On the density of certain languages with p^r letters

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22 de febrero del 2016
ELACTAM 2016-Universidad de la Habana
La Habana, Cuba

The sequence $1, 2, 5, 15, 51, 187, 715, 2795, 11051, 43947, \cdots$ with the form:

$$g(n) = \frac{(2^{n-1}+1)(2^{n-2}+1)}{3},$$

is found in the web OEIS with the name A007581:

(1) The density of a language with four letters.

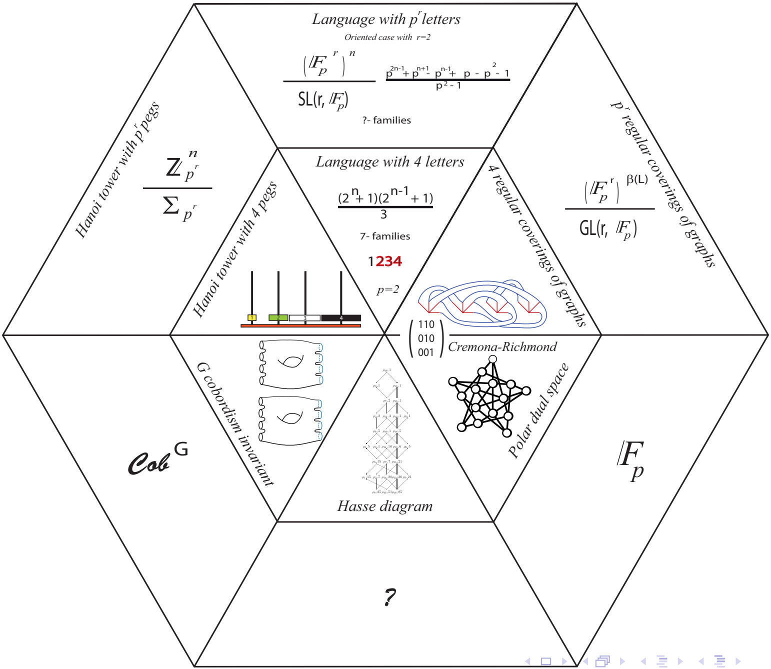
(2) The dimension of the universal embedding of the polar dual space.

(3) The number of non-equivalent states of Hanoi graph H_4^n.

(4) The dimension of the centralized algebra $\mathrm{End}_{H_1}\left(V_{10}^{\otimes k}\right)$ where H_1 is a group of order 96.

(5) The number of isomorphism classes of regular-four-fold-coverings of a graph L with Betti number $n = \beta(L)$ and voltage group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(6) The $\mathbb{Z}_2^n$-cobordism invariant.


Language with p^r letters
Oriented case with r=2
$\frac{(\mathbb{F}_p^r)^n}{SL(r, \mathbb{F}_p)}$
$\frac{p^{2n-1} + p^{n+1} - p^{n-1} + p - p^2 - 1}{p^2 - 1}$
?- families
Hanoi tower with p^r pegs
$\frac{\mathbb{Z}_{p^r}^n}{\Sigma_{p^r}}$
Language with 4 letters
$\frac{(2^n + 1)(2^{n-1} + 1)}{3}$
7- families
1234
p=2
Hanoi tower with 4 pegs
4 regular coverings of graphs
p^r regular coverings of graphs
$\frac{(\mathbb{F}_p^r)^{\beta(L)}}{GL(r, \mathbb{F}_p)}$
110
010
001
Cremona-Richmond
Polar dual space
G cobordism invariant
Cob^G
Hasse diagram
$\mathbb{F}_p$
?

Languages with $2^2 = 4$ letters

Consider the following game: elaborate words $a = a_1 a_2 \cdots a_n$ with four letter $a_i \in \{1, 2, 3, 4\}$ satisfying

$$0 < a_i \leq \max_{j<i}\{a_j\} + 1.$$

Set L^n = the set of words of length n.

Examples:

- $\boxed{n = 2}$ 2 words: 11, 12.
- $\boxed{n = 3}$ 5 words: 111, 112, 121, 122, 123.
- $\boxed{n = 4}$ 15 words:

1111	1112	1121	1122	1123
1211	1212	1213	1221	1222
1223	1231	1232	1233	1234 .

N. Moreira and R. Reis 2005

Density of a language with four letter in degree n $:= |L^n|$.

Language with p^r letters

Set p prime, r positive integer. The division algorithm gives the bijection

$$\phi : \mathbb{Z}_{p^r} \longrightarrow \mathbb{F}_p^r .$$

defined $a \longrightarrow (u_0, u_1, \cdots, u_{r-1})^t$ with a uniquely expressed $a = u_0 + u_1 p + \cdots + u_{r-1} p^{r-1}$.
The set of words of length n with letters in $\mathbb{Z}_{p^r} = \{0, \cdots, p^r - 1\}$ denoted by T_n, has the bijection

$$\Phi : T^n \longrightarrow M(r \times n, \mathbb{F}_p) ,$$

defined by $a_1 a_2 \cdots a_n \longmapsto (\phi(a_1)\phi(a_2) \cdots \phi(a_n))$.
We apply Gauß elimination to produce the Bruhat decomposition

$$\begin{pmatrix} 0 & 1 & * & * & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition

Take letters $\{0, \cdots, p^r - 1\}$, with p prime, r positive integer. Define the language in degree n denoted W_p^n with words $w = a_0 a_1 \cdots a_{n-1}$ such that there exist itegers $0 \leq k_0 < k_1 < \cdots < k_{r-1}$ with rules:

(R1) $a_i = 0$ for $i < k_0$;

(R2) $a_{k_i} = p^i$ for all k_i ;

(R3) $a_j \in \{0, 1, \cdots, p^l - 1\}$ for $k_{l-1} < j < k_l$.

Example (Moreira and Reis)

Consider $p = 2$, the letters $0, 1, 2, 3$,

- $W_2^1 = \{0, 1\}$,
- $W_2^2 = \{00, 01, 10, 11, 12\}$,
- $W_2^3 = \{000, 001, 010, 011, 012, 100, 101, 102,$
  $110, 111, 112, 120, 121, 122, 123\}$.

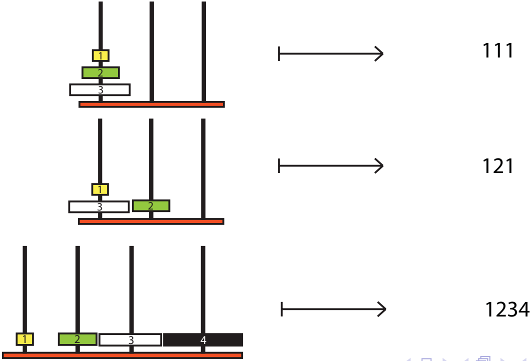
Hanoi graphs

Consider the Hanoi tower with m pegs and n discs.

H_m^n = the graph of states of the tower,

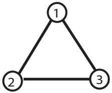
- vertices = states of the tower, with labels $a_1 a_2 \cdots a_n$ where a_i = the position of the disc i.
- edges = Hanoi moves.

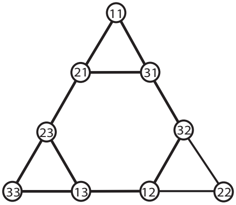
Examples:



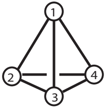
Hanoi graphs

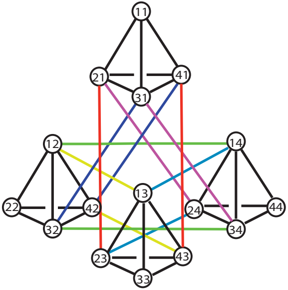
H_3^1 y H_3^2



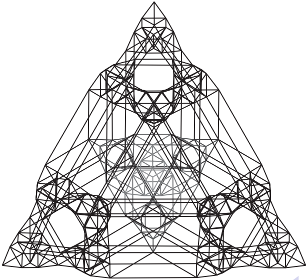


H_4^1 y H_4^2

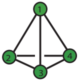




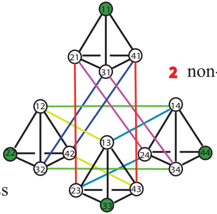
H_4^4



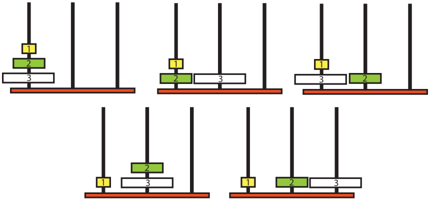
The non-equivalent classes are the quotient by the action of the permutation group Σ_m in the m pegs .



1 non-equivalent class



2 non-equivalent classes



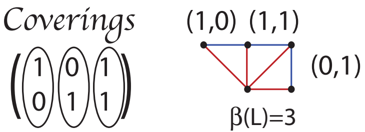
111, 112, 121, 122, 123

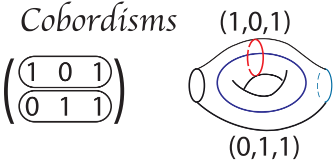
Coverings of graphs and cobordisms

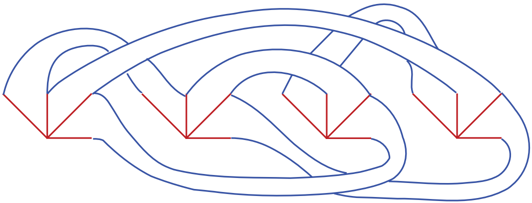
$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$

1 **2** **3** **4**

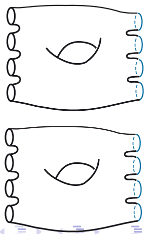
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$







(0,0) (1,0) (0,1) (1,1)



Theorem (Segovia)

The language with p^r letters is defined by the quotient $\mathbb{Z}_{p^r}^n / \mathrm{GL}(r, \mathbb{F}_p)$.

Theorem (Segovia)

The non-equivalence classes of a Hanoi tower $H_{p^r}^n$ is given by the quotient $\mathbb{Z}_{p^r}^n / \Sigma_n$.

Theorem (Hofmeister)

The isomorphism classes of regular coverings over a graph L, with voltage group $\mathbb{Z}_p^r$. is the cardinality of $\mathbb{Z}_{p^r}^{\beta(L)} / \mathrm{GL}(r, \mathbb{F}_p)$.

Theorem (Segovia-Winklmeier)

The $\mathbb{Z}_p^n$-cobordism invariant is the cardinality of $|\mathbb{Z}_{p^2}^n / \mathrm{SL}(2, \mathbb{F}_p)| = \frac{p^{2n-1}+p^{n+1}-p^{n-1}+p^2-p-1}{p^2-1}$.

Dual polar space

Consider a symplectic vector space over $\mathbb{F}_2$ with dimension $2n$.

$$< v, w >:= vJw^t \text{ with } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

The partial linear space $\mathcal{G}_n = (\mathcal{P}_n, \mathcal{L}_n)$:

points $\mathcal{P}_n = \{$ the maximal totally isotropic subspaces$\}$

lines $\mathcal{L}_n = \{$ the totally isotropic subspaces of dimension n-1$\}$

- All the maximal totally isotropic spaces have the same dimension equal to n.
- The incidence is given by inclusion.
- Every line has exactly three points.
- There is a linear map $\theta : \mathbb{F}_2(\mathcal{L}_n) \longrightarrow \mathbb{F}_2(\mathcal{P}_n)$ given by $\{p, q, r\} \longmapsto p + q + r$.

The universal dimension of the polar dual space, denoted by $\dim U(\mathcal{G}_n)$ in degree n, is the dimension of the quotient $\mathbb{F}_2(\mathcal{P}_n)/\operatorname{Im}(\theta)$.

Examples:

For $n = 1$, $\mathcal{P}_1 = \{10, 01, 11\}$ and $\mathcal{L}_1 = \{0\}$.

Hence

$$\dim U(\mathcal{G}_1) = 2\,.$$

Examples:

For $n = 2$, $\mathcal{P}_2$ is

$$A\begin{pmatrix} 0001 \\ 0010 \end{pmatrix} B\begin{pmatrix} 0001 \\ 1000 \end{pmatrix} C\begin{pmatrix} 0001 \\ 1010 \end{pmatrix} D\begin{pmatrix} 0010 \\ 0100 \end{pmatrix} E\begin{pmatrix} 0010 \\ 0101 \end{pmatrix}$$
$$F\begin{pmatrix} 0100 \\ 1000 \end{pmatrix} G\begin{pmatrix} 0100 \\ 1010 \end{pmatrix} H\begin{pmatrix} 0101 \\ 1000 \end{pmatrix} I\begin{pmatrix} 0101 \\ 1010 \end{pmatrix} J\begin{pmatrix} 0110 \\ 1001 \end{pmatrix}$$
$$K\begin{pmatrix} 0011 \\ 1100 \end{pmatrix} L\begin{pmatrix} 0011 \\ 1101 \end{pmatrix} M\begin{pmatrix} 0110 \\ 1011 \end{pmatrix} N\begin{pmatrix} 0111 \\ 1011 \end{pmatrix} O\begin{pmatrix} 0111 \\ 1001 \end{pmatrix}$$

and $\mathcal{L}_2$ is

A-B-C	J-D-M	C-M-N
A-K-L	E-O-N	F-N-K
D-A-E	B-H-F	M-H-L
D-G-F	J-B-O	G-O-L
E-I-H	C-G-I	J-I-K

Examples:

For $n = 2$, $\mathcal{P}_2$ is

$$
A\begin{pmatrix} \mathbf{0001} \\ 0010 \end{pmatrix}
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C\begin{pmatrix} \mathbf{0001} \\ 1010 \end{pmatrix}
D\begin{pmatrix} 0010 \\ 0100 \end{pmatrix}
E\begin{pmatrix} 0010 \\ 0101 \end{pmatrix}
$$
$$
F\begin{pmatrix} 0100 \\ 1000 \end{pmatrix}
G\begin{pmatrix} 0100 \\ 1010 \end{pmatrix}
H\begin{pmatrix} 0101 \\ 1000 \end{pmatrix}
I\begin{pmatrix} 0101 \\ 1010 \end{pmatrix}
J\begin{pmatrix} 0110 \\ 1001 \end{pmatrix}
$$
$$
K\begin{pmatrix} 0011 \\ 1100 \end{pmatrix}
L\begin{pmatrix} 0011 \\ 1101 \end{pmatrix}
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Examples:

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Examples:

For $n = 2$, $\mathcal{P}_2$ is

$$
A\begin{pmatrix} 0001 \\ \mathbf{0010} \end{pmatrix} B\begin{pmatrix} 0001 \\ 1000 \end{pmatrix} C\begin{pmatrix} 0001 \\ 1010 \end{pmatrix} D\begin{pmatrix} \mathbf{0010} \\ 0100 \end{pmatrix} E\begin{pmatrix} \mathbf{0010} \\ 0101 \end{pmatrix}
$$

$$
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and $\mathcal{L}_2$ is

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$$F\begin{pmatrix} 0100 \\ 1000 \end{pmatrix} G\begin{pmatrix} 0100 \\ 1010 \end{pmatrix} H\begin{pmatrix} \mathbf{0101} \\ 1000 \end{pmatrix} I\begin{pmatrix} \mathbf{0101} \\ 1010 \end{pmatrix} J\begin{pmatrix} 0110 \\ 1001 \end{pmatrix}$$
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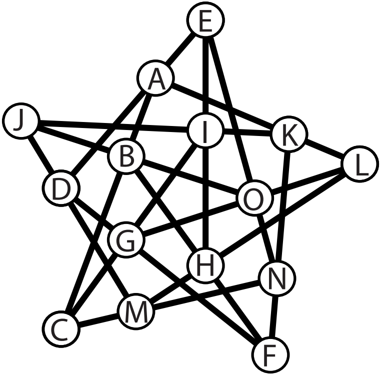
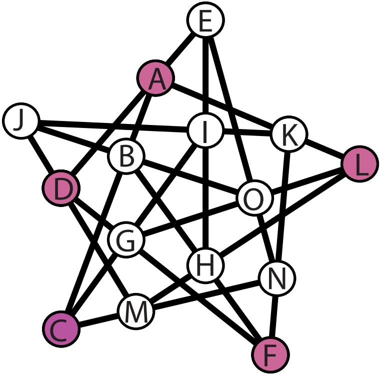
D-A-E	B-H-F	M-H-L
D-G-F	J-B-O	G-O-L
E-I-H	C-G-I	J-I-K

The *Cremona-Richmond* configuration



$\dim U(\mathcal{G}_2) = 5$



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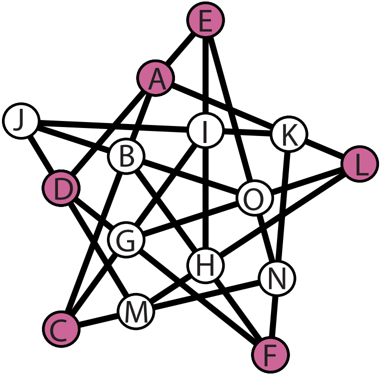

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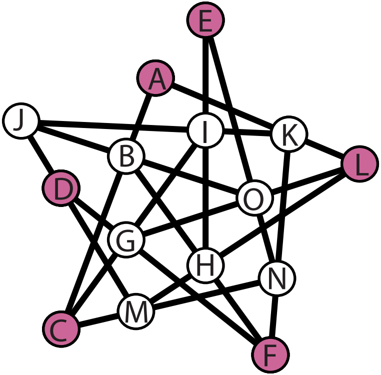

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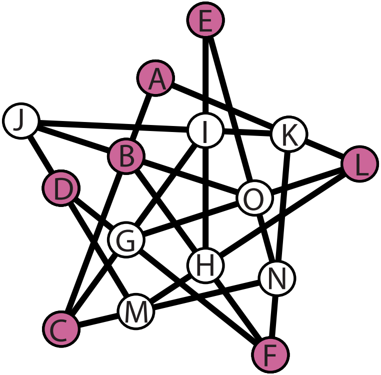

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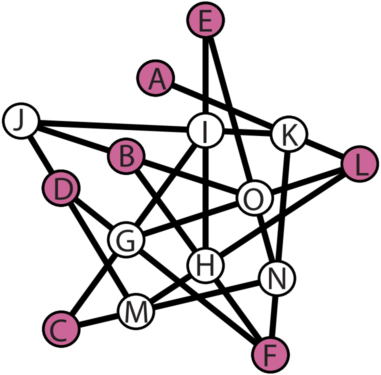

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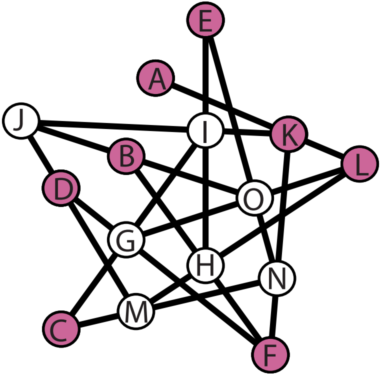
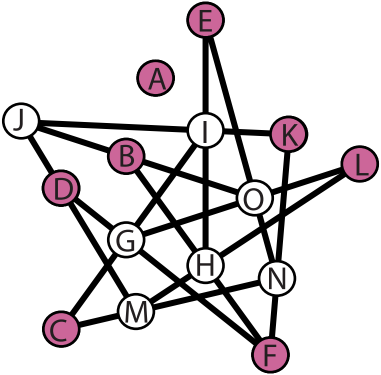


dim $U(\mathcal{G}_2) = 5$



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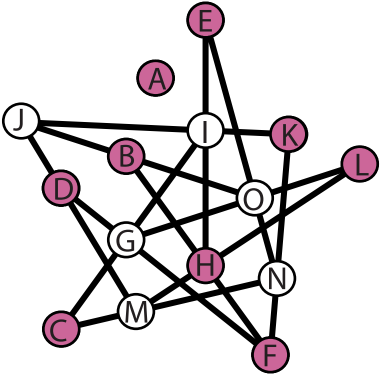

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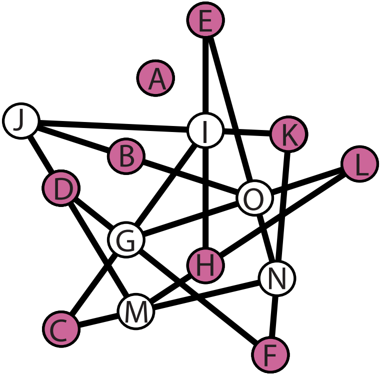

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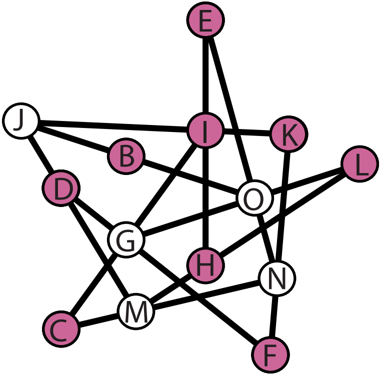

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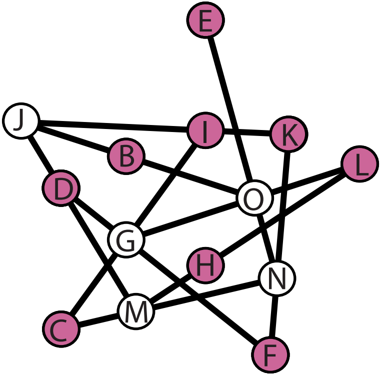

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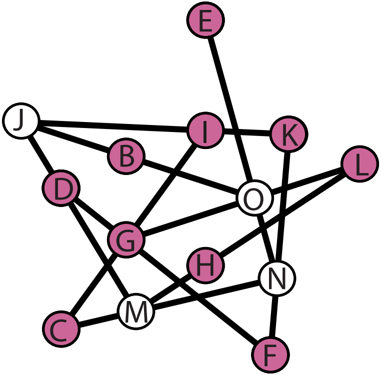

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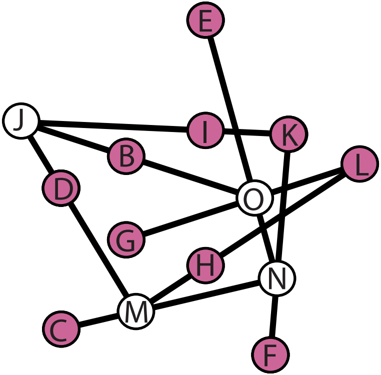

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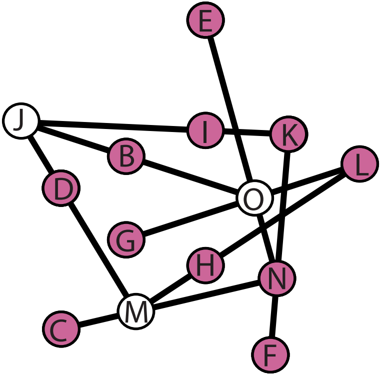

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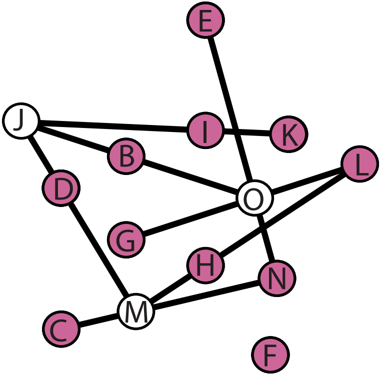
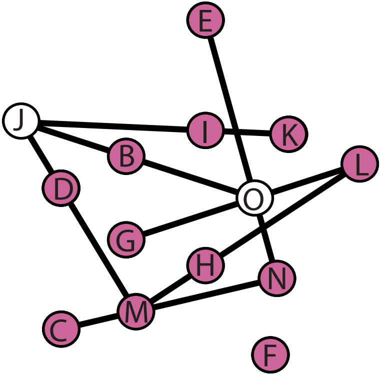
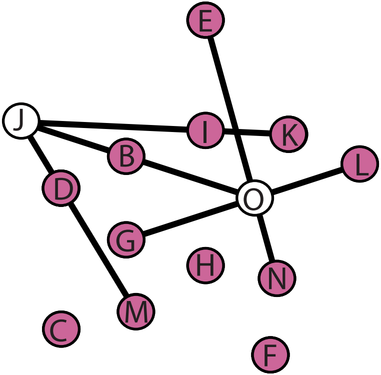


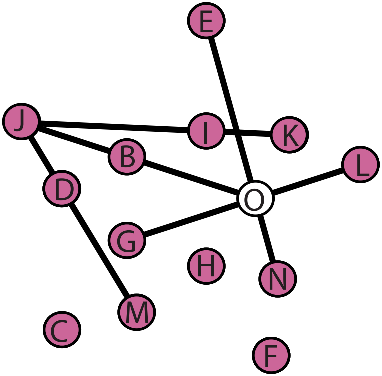
dim $U(\mathcal{G}_2) = 5$



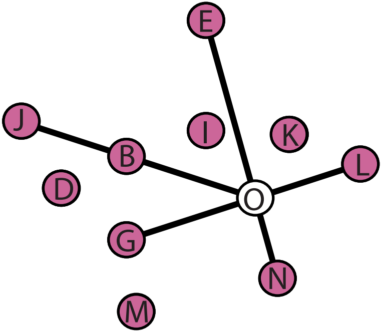
$\dim U(\mathcal{G}_2) = 5$



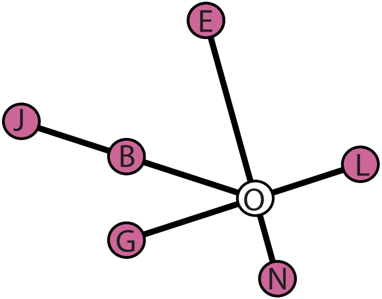
$\dim U(\mathcal{G}_2) = 5$



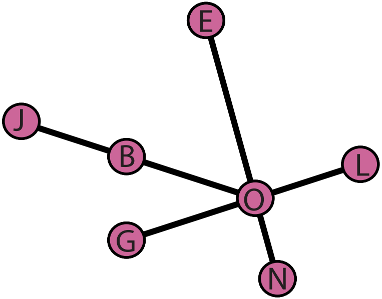
$\dim U(\mathcal{G}_2) = 5$



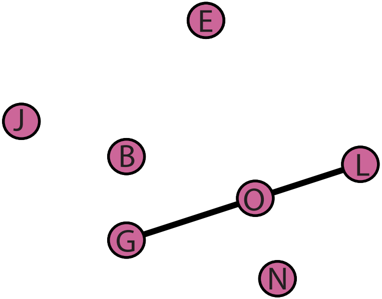
dim $U(\mathcal{G}_2) = 5$



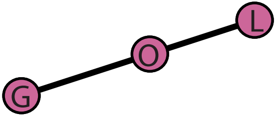
$\dim U(\mathcal{G}_2) = 5$



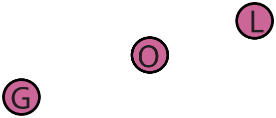
$\dim U(\mathcal{G}_2) = 5$



$\dim U(\mathcal{G}_2) = 5$

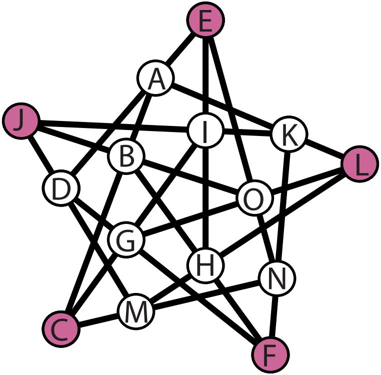


$\dim U(\mathcal{G}_2) = 5$



$\dim U(\mathcal{G}_2) = 5$

Counterexample



Languages and polar dual spaces

- $n = 2$

$$\mathbf{1}1, \mathbf{1}2\,.$$

- $n = 3$

$$\mathbf{1}11, \mathbf{1}12, \mathbf{1}21, \mathbf{1}22, \mathbf{1}23\,.$$

- $n = 4$

$$\begin{array}{lllll} \mathbf{1}111 & \mathbf{1}112 & \mathbf{1}121 & \mathbf{1}122 & \mathbf{1}123 \\ \mathbf{1}211 & \mathbf{1}212 & \mathbf{1}213 & \mathbf{1}221 & \mathbf{1}222 \\ \mathbf{1}223 & \mathbf{1}231 & \mathbf{1}232 & \mathbf{1}233 & \mathbf{1}234\,. \end{array}$$

Languages and polar dual spaces

- $n = 2$

$$0, 1\,.$$

- $n = 3$

$$00, 01, 10, 11, 12\,.$$

- $n = 4$

$$\begin{array}{ccccc} 000 & 001 & 010 & 011 & 012 \\ 100 & 101 & 102 & 110 & 111 \\ 112 & 120 & 121 & 122 & 123\,. \end{array}$$

Languages and polar dual spaces

- $n = 2$

$$0, 1\,.$$

- $n = 3$

$$00, 01, 10, 11, 12\,.$$

- $n = 4$

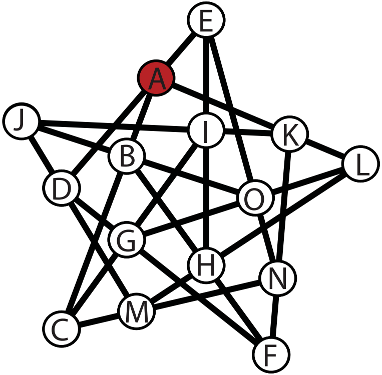
000	001	010	011	012
100	101	102	110	111
112	120	121	122	123 .

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
00 10	01 12	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	11

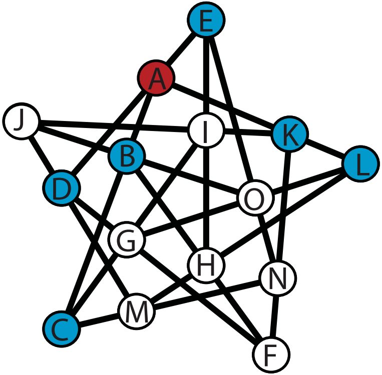
Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
000 010 100 110 120	001 012 102 112 123	101	111	121	$\emptyset$	011 122

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
000**0**	000**1**	10**0**1	10**1**1	10**2**1	12**3**1	**0**011
001**0**	001**2**	12**0**1	12**1**1	12**2**1	12**3**2	01**22**
010**0**	010**2**	01**0**1	01**1**1	01**2**1		10**22**
011**0**	011**2**	11**0**1	11**1**1	11**2**1		11**22**
012**0**	012**3**	12**0**2	12**1**2	12**2**2		
100**0**	100**2**					
101**0**	101**2**					
102**0**	102**3**					
110**0**	110**2**					
111**0**	111**2**					
112**0**	112**3**					
120**0**	120**3**					
121**0**	121**3**					
122**0**	122**3**					
123**0**	123**3**					

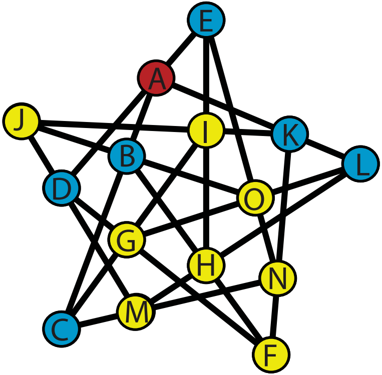
Take $x_0 = A$



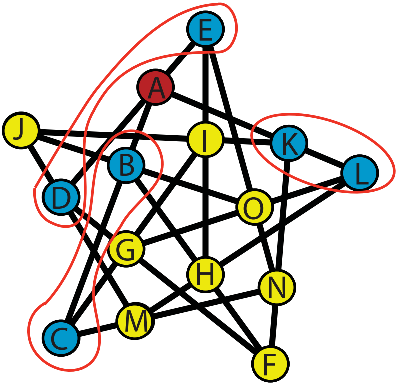
Take $x_0 = A$



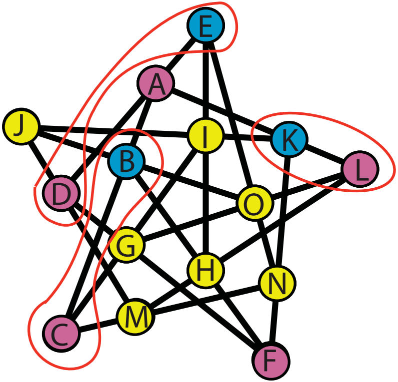
Take $x_0 = A$



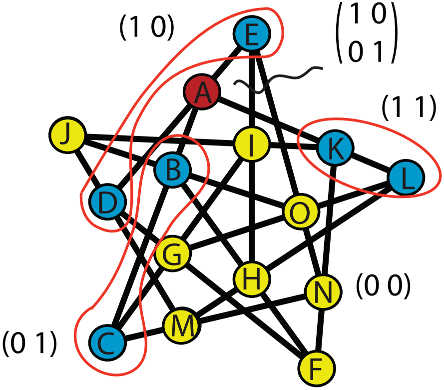
Take $x_0 = A$



Take $x_0 = A$



Case 1	Case 2	Case 3, 4, 5, 6	Case 7
0**0** (0 0)	0**1** (0 1)	$\emptyset$	11 (1 1)
1**0** (1 0)	1**2** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		



$$234 \longmapsto \begin{array}{ccc} 1 & 1 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ & & \mathbf{1} \end{array} \longmapsto \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}$$

$$23 \longmapsto \begin{array}{cc} 1 & \mathbf{0} \\ 0 & \mathbf{1} \end{array} \longmapsto 1$$

$$2 \longmapsto 1$$

Theorem (Segovia)

There is a bijection between the elements of the language with four letters and the Li sub-spaces from the universal embedding of the polar dual space.

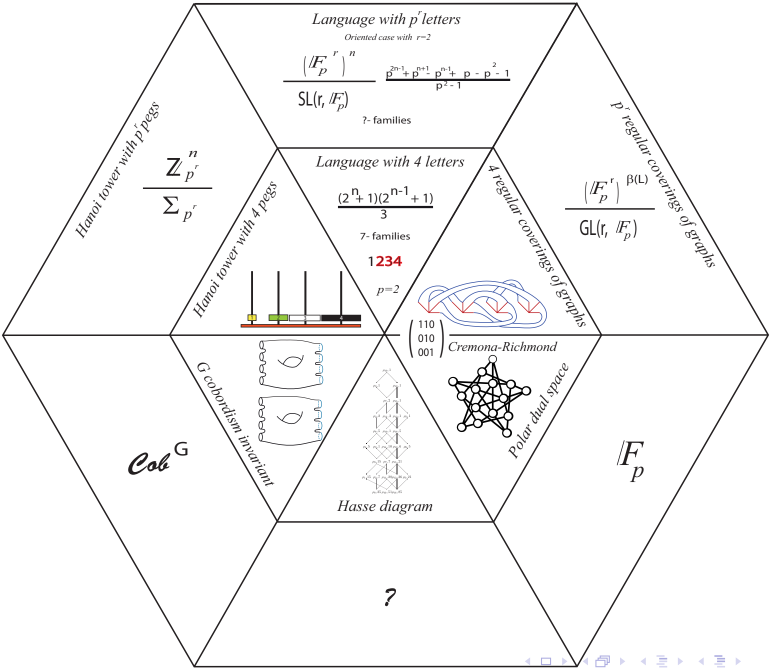
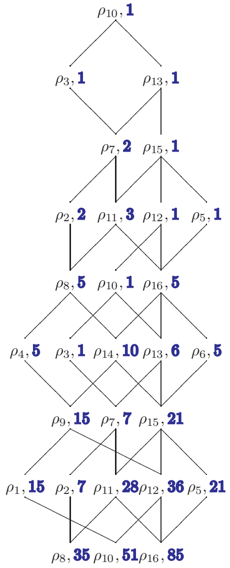

Language with p^r letters
Oriented case with r=2
SL(r, F_p)
p^{2n-1} + p^{n+1} - p^{n-1} + p - p^2 - 1 / p^2 - 1
?- families
Hanoi tower with p^r pegs
Z_{p^r}^n / Σ_{p^r}
p^r regular coverings of graphs
(F_p^r)^{β(L)} / GL(r, F_p)
Language with 4 letters
(2^n + 1)(2^{n-1} + 1) / 3
7- families
1234
p=2
Hanoi tower with 4 pegs
4 regular coverings of graphs
110
010
001
Cremona-Richmond
Polar dual space
G cobordism invariant
Cob^G
Hasse diagram
F_p
?

Diagrama de Hasse



$1^2 = \mathbf{1}$

$1^2 + 1^2 = \mathbf{2}$

$2^2 + 1^2 = \mathbf{5}$

$2^2 + 3^2 + 1^2 + 1^2 = \mathbf{15}$

$5^2 + 1^2 + 5^2 = \mathbf{51}$

$5^2 + 1^2 + 10^2 + 6^2 + 5^2 = \mathbf{187}$

$15^2 + 7^2 + 21^2 = \mathbf{715}$

$15^2 + 7^2 + 28^2 + 36^2 + 21^2 = \mathbf{2795}$

$35^2 + 51^2 + 85^2 = \mathbf{11051}$

$H_1 \leq U_2$ generated

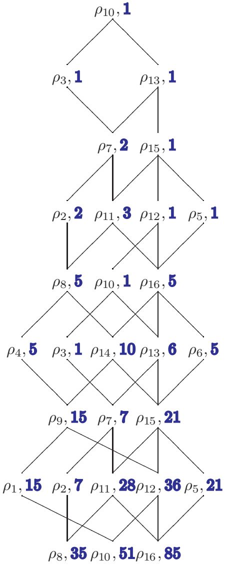
$$T = \frac{1+i}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and ρ_{10} the natural representation. We take the Hasse diagram of the decomposition of $(\rho_{10}^k, V_{10}^{\otimes k})$ in irreducible representations. The centralized algebra of H_1 in $V_{10}^{\otimes k}$, where H_1 acts diagonally is

$$\mathrm{End}_{H_1}(V_{10}^{\otimes k}) := \left\{ f : V_{10}^{\otimes k} \longrightarrow V_{10}^{\otimes k} \;\; H_1 - \text{lineal maps} \right\}$$

then $f(h \cdot v) = h \cdot f(v)$.
The dimension of this algebra is the values of the sequence.

Gracias



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