1. The dimension of the universal embedding of the symplectic polar space

Consider a $\mathbb{Z}_2$-vector space of dimension $2n$ with a symplectic form $\omega$. Consider the geometry with lines of three elements defined as follows. The points are the maximal totally isotropic subspaces of dimension $n$, i.e., $w(U) = 0$ for $U$ a subspace. The lines are given by the totally isotropic subspaces of dimension $n-1$. Denote $X$ and $L$ the sets of points and lines respectively. We consider the linear map $\omega : X \times L \to \mathbb{Z}_2$, such that $\omega(x, l) = 1$ if $l$ is contained in $x$. The dimension of the universal embedding of the symplectic polar space is the dimension of the module $\mathbb{Z}_2 X/\langle \mathbb{Z}_2 X, L \rangle$. For example for $n = 2$ we have $X = \{(0,1), (1,1)\}$ with only one line. For $n = 3$ the geometry gives the Cremona-Richmond configuration.

2. The density of a language with four letters

The density of a language with four letters is defined by counting words as follows. Take words with letters $1, 2, 3, 4$ of length $n$ with the property that from left to right each letter satisfies $0 \leq n_i \leq \max_{j \neq i} (n_j) + 1$. Thus the first letter is always 1, so we can dismiss it. Consequently for $n = 2$ there are two words 1 and 2. For $n = 3$ the words are 11, 12, 21, 22, 23 while for $n = 4$ we have 15 words.

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3. The rank of the $\mathbb{Z}_2^n$-cobordism category in dimension 1 + 1

We consider the cardinality of the quotient of $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$ under the action of the special linear group $\text{SL}(2, \mathbb{Z})$. This group is generated by two matrices which produce essentially two basic equations $g(k) \sim (k, -\bar{k})$ and $g(k) \sim (\bar{k}, k)$. The orbits of this quotient give a set of generators for the monoid of principal $\mathbb{Z}_2^n$-bundles over closed surfaces with two boundary circles up to a homeomorphism identification. For $n = 1$ we get 2 orbits (0,0) and (0,1) ~ (1,0) ~ (1,1). For $n = 2$ we get 5 orbits.

[(N1) $wt(v_i) \leq 2$ for every $i \in \{1, \ldots, k\}$]
[(N2) If $v_i > v_j$ (i.e., $i < j$) and $wt(v_i) \geq 2$, then $\beta(v_i) \leq \beta(v_j)$.]
[(N3) If $v_i > v_j > v_k$, $wt(v_i) = 2$, then $\beta(v_i) < \beta(v_k)$.]
[(N4) There do not exist $v_i > v_j > v_k$ such that $wt(v_i) = wt(v_j) = wt(v_k) = 2$ and $\beta(v_i) = \beta(v_j) = \beta(v_k)$.]

Reference: