed smooth $(n-1)$-dimensional manifolds, and the morphisms are the oriented smooth n-dimensional manifolds(n-cobordism). We need to address whether the composition of two cobordisms of the same dimension is a smooth manifold. Up to a smoothing process the answer is yes (see [11]). Let be $nCob' = \widetilde{nCob}/\sim$ where $\sim$ is the relation of diffeomorphism equivalence. Let Σ be a closed submanifold of M of codimension 1. We assume that both are oriented. At a point $x \in \Sigma$, let $[v_1, \ldots, v_{n-1}]$ be a positive basis for $T_x\Sigma$. A vector $w \in T_xM$ is called a *positive normal* if $[v_1, \ldots, v_{n-1}, w]$ is a positive basis for T_xM. Now suppose Σ is a connected component of the boundary of M with an specific orientation; then it makes sense to ask whether the positive normal w points inward or it points outward as compared to M. Locally the situation is the following, a vector in $\mathbb{R}^n$ either points inward or outward with respect to the half-space $\mathbb{H}^n$ ($\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq 0)\}$). If a positive normal points inward we call Σ an *in-boundary*, and if it points outward we call it an *out-boundary*. To see that this makes sense we have to check that this does not depend on the choice of positive normal (or the choice of the point $x \in \Sigma$). If some positive normal points inward, it is easy to verify that every other positive normal at any other point $y \in \Sigma$ points inward as well. This follows from the fact that the normal bundle is a trivial line bundle on Σ. This in turn is a consequence of the assumption that both M and Σ are orientable (see Hirsch [10], theorem 4.4.2.). Thus the boundary of a manifold M is the union of various in-boundaries and out-boundaries. The in-boundary of M may be empty, and the out-boundary may also be empty. Note that if we reverse the orientation of both M and its boundary Σ, then the notion of what is in-boundary or out-boundary remains the same. We will denote by nCob the category $nCob'$ where every object is given an orientation (therefore any cobordism has a direction).

In the next definition we will assume that the reader is familiar with the concept of monoidal category, if this is not the case we refer the reader to the Appendix.

Definition 3.2. An *n-dimensional topological field theory* is a symmetric monoidal functor Z^C, from (nCob,$\sqcup$,$\emptyset$,T) to ($\mathrm{Vect}_{\Bbbk}$,$\otimes$,$\Bbbk$,σ).

Proposition 3.3. *Atiyah's definition and the categorical definition of TFT coincide.*

Proof. Suppose Z^A is a TFT in the sense of Atiyah, then for M an oriented n-dimensional manifold, the next isomorphism gives the correspondence

$$\begin{array}{cccc} \Psi & \mathrm{Z}^A(\Sigma_1)^* \otimes \mathrm{Z}^A(\Sigma_2) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{Z}^A(\Sigma_1), \mathrm{Z}^A(\Sigma_2)) \\ & \mathrm{Z}^A(M) & \longmapsto & \mathrm{Z}^C(M) \end{array} \tag{1}$$

where $\partial M = \Sigma_1^* \sqcup \Sigma_2$. Set $\mathrm{Z}^C(M) := \mathrm{Z}^A(M)$, if we identify the image of the idempotent element $\mathrm{Z}^A(\Sigma \times I)$ with the identity $1_{\mathrm{Z}^A(\Sigma)}$, then we get a functor $\mathrm{Z}^C : \mathrm{nCob} \to \mathrm{Vect}_{\Bbbk}$. This functor is well defined by the *functorial* and *multiplicative* axioms. Moreover, the monoidal structure is given by $\sqcup \to \otimes$ and it is symmetrical since $\mathrm{Z}^C(T_{\Sigma,\Sigma'}) = \sigma_{\mathrm{Z}^C(\Sigma),\mathrm{Z}^C(\Sigma')}$.

Conversely, given a symmetrical monoidal functor $\mathrm{Z}^C : \mathrm{nCob} \to \mathrm{Vect}_{\Bbbk}$, if Σ is a closed $(n-1)$-dimensional smooth manifold, set $\mathrm{Z}^A(\Sigma) := \mathrm{Z}^C(\Sigma)$. For M a n-dimensional oriented smooth manifold we take

$$\mathrm{Z}^A(M) = \mathrm{Z}^C(M')(1) \in \mathrm{Z}^C(\Sigma_{In})^* \otimes \mathrm{Z}^C(\Sigma_{Out}),$$

where M' is M reversing the orientation to the in-boundary. By hypothesis, we have $\mathrm{Z}^C(\emptyset) = \Bbbk$. Moreover, the functor Z^C is multiplicative and it is independent of the cut by the correspondence 1. As consequence, the axioms (iii) and (iv) are satisfied. Clearly $\mathrm{Z}^A(\emptyset) = \widehat{1} \otimes 1$. Axiom (v) follows from $\Psi(\mathrm{Z}^A(\emptyset)) = \Psi(\widehat{1} \otimes 1) = \Bbbk$. Axiom (i) is satisfied because Z^C factors through differential homotopy classes. Axiom (ii) is proposition 3.5. $\square$

Corollary 3.4. *For a Topological Field Theory* Z *of any dimension and* Σ *an object in* nCob*, the image of* Σ *under* Z *is a finite dimensional vector space.*

Proof. Let

$$\langle\, ,\, \rangle_{\Sigma} : \mathrm{Z}(\Sigma) \otimes \mathrm{Z}(\Sigma^{*}) \longrightarrow \mathbb{k}$$

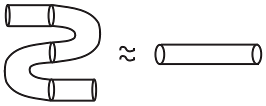
and

$$\theta_{\Sigma} : \mathbb{k} \longrightarrow \mathrm{Z}(\Sigma^{*}) \otimes \mathrm{Z}(\Sigma)$$

the maps associated to  and  respectively. Since Z is a TFT, then the next diagram

$$\begin{tikzcd}
\mathrm{Z}(\Sigma) \arrow[d, "\simeq"'] & (\mathrm{Z}(\Sigma) \otimes \mathrm{Z}(\Sigma)^{*}) \otimes \mathrm{Z}(\Sigma) \arrow[r, "\langle , \rangle_{\Sigma} \otimes id_{\mathrm{Z}(\Sigma)}"] & \mathbb{k} \otimes \mathrm{Z}(\Sigma) \arrow[d, "\simeq"] \\
\mathrm{Z}(\Sigma) \otimes \mathbb{k} \arrow[r, "1_{\mathrm{Z}(\Sigma)} \otimes \theta_{\Sigma}"] & \mathrm{Z}(\Sigma) \otimes (\mathrm{Z}(\Sigma^{*}) \otimes \mathrm{Z}(\Sigma)) \arrow[u, "\simeq"'] & \mathrm{Z}(\Sigma)
\end{tikzcd}$$

is the identity map. Graphically

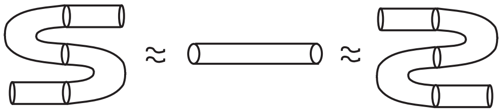


then we have $(\langle\, ,\, \rangle_{\Sigma} \otimes 1_{\mathrm{Z}(\Sigma)}) \circ (1_{\mathrm{Z}(\Sigma)} \otimes \theta_{\Sigma}) = 1_{\mathrm{Z}(\Sigma)}$. For $\theta_{\Sigma}(1) = \sum v_j \otimes w_j$ and $a \in \mathrm{Z}(\Sigma)$ then we have:

$$\begin{aligned}
a \xrightarrow{\sim} a \otimes 1 = (\langle\, ,\, \rangle_{\Sigma} \otimes 1_{\mathrm{Z}(\Sigma)}) \circ (1_{\mathrm{Z}(\Sigma)} \otimes \theta_{\Sigma})(a \otimes 1) \\
= (\langle\, ,\, \rangle_{\Sigma} \otimes 1_{Z(\Sigma)})(\sum a \otimes v_j \otimes w_j) \\
= \sum \langle a, v_j\rangle_{\Sigma} \otimes w_j \xrightarrow{\sim} \sum \langle a, v_j\rangle_{\Sigma} w_j.
\end{aligned}$$

Therefore $a = \sum \langle a, v_j\rangle_{\Sigma} w_j$, and consequently $\{w_j\}$ generates $\mathrm{Z}(\Sigma)$, but since $\mathbb{k}$ is at least a division ring, we can extract a basis from the generating set. Now since every division ring has the property of invariance of dimension then $\mathrm{Z}(\Sigma)$ is finitely generated with $n = rank(A) \leq |\{w_j\}|$. ☐

The simplicity of the definition may be misleading, it is remarkable how much information a TFT encodes. For example the fact that the theory only depends on the topology implies that to the cobordisms



we associate the same linear transformation, which is the identity. In the literature this equivalences are called the *zig-zag* identities. This simple fact implies that for any n-dimensional TFT vector space associated to every object of $nCob$ is a *Frobenius algebra*, see the definition below. The next proposition proves that there exists a nondegenerate pairing, which consequently entails the construction of the product and the unit for the vector space.

Proposition 3.5. *Let* Z *be an* n*-dimensional TFT, and* Σ *an* n*-dimensional oriented closed smooth manifold, then* $\mathrm{Z}(\Sigma)$ *is equipped with a nondegenerate pairing and* $\mathrm{Z}(\Sigma^*) \simeq \mathrm{Z}(\Sigma)^*$.

Proof. Similarly to 3.4 we have that the next diagrams

$$\begin{tikzcd}
\mathrm{Z}(\Sigma) \arrow[d, "\simeq"'] & (\mathrm{Z}(\Sigma) \otimes \mathrm{Z}(\Sigma^*)) \otimes \mathrm{Z}(\Sigma) \arrow[r, "\langle , \rangle_\Sigma \otimes 1_{\mathrm{Z}(\Sigma)}"] & \Bbbk \otimes \mathrm{Z}(\Sigma) \arrow[d, "\simeq"] \\
\mathrm{Z}(\Sigma) \otimes \Bbbk \arrow[r, "1_{\mathrm{Z}(\Sigma)} \otimes \theta_\Sigma"] & \mathrm{Z}(\Sigma) \otimes (\mathrm{Z}(\Sigma^*) \otimes \mathrm{Z}(\Sigma)) \arrow[u, "\simeq"'] & \mathrm{Z}(\Sigma)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Bbbk \otimes \mathrm{Z}(\Sigma^*) \arrow[r, "\theta_\Sigma \otimes 1_{\mathrm{Z}(\Sigma^*)}"] & (\mathrm{Z}(\Sigma^*) \otimes \mathrm{Z}(\Sigma)) \otimes \mathrm{Z}(\Sigma^*) \arrow[d, "\simeq"] & \mathrm{Z}(\Sigma^*) \\
\mathrm{Z}(\Sigma^*) \arrow[u, "\simeq"] & \mathrm{Z}(\Sigma^*) \otimes (\mathrm{Z}(\Sigma) \otimes \mathrm{Z}(\Sigma^*)) \arrow[r, "1_{\mathrm{Z}(\Sigma^*)} \otimes \langle , \rangle_\Sigma"] & \mathrm{Z}(\Sigma^*) \otimes \Bbbk \arrow[u, "\simeq"']
\end{tikzcd}$$

are the identity maps of $\mathrm{Z}(\Sigma)$ and $\mathrm{Z}(\Sigma^*)$ respectively, i.e.

$$1_{\mathrm{Z}(\Sigma)} = (\langle\, ,\, \rangle_\Sigma \otimes 1_{\mathrm{Z}(\Sigma)}) \circ (1_{\mathrm{Z}(\Sigma)} \otimes \theta_\Sigma)$$

and

$$1_{\mathrm{Z}(\Sigma)} = (1_{\mathrm{Z}(\Sigma^*)} \otimes \langle\, ,\, \rangle_\Sigma) \circ (\theta_\Sigma \otimes 1_{\mathrm{Z}(\Sigma^*)})$$

An easy algebraic exercise proves that $\langle\, ,\, \rangle_\Sigma$ is a nondegenerate pairing and that the map

$$\begin{array}{rccc} \lambda_{\text{left}}: & \mathrm{Z}(\Sigma^*) & \longrightarrow & \mathrm{Z}(\Sigma)^* \\ & y & \longmapsto & \langle x, y \rangle_\Sigma \end{array}$$

is an isomorphism (hint:use that $\mathrm{Z}(\Sigma)$ and $\mathrm{Z}(\Sigma^*)$ are finitely generated). □

4. FROBENIUS ALGEBRAS

4.1. **Definitions.** The concept of a Frobenius algebra is quite important for topological field theories, so we will review the basic definition. We start by giving a serie of equivalent definitions of Frobenius algebras.

Definition 4.1. A *Frobenius algebra* is a $\Bbbk$-algebra $\mathcal{A}$ whit a non-degenerate bilinear form $\langle\, , \,\rangle : \mathcal{A} \otimes \mathcal{A} \to \Bbbk$ which is associative, in the sense $\langle ab, c\rangle = \langle a, bc\rangle$.

Definition 4.2. A *Frobenius algebra* is a $\Bbbk$-algebra $\mathcal{A}$ with a linear function $\varepsilon : \mathcal{A} \to \Bbbk$ called **counit**, such that the $\ker(\varepsilon)$ do not have non trivial ideals.

Definition 4.3. A *Frobenius algebra* is a $\Bbbk$-algebra $\mathcal{A}$ with an $\mathcal{A}$-module isomorphism $\lambda : \mathcal{A} \to \mathcal{A}^*$, where the dual space $\mathcal{A}^*$ is an $\mathcal{A}$-module with the action $a\cdot\varphi = \varphi\circ\overline{m}(a)$, where $\overline{m} : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ is the multiplication by $a \in \mathcal{A}$.

The proof of this fact is as follows. Given $\langle\, , \,\rangle : \mathcal{A} \otimes \mathcal{A} \to \Bbbk$ we define $\varepsilon : \mathcal{A} \to \Bbbk$ by $\varepsilon(a) = \langle 1_{\mathcal{A}}, a\rangle$. If we have $\varepsilon : \mathcal{A} \to \Bbbk$ we define $\lambda : \mathcal{A} \to \mathcal{A}^*$ by $\lambda(a)(b) = \varepsilon(ab)$ and finally, given $\lambda : \mathcal{A} \to \mathcal{A}^*$ let $\langle a, b\rangle = \lambda(1_{\mathcal{A}})(ab)$.

The next theorem is due to Lowell Abrams [1] and Aaron D. Lauda in [14]. They give two additional definitions of a Frobenius algebra.

Theorem 4.4. *A commutative algebra $\mathcal{A}$ of finite dimension with product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and unit $u : \Bbbk \to \mathcal{A}$ is a Frobenius algebra if and only if it satisfies one of the next conditions*

i) (Abrams) There is a coproduct $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, with a counit $\varepsilon : \mathcal{A} \to \Bbbk$ satisfying the Frobenius identities which define a coalgebra structure on $\mathcal{A}$. Explicitly the following diagrams commute:

- *The coalgebra axioms*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\delta\otimes 1"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes\delta"'] & \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \otimes \Bbbk & \mathcal{A} \otimes \mathcal{A} \arrow[l, "1\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes 1"] & \Bbbk \otimes \mathcal{A} \\
 & \mathcal{A} \arrow[ul, "\cong"] \arrow[u, "\delta"'] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

- *The Frobenius identities*

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] \arrow[d, "1\otimes\delta"'] & \mathcal{A} \arrow[d, "\delta"] \\
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "m\otimes 1"'] & \mathcal{A} \otimes \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] \arrow[d, "\delta\otimes 1"'] & \mathcal{A} \arrow[d, "\delta"] \\
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A} \otimes \mathcal{A}
\end{tikzcd}$$

ii) (Lauda) There exists a co-pairing $\theta : \Bbbk \to \mathcal{A} \otimes \mathcal{A}$ that equips $\mathcal{A}$ with two equivalent coproducts and units. This is to say, the following diagrams commute:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "1\otimes\theta"] \arrow[d, "\theta\otimes 1"'] \arrow[dr, "\delta"] & \mathcal{A} \otimes A \otimes A \arrow[d, "m\otimes 1"] \\
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes m"'] & \mathcal{A} \otimes \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \arrow[r, "\theta"] \arrow[d, "\theta"'] \arrow[dr, "u"] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\varepsilon\otimes 1"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "1\otimes\varepsilon"'] & \mathcal{A}
\end{tikzcd}$$

Proof. i) We define the coproduct as follows

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\delta"] \arrow[d, "\lambda"'] & \mathcal{A} \otimes \mathcal{A} \\
\mathcal{A}^* \arrow[r, "m^*"'] & \mathcal{A}^* \otimes \mathcal{A}^* \arrow[u, "\lambda^{-1}\otimes\lambda^{-1}"']
\end{tikzcd}$$

that is $\delta := (\lambda^{-1} \otimes \lambda^{-1}) \circ m^* \lambda$.
Using that m is a commutative and an associative map we have that δ is a cocommutative and a coassociative map. We need to check that δ is an $\mathcal{A}$-module morphism, for this we construct the next map

$$\begin{array}{ccccc}
\overline{m} : & \mathcal{A} & \longrightarrow & \mathrm{End}(\mathcal{A}) & \cong & \mathcal{A} \otimes \mathcal{A}^* \\
 & a & \longmapsto & a\cdot & \mapsto & a \sum_i e_i \otimes e_i^*
\end{array}$$

where $\{e_1, \ldots, e_n\}$ is a basis of $\mathcal{A}$ and $\{e_1^*, \ldots, e_n^*\}$ is the dual basis.
It is easy to prove that the next diagrams commute.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\lambda"] \arrow[d, "\delta"'] & \mathcal{A}^* \arrow[r, "\lambda^{-1}"] \arrow[d, "m^*"'] & \mathcal{A} \arrow[d, "\overline{m}"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "\lambda^{-1}\otimes\lambda^{-1}"'] & \mathcal{A}^* \otimes \mathcal{A}^* \arrow[r, "\lambda\otimes 1^*"'] & \mathcal{A} \otimes \mathcal{A}^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, "\delta"'] \arrow[dr, "\overline{m}"] & \\
\mathcal{A} \otimes \mathcal{A} & \mathcal{A} \otimes \mathcal{A}^* \arrow[l, "1\otimes\lambda^{-1}"]
\end{tikzcd}$$

We consider the next diagram.

$$\begin{tikzcd}
& \mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[dl, "1\otimes\overline{m}"'] \arrow[d, "1\otimes\delta"] \arrow[dr, phantom, "\textcircled{1}"] & \mathcal{A} \arrow[d, "\delta"] \arrow[dr, "\overline{m}"] & \\
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}^* \arrow[r, "1\otimes1\otimes\lambda^{-1}"'] \arrow[ur, phantom, "\textcircled{5}"] \arrow[dr, "m\otimes1^*"'] & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m\otimes1"'] \arrow[dr, phantom, "\textcircled{4}"] & \mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes\lambda^{-1}"] \arrow[d, "1\otimes\lambda"] \arrow[ur, phantom, "\textcircled{2}"] \arrow[r, phantom, bend right, "\textcircled{3}"'] & \mathcal{A}\otimes\mathcal{A}^* \arrow[dl, "1\otimes1^*"] \\
& \mathcal{A}\otimes\mathcal{A}^* \arrow[r, "1\otimes1^*"'] & \mathcal{A}\otimes\mathcal{A}^* &
\end{tikzcd}$$

Note that ② and ⑤ commute by definition of $\overline{m}$, ③ and ④ clearly commute. The external diagram commute because

$$\begin{tikzcd}
& x\otimes y \arrow[r, mapsto, "m"] \arrow[dl, mapsto, "1\otimes\overline{m}"'] & xy \arrow[dr, mapsto, "\overline{m}"] & \\
x\otimes\sum_i ye_i\otimes e_i^* \arrow[dr, mapsto, "m\otimes1^*"'] & & & \sum_i xye_i\otimes e_i^* \arrow[dl, mapsto, "1\otimes1^*"] \\
& x\sum_i ye_i\otimes e_i^* \arrow[r, mapsto, "1\otimes1^*"'] & xy\sum_i e_i\otimes e_i^* &
\end{tikzcd}$$

Then the diagram ① commutes and δ is a morphism of $\mathcal{A}$-modules.

Reciprocally, we define $\langle\,,\,\rangle : \mathcal{A}\otimes\mathcal{A}\to\Bbbk$ by $\langle\,,\,\rangle = \varepsilon\circ m$. Using that m and ε are linear maps we have that $\langle\,,\,\rangle$ is also linear. The associativity is a consequence of the associativity of the product. Finally, to prove that the pairing is non-degenerate, we use that the next diagram commute since δ is a $\mathcal{A}$-module morphism.

$$\begin{tikzcd}
& & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[dr, "1\otimes m"] & & \\
\Bbbk\otimes\mathcal{A} \arrow[r, "u\otimes1"] & \mathcal{A}\otimes\mathcal{A} \arrow[ur, "\delta\otimes1"] \arrow[dr, "m"'] & & \mathcal{A}\otimes\mathcal{A} \arrow[r, "1\otimes\varepsilon"] & \mathcal{A}\otimes\Bbbk \\
& & \mathcal{A} \arrow[ur, "\delta"'] & &
\end{tikzcd}$$

The top composition gives

$$1\otimes x\mapsto 1_{\mathcal{A}}\otimes x\mapsto (\sum_j u_j\otimes e_j)\otimes x\mapsto \sum_j u_j\otimes e_jx\mapsto \sum_j\langle e_j,x\rangle u_j\otimes 1.$$

and the under composition gives

$$1 \otimes x \mapsto 1_{\mathcal{A}} \otimes x \mapsto x \mapsto \delta(x) \mapsto (1 \otimes \varepsilon)\delta(x) = x \otimes 1$$

Then $x = \sum_j \langle e_j, x \rangle u_j$, therefore $\{u_j\}$ is a basis of $\mathcal{A}$. In particular if we take $x = u_i$ we have $\langle e_j, u_i \rangle = \delta_{ij}$.
Now we take k_i such that $\langle \sum_i k_i e_i, x \rangle = 0$ for all $x \in \mathcal{A}$. If $x = u_j$ we have $\sum_i k_i \langle e_i, u_j \rangle = 0$, then $k_i = 0$ for all $i = 1, \dots n$. Therefore $\sum_i k_i e_i = 0$ and the pairing $\langle \ , \ \rangle$ is non-degenerate.

ii) It is easy to see that this condition is equivalent to the Abrams condition. Given the coproduct δ we define $\theta : \Bbbk \to \mathcal{A} \otimes \mathcal{A}$ by $\theta = \delta \circ u$. We deduce the commutativity of the diagrams using the $\mathcal{A}$-module properties. If we consider the co-pairing $\theta : \Bbbk \to \mathcal{A} \otimes \mathcal{A}$ we define $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ as follows

$$\delta = (1 \otimes m) \circ (\theta \otimes 1) = (m \otimes 1) \circ (1 \otimes \theta)$$

□

Definition 4.5. A Frobenius algebra $\mathcal{A}$ is called a *symmetric Frobenius algebra* if one (and hence all) of the following equivalent conditions holds.

(i) The Frobenius form $\varepsilon : \mathcal{A} \to \Bbbk$ is *central*; this means that $\varepsilon(ab) = \varepsilon(ba)$ for all $a, b \in \mathcal{A}$.
(ii) The pairing $\langle \ , \rangle$ is symmetric (i.e. $\langle a, b \rangle = \langle b, a \rangle$ for all $a, b \in \mathcal{A}$).
(iii) The left $\mathcal{A}$-isomorphism $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^*$ is also right $\mathcal{A}$-linear.
(iv) The right $\mathcal{A}$-isomorphism $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^*$ is also left $\mathcal{A}$-linear.

Definition 4.6. A *Frobenius algebra homomorphism* $\phi : (\mathcal{A}, \varepsilon) \longrightarrow (\mathcal{A}', \varepsilon')$ between two Frobenius algebras is an algebra homomorphism which is at the same time a coalgebra homomorphism. In particular it preserves the Frobenius form, in the sense that $\varepsilon = \phi\varepsilon'$.

Let $\mathrm{FA}_{\Bbbk}$ denote the category of Frobenius algebras, and let $\mathrm{cFA}_{\Bbbk}$ denote the full subcategory of all commutative Frobenius algebras.

Lemma 4.7. *If a $\Bbbk$-algebra homomorphism ϕ between two Frobenius algebras $(\mathcal{A}, \varepsilon)$ and $(\mathcal{A}', \varepsilon')$ is compatible with the forms in the sence that the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\phi"] \arrow[dr, "\varepsilon"'] & & \mathcal{A}' \arrow[dl, "\varepsilon'"] \\
& \Bbbk &
\end{tikzcd}$$

commutes, then ϕ is injective.

Proof. The kernel of ϕ is an ideal and it is clearly contained in $\ker(\varepsilon)$. But $\ker(\varepsilon)$ contains no nontrivial ideals, so $\ker(\phi) = 0$ and thus ϕ is injective. □

Lemma 4.8. *A Frobenius algebra homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ is always invertible. In other worlds, the category* $\mathrm{FA}_{\Bbbk}$ *is a groupoid and so is* $\mathrm{cFA}_{\Bbbk}$.

Proof. Since ϕ is comultiplicative and respects the counits ε and ε' (as well as the units η and η'), the dual map $\phi^* : \mathcal{A}'^* \to \mathcal{A}^*$ is multiplicative and respects the units and counits. But then the preceding lemma applies and shows that ϕ^* is injective. Since $\mathcal{A}$ is a finite-dimensional vector space this implies that ϕ is surjective. We already know it is injective, hence it is invertible. ☐

In this section we will present a collection of examples of Frobenius algebras. A good reference for this is [11]. The principal example is the Poincaré algebra, it is the principal motivation for the definition presented in the next section, this is because if we consider M a manifold not necessarily compact we do not necessarily have the trace but all the other structures are preserved.

4.2. **The trivial Frobenius algebra.** Let $\mathcal{A} = \Bbbk$, and $\varepsilon : \mathcal{A} \to \Bbbk$ be the identity map of $\Bbbk$. Clearly there are no ideals in the kernel of this map, so we have a Frobenius algebra.

4.3. **Concrete example.** The field of complex number $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$: an obvious Frobenius form is taking the real part

$$\begin{aligned} \mathbb{C} &\to \mathbb{R} \\ a + ib &\mapsto a. \end{aligned}$$

4.4. **Skew-fields.** Let $\mathcal{A}$ be a skew-field (also called division algebra) of finite dimension over $\Bbbk$. Since just like a field, a skew-field has no nontrivial left ideals (or right ideals), any nonzero linear form $\mathcal{A} \to \Bbbk$ will make $\mathcal{A}$ into a Frobenius algebra over $\Bbbk$, for example the quaternions $\mathbb{H}$ form a Frobenius algebra over $\mathbb{R}$.

4.5. **Matrix algebras.** Let $\mathcal{A}$ be the space $\mathrm{Mat}_n(\Bbbk)$ of all $n \times n$ matrices over $\Bbbk$, this is a Frobenius algebra with the usual trace map

$$\begin{aligned} \mathrm{Tr} : \mathrm{Mat}_n(\Bbbk) &\to \quad \Bbbk \\ (a_{ij}) \qquad &\mapsto \sum_i a_{ii} \end{aligned}$$

To see that the bilinear pairing resulting from Tr is nondegenerate, take the linear basis of $\mathrm{Mat}_n(\Bbbk)$ consisting of E_{ij} with only one nonzero entry $e_{ij} = 1$. Clearly E_{ji} is the dual basis element to E_{ij} under this pairing. Note that this is a symmetric Frobenius algebra since two matrix products AB and BA have the same trace. If we twist the Frobenius form by multiplication with a noncentral invertible matrix we obtain a nonsymmetric Frobenius algebra.

As a concrete example, consider $\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ with the usual trace map

$$\begin{array}{rccc} \mathrm{Tr}: & \mathrm{Map}_2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & a+d \end{array}$$

Now twist and take as Frobenius form the composition

$$\begin{array}{ccccc} \mathrm{Mat}_2(\mathbb{R}) & \longrightarrow & \mathrm{Mat}_2(\mathbb{R}) & \xrightarrow{\mathrm{Tr}} & \mathbb{R} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \longmapsto & b+c \end{array}$$

This composition is not a central function, for example if we take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then $AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and finally the map gives, in the first case 1 and in the second 2.

4.6. **Finite group algebras.** Let $G = \{e, g_1, \dots, g_n\}$ be a finite group, the *group algebra* $\mathbb{C}[G]$ is defined as the set of formal linear combinations $\sum_{i=0}^{n} c_i g_i$, where $c_i \in \mathbb{C}$, with multiplication given by the multiplication of G. It can be made into a Frobenius algebra by taking the Frobenius form to be the functional

$$\begin{array}{rccl} \varepsilon: & \mathbb{C}G & \longrightarrow & \mathbb{C} \\ & e & \longmapsto & 1 \\ & g_i & \longmapsto & 0 \quad \text{for } i \neq 0. \end{array}$$

Indeed, the corresponding pairing $g \otimes h \mapsto \varepsilon(gh)$ is nondegenerate since $g \otimes h \mapsto 1$ if and only if $h = g^{-1}$.

4.7. **The ring of group characters.** Assume the group field is $\Bbbk = \mathbb{C}$. Let G be a finite group of order n. A *class function* on G is a function $G \to \mathbb{C}$ which is constant on each conjugacy class; the class functions form a ring denoted $R(G)$. In particular, the characters (traces of representations) are class functions, and in fact every class function is a linear combination of characters. There is a bilinear pairing on $R(G)$ defined by

$$\langle \phi, \psi \rangle := \frac{1}{n} \sum_{t \in G} \phi(t)\psi(t^{-1}).$$

The characters form an orthonormal basis of $R(G)$with respect to this bilinear pairing, so in particular the pairing is nondegenerate and provides a Frobenius algebra structure on $R(G)$.

4.8. **The Poincaré Algebra.** Let be M an oriented, compact, connected smooth manifold of dimension n.

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"'] & M \times M \arrow[d, "1\times\delta"] \\ M \times M \arrow[r, "\delta\times 1"'] & M \times M \times M \end{tikzcd}$$

We have that:

$$(\Delta \times 1)^*(1 \times \Delta)^! = \Delta^! \Delta^*,$$

where $\delta^* : \mathrm{H}^*(M) \otimes \mathrm{H}^*(M) = \mathrm{H}^*(M \times M) \to \mathrm{H}^*(M)$ is the map induced by the diagonal map in cohomology, and $\delta^! : \mathrm{H}^*(M) \to \mathrm{H}^*(M) \otimes \mathrm{H}^*(M)$ is the gysin map of the diagonal map. Therefore

$$(\delta^* \otimes 1)(1 \otimes \delta^!) = \delta^! \delta^*.$$

Then $\mathcal{A} := \mathrm{H}^*(M)$ is an algebra with a coproduct, that is a module homomorphism. In the particular case that M is a compact, connected, oriented manifold of finite dimension we can define a counit map $\varepsilon : \mathrm{H}^*(M) \to \Bbbk$ by

$$\varepsilon(\varphi) = \varphi([M]),$$

where $[M]$ is the fundamental class of M in homology. This map induce the pairing

$$\langle\, , \rangle : \mathrm{H}^*(M) \otimes \mathrm{H}^*(M) \to \Bbbk$$

defined by $\langle \varphi, \psi \rangle = \varepsilon(\varphi \smile \psi) = (\varphi \smile \psi)([M]) = \varphi([M] \frown \psi)$. Remember that we have the next isomorphism induced by Poincaré duality

$$\Phi : \mathrm{H}^{n-k}(M) \xrightarrow{h} \mathrm{Hom}_{\Bbbk}(\mathrm{H}_{n-k}(M), \Bbbk) \xrightarrow{D^*} \mathrm{Hom}_{\Bbbk}(\mathrm{H}^k(M), \Bbbk)$$

where h is the map induced by the evaluation of cochains on chains, and D^* is the dual of Poincaré duality. Then $\Phi(\varphi)(\psi) = \varphi([M] \frown \psi)$, this proves that the pairing is nondegenerate.

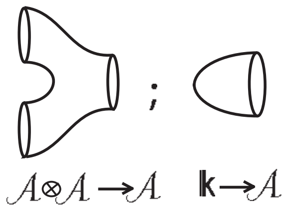
5. $(1+1)$-DIMENSIONAL TQFTS AS FROBENIUS ALGEBRAS.

Theorem 5.1. (Folklore) *There is a canonical equivalence of categories*

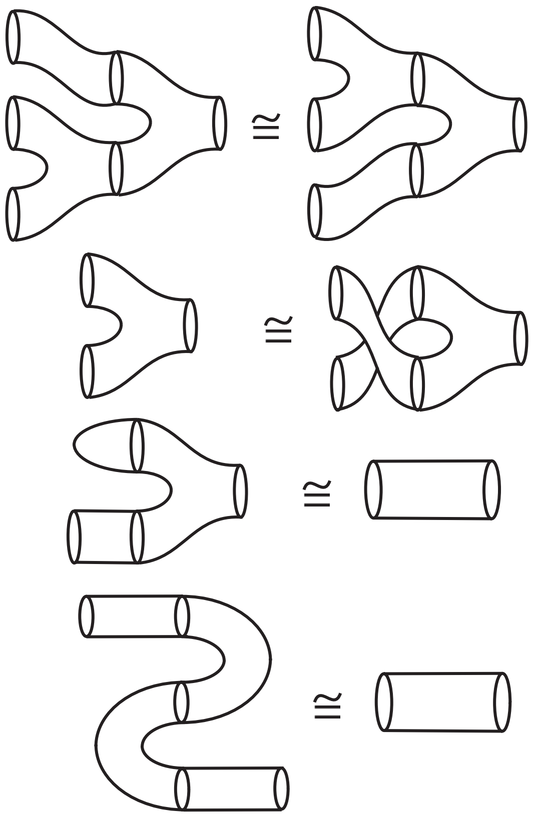
$$\text{2D-TFT}_{\Bbbk} \simeq \text{cFA}_{\Bbbk}$$

where $\text{cFA}_{\Bbbk}$ *is the category of commutative Frobenius algebras.*

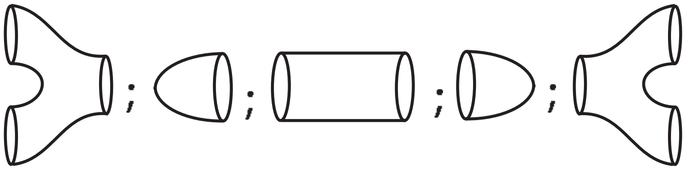
Proof. We follow Moore-Segal [15] for this. It is easy to see that a 2-TFT determines a Frobenius algebra. This is the vector space $\mathcal{A}$ associated to the circle. The next cobordisms induce a product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a unid $u : \mathbb{k} \to \mathcal{A}$.



The next pictures imply respectively the properties of associativity, commutativity, unit and non-degeneracy.



We need to prove that when we have a commutative Frobenius algebra we can assign a well defined functor from $2\,\mathrm{Cob}$ to $\mathrm{Vect}_{\mathbb{k}}$, for this first we note that the category is generated under composition and disjoint unions by the next five elementary cobordisms,



For the matter of argumentation fix a 2-dimensional cobordism Σ.

It is not hard to associate a linear operator to a pair consisting of a cobordism together with a decomposition on the previous five elementary building blocks. The problem is to show that he operator is independent of the chosen decomposition.

The basic idea of the proof is analogous to the proof of the Poincaré-Hopf theorem, where one embeds the discrete space of triangulations on the continuous space of vector fields on a manifold and maving around in the space of vector fields one proves that the Euler characteristic does not depend on the triangulation. Now we will embed the discrete space of possible decompositions of Σ into the continous space of Morse functions on Σ

Given a Morse function $f\colon \Sigma \to \mathbb{R}$ on a 2-dimensional cobordism (with the boundaries attaining constant values corresponding to the max and the min of the function f, and all critical points of Morse type and taking different values) we must associate a decomposition of Σ. This is easily achieved by cutting up sigma along $f^{-1}(t)$ for one choice of t between any two consecutive critical values of f.

Moreover every decomposition in elementary cobordisms can be achieved by a Morse function of this sort. The construction of a well defined functor is possible because there is a path in the space of Morse functions that joins any pair of Morse functions associated to a specific cobordism. According to Cerf's theory [4], two Morse functions can always be connected by a good path in which every element is a Morse function except for a finite set which belongs to one of the two following cases:

1. The function has one degenerate critical point where in local coordinates (x, y) it has the form $\pm x^2 + y^3$.
2. Only two critical values of Morse type coincide.

It is understood that in any of the two cases the remaining critical values are different (for the case 1, they are even different to the degenerate critical point) and of Morse type. The invariance of the operator associated to Σ in the first case is implied by the unit and counit axioms, for the second case we must use the identity for the

Euler number

$$\chi = \sum (-1)^{\lambda} c_{\lambda}$$

with c_λ the number of critical points of index λ of its Morse function. Since every elementary cobordism has at most a critical point of index 0, 1 or 2; then for the case $\chi = 2$ the cobordism corresponding to the two critical values has Euler number -2, 0 or 2. When $\chi = 0$ or 2 the only relevant possibilities are the cylinder and the sphere while for $\chi = -2$ it is just a torus with two holes or the sphere with four holes. In the case $(1, 1, 1)$ (one entry, genus one and one exit) there is nothing to check, because, though a torus with two holes can be cut into two pair of pants by many different isotopy classes of cuts, there is only one possible composite cobordism, and we have only one possible composite map

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

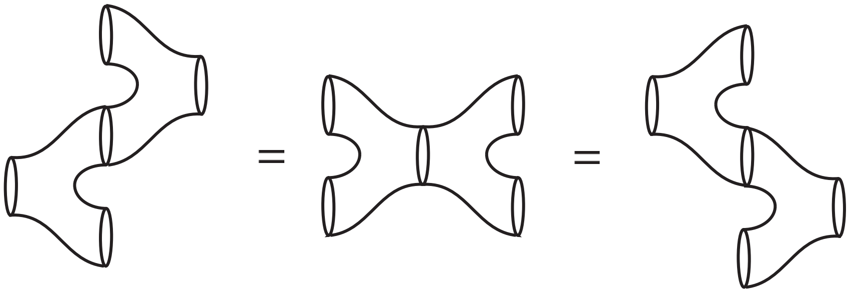
Note that the coproduct is just

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\delta"] \arrow[d, "\lambda"'] & \mathcal{A} \otimes \mathcal{A} \\
\mathcal{A}^* \arrow[r, "m^*"'] & \mathcal{A}^* \otimes \mathcal{A}^* \arrow[u, "\lambda^{-1} \otimes \lambda^{-1}"']
\end{tikzcd}$$

where λ is the corresponding Frobenius isomorphism between $\mathcal{A}$ and its dual. For a commutative algebra is easy to prove that

$$\Delta(a) = \sum a e_i \otimes e_i^{\#} = \sum e_i \otimes e_i^{\#} a$$

with $\{e_i\}$ a basis for $\mathcal{A}$ and $\#$ denotes the dual. For the sphere with four holes when we have $(3, 0, 1)$ and $(1, 0, 3)$ these cases are covered by the associativity of the product and coassociative of the coproduct respectively. Finally for $(2, 0, 2)$ it is enough to prove that it is well defined for all the possible pants decomposition; it is known that for a compact surface (m, g, n) (meaning m input circles, genus g and n output circles,) every pair-of-pants decomposition has $3g - 3 + m + n$ simple closed curves which cut the surface in $2g - 2 + m + n$ pairs of pants, hence for this case we have only a curve dividing in two pair of pants and then the only possibilities are



but this is clearly Abrams' condition 4.4. ☐

6. CONSTRUCTION OF THEORIES

A very important problem in mathematics is the rigorous construction of field theories.

The basic example is afforded to us by Poincaré duality. This model written $(H^M, Z^M)_{1+1} \cong (A_M, \theta_M)$ depends only of a fixed oriented compact closed smooth manifold M and lives in dimension $1+1$. Let $\mathrm{Maps}^{\odot}(Y, M)$ be the space of constant maps from Y to M. Clearly if Y is connected (and non-empty), $\mathrm{Maps}^{\odot}(Y, M) \cong M$ and in fact this last homeomorphism is given by the map

$$\mathrm{ev}_y \colon \mathrm{Maps}^{\odot}(Y, M) \to M$$

that evaluates at $y \in Y$. For $Z \subset Y$ we will write $\mathrm{ev}_Z \colon \mathrm{Maps}^{\odot}(Y, M) \to \mathrm{Maps}^{\odot}(Z, M)$ to be the restriction map defined by $\mathrm{ev}_Z(f) = f|_Z$.

In this theory the fields are

$$\mathcal{F}(Y) = \mathrm{Maps}^{\odot}(Y, M),$$

namely the moduli space of constant maps from Y to M. We consider Y to be $(1+1)$-dimensional. Notice that

$$\mathrm{Maps}^{\odot}(Y, M) = M \times M \times \cdots \times M$$

where the product contains as many copies of M as connected components has Y. Consider now the situation in which $Y = P$ a 2-dimensional pair-of-pants (a 2-sphere with three small discs removed) with two incoming boundary components and one outgoing, and M is an oriented compact closed smooth manifold. Let a,b and c be three boundary components P each one diffeomorphic to S^1.

$$\begin{tikzcd}
 & \mathcal{F}(Y) \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathcal{F}(\partial_0 Y) & & \mathcal{F}(\partial_1 Y)
\end{tikzcd}$$

that is to say
(2)

$$\begin{tikzcd}
 & \mathrm{Maps}^{\odot}(P, M) \arrow[dl, "\mathrm{ev}_a \times \mathrm{ev}_b"'] \arrow[dr, "\mathrm{ev}_c"] & \\
\mathrm{Maps}^{\odot}(S^1, M) \times \mathrm{Maps}^{\odot}(S^1, M) & & \mathrm{Maps}^{\odot}(S^1, M)
\end{tikzcd}$$

which becomes thus

$$\begin{tikzcd}
 & M \arrow[dl, "\triangle"'] \arrow[dr, "="] & \\
M \times M & & M
\end{tikzcd}$$

and indeed, since that is a smooth correspondence of degree $-d$ we have that

$$\triangle_! = ev_c \circ (\mathrm{ev}_a \times \mathrm{ev}_b)_! \colon H_*(M) \otimes H_*(M) \to H_{*-d}(M)$$

is the induced homomorphism of degree $-d$ in homology. Namely, *the Feynman evolution for a pair of pants in this field theory is simply the intersection product in homology.*

We could have used the space $\mathbf{8}$ consisting of the wedge of two copies of S^1 instead of P (they are after all homotopy equivalent, we can define ev_c by choosing a quotient map $c \to \mathbf{8}$ identifying two points of c). Notice that by using pairs-of-pants we can recover any compact oriented 2-dimensional cobordism Y which is not boundaryless. In fact by using correspondences we can recover Ψ_Y^M for all Y that has at least one outgoing boundary component. In a sense correspondences encode a big portion of Poincaré duality this way, the so-called positive boundary sector of the TQFT.

For this model we have,

- $A_M = \mathcal{H}(\bullet) = H_*(M)$ (the homology of M which is graded).
- The mapping associated to the pair of pants

  $$A_M \otimes A_M \to A_M \tag{3}$$

  is the intersection product on the homology of the manifold (and is of degree $-d$).
- The trace is defined as $\theta_M \colon A_M = H_*(M) \to H_*(\bullet) \cong \mathbb{C}$. The nondegeneracy of the trace is a consequence of Poincaré duality.

It may be instructive to see how the Pontrjagin-Thom construction and the Thom isomorphism can be used to induce the map 3. That basic idea is to use the *diagonal map*

$$\triangle\colon M \to M \times M.$$
$$m \mapsto (m, m)$$

The product on A_M is precisely the Gysin map $\triangle_!$ which can be defined using integration over the fiber, or as follows. It is not hard to verify that the normal bundle ν of $M = \triangle(M)$ in $M \times M$ is isomorphic to the tangent bundle TM of M. Let us write M_ϵ a small neighborhood of M in $M \times M$, and M^{TM} the Thom space on TM. Then we have a natural map

$$M \times M \longrightarrow M \times M/(M \times M - M_\epsilon) = M^{TM}$$

which by the use of the Thom isomorphism induces

$$\triangle_!\colon H_*(M) \otimes H_*(M) \longrightarrow H_{*-d}(M)$$

as desired.

Example 6.1. This is a famous example due to Chas and Sullivan [5]. Following Cohen and Jones [7] we do something rather drastic now and let the maps roam free, namely we write the correspondence 2 but with the whole mapping spaces rather than just the constant maps.

$$\begin{tikzcd}
& \mathrm{Maps}(\mathbf{8}, \mathrm{M}) \arrow[dl, "\mathrm{ev}_a \times \mathrm{ev}_b"'] \arrow[dr, "\mathrm{ev}_c"] & \\
(\mathcal{L}\mathrm{M})^2 = \mathrm{Maps}(\mathrm{S}^1, \mathrm{M}) \times \mathrm{Maps}(\mathrm{S}^1, \mathrm{M}) & & \mathrm{Maps}(\mathrm{S}^1, \mathrm{M}) = \mathcal{L}\mathrm{M}
\end{tikzcd}$$

which *is a degree $-d$ smooth correspondence*. We must replace the pair of pants P for the figure eight space $\mathbf{8}$ in order to ensure that $\mathrm{Maps}(\mathbf{8}, M) \to \mathcal{L}M \times \mathcal{L}M$ is a *finite* codimension embedding. This in turns implies the existence of the Gysin map

$$(\mathrm{ev}_a \times \mathrm{ev}_b)_!\colon H_*(\mathcal{L}M \times \mathcal{L}M) \to H_{*-d}(\mathrm{Maps}(\mathbf{8}, M)).$$

The induced map in homology

$$\bullet\colon H_*(\mathcal{L}M) \otimes H_*(\mathcal{L}M) \to H_{*-d}(\mathcal{L}M)$$

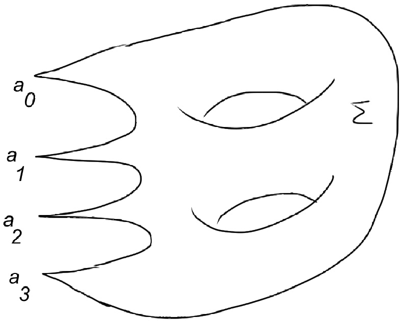
is called the *Chas-Sullivan* product on the homology of the free loop space of M. From the functoriality of correspondences it isn't hard to verify that the product is associative.

Chas and Sullivan proved more, by defining a degree one map $\Delta\colon H_*(\mathcal{L}M) \to H_{*+1}(\mathcal{L}M)$ given by $\Delta(\sigma) = \rho_*(\theta \otimes \sigma)$ where $\rho\colon S^1 \times \mathcal{L}M \to \mathcal{L}M$ is the evaluation map and θ is the generator of $H^1(S^1, \mathbb{Z})$, they proved that $(H_*(M), \bullet, \Delta)$ is a Batalin-Vilkovisky algebra, namely

- $(H_{*-d}(M), \bullet)$ is a graded commutative algebra.
- $\Delta^2 = 0$
- The bracket $\{\alpha, \beta\} = (-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta)$ makes $H_{*-d}(M)$ into a graded Gerstenhaber algebra (namely it is a Lie bracket which is a derivation on each variable).

This statement amounts essentially to the construction of $\Psi_Y^{\mathcal{L}M}$ for all positive boundary genus zero $(1+1)$-dimensional cobordisms Y due to a theorem of Getzler (cf. [9]). The case of positive genus has been studied by Cohen and Godin [6].

Example 6.2. The Gromov-Witten invariants introduced by Ruan in [17] can be understood in terms of a field theory [16]. Now we consider a Riemmann surface $Y = \Sigma_g$ of genus g with k marked points. These marked points will take the place of $\partial_0 Y$ and for simplicity we will not consider outgoing boundary for now.



In this (1+1)-dimensional quantum field theory we start by considering a fixed symplectic manifold (M, ω). The space of fields is given (roughly speaking) by the space of J-holomorphic maps on the class $\beta \in H_2(M)$,

$$\mathcal{F}(Y) = \mathcal{M}_\Sigma = \mathrm{Hol}_\beta(\Sigma, M) = \{f \in \mathrm{Hol}(\Sigma, M) | f_*[\Sigma] = \beta\},$$

If we denote by $\mathrm{ev}_i \colon \mathcal{M}_\Sigma \to M$ the evaluation map at $a_i \in \Sigma$, then we have the correspondence diagram

$$\begin{tikzcd}
& \mathcal{M}_\Sigma \arrow[dl, "\times_i \mathrm{ev}_i"'] \arrow[dr] & \\
M^k = \mathcal{F}(\amalg_i a_i) & & \mathcal{F}(\varnothing) = \bullet
\end{tikzcd}$$

Given k cohomology classes $u_1, \ldots, u_k \in H^*(M)$ we can let them evolve according to Feynman's pull-push formalism to obtain the corresponding *Gromov-Witten invariant*

$$\Phi_{g,\beta,k}(u_1, \ldots, u_k) = \int_{\mathcal{M}_\Sigma} \mathrm{ev}_1^* u_1 \wedge \ldots \wedge \mathrm{ev}_k^* u_k$$

Here we should mention two important technical points regarding the moduli space $\mathcal{M}_\Sigma$. Firstly Kontsevich [13] discovered that the most convenient space for defining this field theory is the moduli space of stable maps (where at most ordinary double points are allowed, and with finite automorphism groups). The moduli space turns out to be an orbifold, not a manifold. We will return to the definition of an orbifold later.

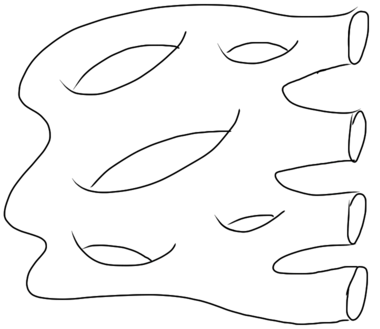
Secondly, the moduli space does not quite have a fundamental class (that we require to do the integration). The problem is that roughly speaking $\mathcal{M}$ is given as the intersection of two submanifolds (equations) N_1 and N_2 of a larger manifold V (taking only two is possible by using the diagonal map trick, namely $N_1 \cap \ldots \cap N_r = (N_1 \times \cdots \times N_r) \cap \triangle(V^r)$). Often this intersection is not transversal. Therefore rather than a tangent we have a *virtual* tangent bundle (in K-theory)

$$[T\mathcal{M}]^{\mathrm{virt}} = [TN_1]|_{\mathcal{M}} + [TN_2]|_{\mathcal{M}} - [TV]|_{\mathcal{M}}$$

whose orientation (in cohomology, K-theory, complex cobordism) is called the *virtual fundamental class* $[\mathcal{M}]^{\mathrm{virt}}$. The corrected formula for the Gromov-Witten invariants is then

$$\Phi_{g,\beta,k}(u_1, \ldots, u_k) = \int_{[\mathcal{M}_\Sigma]^{\mathrm{virt}}} \mathrm{ev}_1^* u_1 \wedge \ldots \wedge \mathrm{ev}_k^* u_k.$$

Example 6.3. Floer theory is also a quantum field theory. Now we consider $Y = \Sigma_{g,k}$ to be a genus g Riemann surface with k small discs removed.



The fields are again holomorphic mappings $\mathcal{F}(Y) = \mathcal{M}_\Sigma$.

$$\begin{tikzcd}
 & \mathcal{M}_\Sigma \arrow[dl, "\times_i \mathrm{ev}_i"'] \arrow[dr] & \\
\mathcal{F}(\varnothing) = \bullet & & (\mathcal{L}M)^k = \mathcal{F}(\partial_1 \Sigma)
\end{tikzcd}$$

In this case rather than simply considering the homology of $\mathcal{L}M$ we consider its semi-infinite (co)homology. This means that we consider the homology of cycles that are half-dimensional in $\mathcal{L}M$. The semi-infinite (co)homology $H_*^{\mathrm{si}}(\mathcal{L}M)$ is also known as the *Floer* (co)homology $HF_*(M)$.

Finally we mention that a very important generalizations of the topics described here can be found in [12], [15], [8]. We refer the reader to those excellent papers to learn more about them.

7. APPENDIX: MONOIDAL CATEGORIES.

Definition 7.1. A monoidal category (or tensor category) consists of the following data: a category $\mathscr{C}$, a covariant functor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$, called the monoidal product(or tensor product), an object $u \in \mathrm{Ob}(\mathscr{C})$, called the unit and natural isomorphisms

- $\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z$,
- $\lambda_x : u \otimes x \longrightarrow x$,
- $\rho_x : x \otimes u \longrightarrow x$,

called *associativity*, *left unit* and *right unit*. This natural isomorphisms satisfy the following axioms:

$$\begin{tikzcd}
x \otimes (y \otimes (w \otimes z)) \arrow[r, "\alpha_{x,y,w\otimes z}"] \arrow[d, "1\otimes\alpha_{y,w,z}"'] & (x \otimes y) \otimes (w \otimes z) \arrow[r, "\alpha_{x\otimes y,w,z}"] & ((x \otimes y) \otimes w) \otimes z \\
x \otimes ((y \otimes w) \otimes z) \arrow[rr, "\alpha_{x,y\otimes w,z}"'] & & (x \otimes (y \otimes w)) \otimes z \arrow[u, "\alpha_{x,y,w}\otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (u \otimes y) \arrow[rr, "\alpha_{x,u,y}"] \arrow[dr, "1\otimes\lambda_y"'] & & (x \otimes u) \otimes y \arrow[dl, "\rho_x\otimes 1"] \\
 & x \otimes y &
\end{tikzcd}$$

for $x, y, w, z \in \mathrm{Ob}(\mathscr{C})$, and also

$$\lambda_u = \rho_u : u \otimes u \longrightarrow u.$$

A monoidal category is called *strict monoidal category*, if the morphisms α, λ, ρ are the identity morphisms.

7.1. **Monoidal Functors.**

Definition 7.2. Let $(\mathscr{C}, \otimes)$ and $(\mathscr{D}, \otimes)$ be monoidal categories. A *monoidal functor* is a functor $F : \mathscr{C} \longrightarrow \mathscr{C}$ together with natural isomorphisms

- $\xi_{x,y} : F(x) \otimes F(y) \longrightarrow F(x \otimes y)$
- $\xi_0 : u_{\mathscr{D}} \longrightarrow F(u_{\mathscr{D}})$

which satisfy the following commutative diagrams:

$$\begin{tikzcd}
F(x) \otimes (F(y) \otimes F(z)) \arrow[r, "1\otimes\xi"] \arrow[d, "\alpha"'] & F(x) \otimes F(y \otimes z) \arrow[r, "\xi"] & F(x \otimes (y \otimes z)) \arrow[d, "F(\alpha)"] \\
(F(x) \otimes F(y)) \otimes F(z) \arrow[r, "\xi\otimes 1"'] & F((x \otimes y) \otimes F(z) \arrow[r, "\xi"'] & F((x \otimes y) \otimes z)
\end{tikzcd}$$

$$\begin{tikzcd}
u \otimes F(x) \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\lambda"'] & F(u) \otimes F(x) \arrow[r, "\xi"] & F(u \otimes x) \arrow[dl, "F(\lambda)"] \\
& F(x) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(x) \otimes u \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\rho"'] & F(x) \otimes F(u) \arrow[r, "\xi"] & F(x \otimes u) \arrow[dl, "F(\rho)"] \\
& F(x) &
\end{tikzcd}$$

A monoidal functor is called *stric monoidal functor* if ξ and ξ_0 are the identity morphisms.

In all that follows we will further assume that the topological cylinder $\Sigma_0 := S^1 \times [0,1]$ seen as a cobordism between a circle and itself gets assigned the identity map by the functor, namely $\mathrm{Z}^C(\Sigma_0) = \mathrm{id}$.

Remark 7.3. For any monoidal functors $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{E}$. Let (ξ, ξ_0) and (ξ', ξ_0') the natural isomorphisms of F and G, respectively. The natural isomorphisms (ξ'', ξ_0'') for the composition $F \circ G : \mathscr{C} \longrightarrow \mathscr{E}$ are defined by

$$\begin{tikzcd}
G \circ F(x) \otimes G \circ F(y) \arrow[r, "\xi'"] \arrow[rr, bend right, "\xi''"'] & G(F(x) \otimes F(y)) \arrow[r, "G(\xi)"] & G \circ F(x \otimes y)
\end{tikzcd}$$

$$\begin{tikzcd}
u_{\mathcal{E}} \arrow[r, "\xi_0'"] \arrow[rr, bend right, "\xi_0''"'] & G(u_{\mathscr{D}}) \arrow[r, "G(\xi_0)"] & G \circ F(u_{\mathscr{C}})
\end{tikzcd}$$

Example 7.4. The most important ones are

$(\mathit{Set}, \times, \{*\})$,	the category of sets with the cross product.
$(\mathit{Set}, \sqcup, \emptyset)$,	the category of sets with the disjoint union.
$(\mathrm{Vect}_{\Bbbk}, \otimes, \Bbbk)$,	the category of vector spaces with the tensor product over $\Bbbk$.
$(\mathrm{Top}, \times, *)$,	the category of topological spaces with the cross product.
$(\mathrm{Ab}, \otimes, \mathbb{Z})$,	the category of abelian groups with the usual tensor product over $\mathbb{Z}$.
$(\mathrm{nCob}, \sqcup, \emptyset)$,	the category of n-cobordisms whit the disjoint union.

7.2. **Monoidal Natural Transformations.**

Definition 7.5. A natural transformation $\sigma : F \longrightarrow F'$ between two monoidal functors is called a *monoidal natural transformation* if the diagrams

$$\begin{tikzcd}
F(x) \otimes F(y) \arrow[r, "\xi"] \arrow[d, "\sigma\otimes\sigma"'] & F(x \otimes y) \arrow[d, "\sigma"] \\
F'(x) \otimes F'(y) \arrow[r, "\xi"'] & F'(x \otimes y)
\end{tikzcd}$$

$$\begin{tikzcd}
u \arrow[r, "\xi_0"] \arrow[dr, "\xi_0'"'] & F(u) \arrow[d, "\sigma"] \\
& F'(u)
\end{tikzcd}$$

commute.

Let $\mathscr{C}$ and $\mathscr{D}$ monoidal categories. A monoidal functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is called a *monoidal equivalence* if there exists a monoidal functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ and monoidal natural isomorphisms $\varphi : G \circ F \cong 1_{\mathscr{C}}$ and $\psi : F \circ G \cong 1_{\mathscr{D}}$.

7.3. **Braided Monoidal Categories.** A *braided monoidal category* consists of a monoidal category $\mathscr{M}$ together with a *braiding*, which is defined by a family of isomorphisms

$$\varsigma_{x,y} : x \otimes y \longrightarrow y \otimes x.$$

They are natural for x and y in $\mathscr{M}$, and satisfy for the unit u the commutative diagram

$$\begin{tikzcd}
x \otimes u \arrow[rr, "\varsigma"] \arrow[dr, "\rho"'] & & u \otimes x \arrow[dl, "\lambda"] \\
& x, &
\end{tikzcd}$$

Moreover the maps $\varsigma_{x,y}$, together with the associativity α make commutative the following hexagonal diagrams:

$$\begin{tikzcd}
 & (x \otimes y) \otimes z \arrow[r, "\varsigma"] \arrow[dl, "\alpha^{-1}"'] & z \otimes (x \otimes y) \arrow[dr, "\alpha"] & \\
x \otimes (y \otimes z) \arrow[dr, "1 \otimes \varsigma"'] & & & (z \otimes x) \otimes y \arrow[dl, "\varsigma \otimes 1"] \\
 & x \otimes (z \otimes y) \arrow[r, "\alpha"'] & (x \otimes z) \otimes y, &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x \otimes (y \otimes z) \arrow[r, "\varsigma"] \arrow[dl, "\alpha"'] & (y \otimes z) \otimes x \arrow[dr, "\alpha^{-1}"] & \\
(x \otimes y) \otimes z \arrow[dr, "\varsigma \otimes 1"'] & & & y \otimes (z \otimes x) \arrow[dl, "1 \otimes \varsigma"] \\
 & (y \otimes x) \otimes z \arrow[r, "\alpha^{-1}"'] & y \otimes (x \otimes z). &
\end{tikzcd}$$

7.4. **Symmetric Monoidal Categories.** A *symmetric monoidal category* is a monoidal category with a braiding, which satisfies the identity

$$\varsigma_{y,x} \circ \varsigma_{x,y} = 1.$$

Proposition 7.6. *For $\mathscr{M}$ a symmetric monoidal category we have the identity*

$$(1 \otimes \varsigma) \circ \varsigma \circ \alpha^{-1} = \alpha \circ \varsigma \circ (1 \otimes \varsigma).$$

Proof.

$$\begin{tikzcd}
 & \cdot \arrow[r, "\alpha"] \arrow[dl, "\varsigma"'] & \cdot \arrow[dr, "\varsigma\otimes 1"] & & & \\
\cdot \arrow[dr, "\alpha^{-1}"'] & & & \cdot \arrow[r, "\varsigma"] \arrow[dl, "\alpha^{-1}"] & \cdot \arrow[dr, "\alpha"] & \\
 & \cdot \arrow[r, "1\otimes\varsigma"'] & \cdot \arrow[dr, "1\otimes\varsigma"'] & & & \cdot \arrow[dl, "\varsigma\otimes 1"] \\
 & & & \cdot \arrow[r, "\alpha"'] & \cdot, &
\end{tikzcd}$$

then

$$\varsigma = (\varsigma \otimes 1) \circ \alpha \circ \varsigma \cdot (\varsigma \otimes 1) \cdot \alpha, \tag{4}$$

$$\Rightarrow \quad \alpha^{-1} \cdot (\varsigma \otimes 1) \cdot \varsigma = \varsigma \cdot (\varsigma \otimes 1) \cdot \alpha, \tag{5}$$

$$\Rightarrow (1 \otimes \varsigma) \circ \varsigma \circ \alpha^{-1} = \alpha \circ \varsigma \circ (1 \otimes \varsigma). \tag{6}$$

□

REFERENCES

[1] Lowell Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5** (5) (1996), 569–587. MR MR1414088 (97j:81292).

[2] Michael F. Atiyah, *Topological quantum field theory*, Publ. Math. IHES **68** (1988), 175–186.

[3] Michael F. Atiyah, "The geometry and physics of knots", *Cambridge University Press*, 1990.

[4] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théoreme de la pseudo-isotopie*, Publ. Math. IHES **39** (1970), 5-173.

[5] M. Chas and D. Sullivan, String topology, arXiv:math.GT/9911159.

[6] R. Cohen and V. Godin, "A polarized view of string topology, Topology, geometry and quantum field theory", Proceedings of the 2002 Oxford symposium in honour of the 60th birthday of Graeme Segal (Ulrike Tillmann, ed.) London Mathematical Society Lecture Note Series, no. 308, *Cambridge University Press* (2004), 127-154. arXiv:math.AT/0303003

[7] R. Cohen and J. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (4) (2002), 773-798. MR 2004c:55019.

[8] K. Costello, *Topological conformal field theories and calabi-yau categories*, Adv. Math. **210** (1) (2007), 165–214.

[9] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (2) (1994), 265–285. MR 95h:81099

[10] Morris W. Hirsch, "Diferential topology", Graduate Texts in Mathematics, *Springer* 33, 1995.

[11] Joachim Kock, "Frobenius algebras and 2d topological quantum field theory", *Cambridge University Press*, 2003.

[12] M. Kontsevich and Y. Manin, *Gromov-witten classes, quantum cohomology, and enumerative geometry*, Commun. Math. Phys. **164** (3) (1994), 525–562.

[13] Maxim Kontsevich, "Enumeration of rational curves via torus actions, The moduli space of curves" (Texel Island, 1994), Progr. Math. **129**, *Birkhauser* Boston, Boston, MA (1995),335–368. MR MR1363062 (97d:14077).

[14] Aaron D. Lauda, *Frobenius algebras and planar open string topological field theories*, arXiv:math/0508349 (2008).

[15] Gregory Moore and Graeme Segal, *D-branes and k-theory in 2d topological field theory*, http://arxiv.org/abs/hep-th/0609042.

[16] S. Piunikhin, D. Salamon, and M. Schwarz, "Symplectic Floer-Donaldson theory and quantum cohomology, Contact and symplectic geometry" (Cambridge, 1994). Publ. Newton Inst. **8**, *Cambridge Univ. Press*, Cambridge (1996), 171–200. MR MR1432464 (97m:57053)

[17] Yongbin Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. J. **83** (2), (1996), 461–500. MR MR1390655 (97d:58042)

[18] G. Segal, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (1988), 165–171.

[19] Graeme Segal, "The Stanford Lectures on QFT", *Stanford Lectures*, (2000).

[20] E. Witten, *Quantum field theory and the jones polynomial*, Commun. Math. Phys. **121**(3) (1989), 351–399.

(A. Gonzalez) CENTRO DE MATEMÁTICA DE LA FACULTAD DE CIENCIAS, MONTEVIDEO, URUGUAY
E-mail address: `ana@cmat.edu.uy`

(E. Lupercio, C. Segovia and M. Xicotencatl) DEPARTAMENTO DE MATEMÁTICAS, CINVESTAV, APDO. POSTAL 14-740, 07000, MÉXICO, D.F.
E-mail address: `lupercio@math.cinvestav.mx`
E-mail address: `csegovia@math.cinvestav.mx`
E-mail address: `xico@math.cinvestav.mx`