Categorical approach to equivariant Morse theory

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Morse theory

Let $M$ be $C^\infty$, compact, Riemannian manifold (without boundary), and

$$f : M \rightarrow \mathbb{R}$$

a $C^\infty$ map. A critical point $p \in M$ is Morse if the bilinear form

$$Hess_p(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

is non-degenerate.

The gradient flow lines $\gamma : \mathbb{R} \rightarrow M$ satisfying

$$\frac{d\gamma}{dt} + \nabla_\gamma(f) = 0.$$

For a critical point, the stable and unstable manifold are

$$W^s(a) = \{ x \in M : \lim_{t \rightarrow +\infty} \gamma_x(t) = a \}$$

$$W^u(a) = \{ x \in M : \lim_{t \rightarrow -\infty} \gamma_x(t) = a \}$$
Classifying spaces

**Definition**

For $\mathcal{C}$ a category, the classifying space $B\mathcal{C}$ is the realization of the nerve $N\mathcal{C}$. Where for a simplicial set (space) $X$ the realization is defined as the quotient $\coprod_{n \geq 0} \Delta_n \times X_n / \sim$ with $(s, X(f)a) \sim (\Delta_f(s), a)$ and we get:

- Category $\mathcal{C} \mapsto \text{Topological space } B\mathcal{C}$
- Functor $F : \mathcal{C} \to \mathcal{D} \mapsto \text{Continuous function } BF : B\mathcal{C} \to B\mathcal{D}$
- Natural transformation $\alpha : F \Rightarrow G \mapsto \text{Homotopy } H_\alpha : B\mathcal{C} \times I \to B\mathcal{D}$

**Definition**

Let $F : \mathcal{C} \to \mathcal{D}$ functor, $y$ in $\mathcal{D}$. The category $y \setminus F$ has objects $(x, v)$, $v : y \to F(x)$, morphisms from $(x, v)$ to $(x', v')$ is $u : x \to x'$, $v' = F(u)v$.

**Theorem A (Quillen)**

*If the category $y \setminus F$ is contractible for every object $y$ of $\mathcal{D}$, then the functor $F$ is a homotopy equivalence.*
Morse theory and classifying spaces

For $f : M \to \mathbb{R}$ a Morse function we define the “flow category” $\mathcal{C}_f$ as follows:

- the objects, are just the union of all the critical points
  \[
  \text{Obj } \mathcal{C}_f = \bigsqcup_{p \in \text{Crit}_f} p
  \]
- for two critical points $a$ and $b$ we define the space of objects $\text{Hom}_{\mathcal{C}_f}(a, b)$ as the compactification of the moduli space $\mathcal{M}(a, b) = (W^u(a) \cap W^s(b))/\mathbb{R}$. We denote this space by $\overline{\mathcal{M}}(a, b)$.

**Theorem (Cohen-Jones-Segal)**

- For $f : M \to \mathbb{R}$ a Morse function, the classifying space of $\mathcal{C}_f$ is of the homotopy type of $M$.
- For $f : M \to \mathbb{R}$ a Morse-Smale function, the classifying space of $\mathcal{C}_f$ is homeomorphic with $M$. 
For a category $\mathcal{C}$ there are pair of functors

$$
\begin{array}{c}
\text{s}(
\mathcal{C})
\end{array}
\xymatrix{
\text{s} 
\ar@{<-}[r] & \text{T} \\
\mathcal{C} 
\ar@{<-}[u] 
\ar@{<-}[d] \\
\mathcal{C}^o
}
$$

where $\text{s}(\mathcal{C})$ has objects $a \xrightarrow{\gamma} b$ and morphism from $a_1 \xrightarrow{\gamma_1} b_1$ to $a_2 \xrightarrow{\gamma_2} b_2$ pairs $a_2 \xrightarrow{\alpha} a_1$ and $b_1 \xrightarrow{\beta} b_2$ with $\gamma_2 = \beta \gamma_1 \alpha$. This functors are prefibred and

$$
S^{-1}(x) = x \setminus \mathcal{C}, \quad T^{-1}(y) = (\mathcal{C}/y)^o
$$

There is a projection functor for the flow category $\overline{s}(\mathcal{C}_f) \to s(\mathcal{C}_f)$ with $\overline{s}(\mathcal{C}_f)$ the category with pais $(\gamma, x)$ as objects with $x$ in $\gamma$ and morphism as in $s(\mathcal{C}_f)$ but with the same $x$. The induced map in $n$-chains has contractible fiber, so it is a homotopy equivalence.

Let $\underline{M}$ the category with objects the elements of $M$ and morphism only identities, so $\underline{BM} = M$. There are functors $\underline{M} \xleftarrow{\text{proj}} \overline{s}(\mathcal{C}_f)$, defined by $x \mapsto (\gamma_x, x)$ and projection. This categories are homotopy equivalent.
Suppose $G$ acts on $C$, the semi-direct product $C \rtimes G$ is a category with:

- the objects of $C$;
- the morphisms are pairs $(\gamma, g) : x \rightarrow y$ with $g \in G$ and $\gamma : gx \rightarrow y$ a morphism in $C$; and
- the composition of $(\gamma, g) : x \rightarrow y$ with $(\delta, h) : y \rightarrow z$ is $(\delta h \gamma, hg)$.

This is described as follows.
Theorem

The classifying space of the semi-direct product $\mathbb{C} \rtimes G$ has the weak homotopy type of the Borel construction $B\mathbb{C} \times^G EG$.

Theorem A (Quillen-Moerdijk)

Let $F : \mathcal{D} \to \mathcal{C}$ be a $G$-invariant continuous functor between topological categories. If for $n \geq 0$, the quotient map

$$B (\text{Nerve}_n(\mathcal{C}) \setminus F) / G \to \text{Nerve}_n(\mathcal{C})$$

is a weak homotopy equivalence, then

$$\widehat{BF} : B\mathcal{D} / G \to B\mathcal{C}$$

is a weak homotopy equivalence.
For $F : \mathcal{D} \rightarrow \mathcal{C}$ functor and $\varphi : X \rightarrow \mathcal{C}_0$ continuous map, with $\mathcal{C}_0$ the objects. The objects of $X \setminus F$ are triples $(x, u, y)$ with $x \in X$, $y \in \mathcal{D}_0$ and $u : \varphi(x) \rightarrow F(y)$; the morphisms

$\gamma : (x, u, y) \rightarrow (x', u', y')$ for $x = x'$ are arrows $\gamma : y \rightarrow y'$ with $F(\gamma) \circ u = u'$.

Let $\overline{G}$ be the category with objects $G$ and only one morphism between any pair of objects. Consider the functor $T : \mathcal{C} \times \overline{G} \rightarrow \mathcal{C} \rtimes G$ defined in objects $(x, g) \mapsto g^{-1}x$ and in morphisms

$(x, g) \xrightarrow{(\gamma, h^{-1}g)} (y, h)$ the image by $T$ is $(h^{-1}\gamma, h^{-1}g)$. Denote the category $\mathcal{T} := Nerve_n(\mathcal{C} \rtimes G)/T$

$$B\mathcal{T} = \coprod_{Nerve_n(\mathcal{C} \rtimes G)} B\mathcal{T}_x \simeq \coprod_{Nerve_n(\mathcal{C} \rtimes G)} \coprod_{k \in G} B\mathcal{T}_k \simeq \coprod_{Nerve_n(\mathcal{C} \rtimes G)} \coprod_{g \in G} EG$$

where we have the action relates $\mathcal{T}_k \xrightarrow{g} \mathcal{T}_{gk}$ and the inclusion $\mathcal{T}_k \hookrightarrow \overline{G}$ is a homotopy equivalence. Thus $B\mathcal{T}/G$ is of the (weak) homotopy type of $Nerve_n(\mathcal{C} \rtimes G)$ and hence $B(\mathcal{C} \times \overline{G})/G \simeq B(\mathcal{C} \rtimes G)$. 

Equivariant Morse category

Let $M$ be a compact manifold with an action of a Lie group $G$, that is

$$M \times G \rightarrow M.$$ 

Furthermore, if $N_1, N_2$ are two Morse submanifolds, then we have an action

$$G \times W(N_1, N_2) \times \mathbb{R} \rightarrow W(N_1, N_2)$$

given by $(g, x, t) \mapsto g \gamma_x(t)$ where suppose $g \gamma_x = \gamma_{gx}$ as sets. Thus we have an action of $G$ over the flow category $\mathcal{C}_f$ and we get the following result.

**Theorem**

- For a $G$-invariant Morse function we get

$$B(\mathcal{C}_f \rtimes G) \simeq B\mathcal{C}_f \times_G EG.$$ 

- For $G$ a finite group we get

$$B(\mathcal{C}_f \rtimes G) \simeq B\mathcal{C}_f \times_G EG \simeq B(B\mathcal{C}_f \rtimes G).$$
Corollary

For $G$ a group acting free over a manifold $M$ and $f : M \longrightarrow \mathbb{R}$ a $G$-invariant function, we get the (weak) homotopy equivalence

$$B(\mathbb{C}_f \rtimes G) \simeq M/G.$$  

Thanks!!