Unexpected relations of cobordism categories with another subjects

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The sequence $1, 2, 5, 15, 51, 187, 715, 2795, 11051, 43947, \cdots$ with the form:

$$g(n) = \frac{(2^n + 1)(2^{n-1} + 1)}{3}.$$

This sequence have the number A007581 in the webpage *The On-Line Encyclopedia of Integer Sequences* and have the following interpretations:

(1) The density of a language with four letters.

(2) The dimension of the universal embedding of the dual polar space.

(3) The number of isomorphy classes of regular fourfold covering of a graph L with Betti number $n = \beta(L)$ and with voltage group $\mathbb{F}_2 \times \mathbb{F}_2$.

(4) The rank of the fundamental group of the classifying space of the $\mathbb{F}_2^n$-cobordism category in dimension 1+1.

Languages

Consider the following game: We form words ($a = a_1 a_2 \cdots a_m$) made with letters $a_i \in \{1, 2, 3, 4\}$ satisfying the property

$$0 < a_i \leq \max_{j<i}\{a_j\} + 1$$

Thus $a_1 = 1$ and we are not going to write it. Let L^n be the set of words of length n. For $n = 1$ there are two words 1 and 2, for $n = 2$ the words are 11, 12, 21, 22, 23, while for $n = 3$ we have 15 words

$$\begin{array}{ccccc} 111 & 112 & 121 & 122 & 123 \\ 211 & 212 & 213 & 221 & 222 \\ 223 & 231 & 232 & 233 & 234\,. \end{array}$$

The density of a language with four letters in degree n is the number of elements of L^n. Nelma Moreira and Rogério Reis consider partition of a set of n elements in subsets.

For $n = 2$,

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
1**1**	1**2**	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	22
2**1**	2**3**					

For $n = 3$,

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
11**1**	11**2**	2**1**2	2**2**2	2**3**2	$\emptyset$	**1**22
12**1**	12**3**					2**33**
21**1**	21**3**					
22**1**	22**3**					
23**1**	23**4**					

For $n = 4$,

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
111**1**	111**2**	21**1**2	21**2**2	21**3**2	23**4**2	**1**122
112**1**	112**3**	23**1**2	23**2**2	23**3**2	23**4**3	12**33**
121**1**	121**3**	12**1**2	12**2**2	12**3**2		21**33**
122**1**	122**3**	22**1**2	22**2**2	22**3**2		22**33**
123**1**	123**4**	23**1**3	23**2**3	23**3**3		
211**1**	211**3**					
212**1**	212**3**					
213**1**	213**4**					
221**1**	221**3**					
222**1**	222**3**					
223**1**	223**4**					
231**1**	231**4**					
232**1**	232**4**					
233**1**	233**4**					
234**1**	234**4**					

Let $L^n(j)$ the words in L^n of the case $j \in \{1, ..., 7\}$ and we denote by $L^n(1,2) = L^n - (L^n(1) \cup L^n(2))$, we have bijections:

1. $A_1 : L^{n-1} \longrightarrow L^n(1)$, $A_2 : L^{n-1} \longrightarrow L^n(2)$.
2. $A_j : L^{n-1}(1,2) \longrightarrow L^n(j)$ $(j = 3, 4, 5)$.
3. $A_6 : L^{n-1}(1,2) \longrightarrow (L^n(6) \cup L^n(7)) - \{11 \cdots 122\}$,

$$a = a' a_{n-1} \longmapsto \begin{cases} a' 4 a_{n-1} \in L^n(6) & \text{if } a' \text{ has a } 3 \\ a' 33 \in L^n(7) & \text{else.} \end{cases}$$

Thus $|L^n(1)| = |L^n(2)| = g(n-1)$, $|L^n(j)| = g(n-1) - 2g(n-2)$ $(j = 3, 4, 5)$ and $|L^n(6)| + |L^n(7)| = g(n-1) - 2g(n-2) + 1$
Therefore,

$$\begin{aligned} |L^n| =& 2g(n-1) + 4\left(g(n-1) - 2g(n-2)\right) + 1 = \\ =& 6g(n-1) - 8g(n-2) + 1 \\ =& \frac{(2^n+1)(2^{n-1}+1)}{3}. \end{aligned}$$

Dual polar space

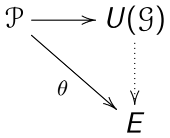
Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a partial linear vector space ($\mathcal{P}$ points and $\mathcal{L}$ lines). Hence they satisfy:

- any line is at least incident with two points, and
- any pair of distinct points is incident with at most one line.

We assume that every line contain exactly three different points. An *embedding* of $\mathcal{G}$ in an $\mathbb{F}_2$-vector space E is a function $\theta : \mathcal{P} \longrightarrow E$ such that:

- E is spanned by the image of θ, i.e. $E = \langle \mathcal{P}^\theta \rangle$; and
- for every line $\{p, q, r\} \in \mathcal{L}$, the vectors p^θ, q^θ, r^θ form a projective line in E, i.e. $p^\theta + q^\theta + r^\theta = 0$.

The universal embedding satisfies



"largest"

$\forall \theta : \mathcal{P} \longrightarrow E$

The $Sp_{2n}(2)$ dual polar space is the partial linear space $\mathcal{G}_n = (\mathcal{P}_n, \mathcal{L}_n)$ constructed from a $2n$-dimensional nondegenerate symplectic space over $\mathbb{F}_2$ with:

- $\mathcal{P}_n = \{$ the maximal totally isotropic subspaces$\}$,
- $\mathcal{L}_n = \{$ the totally isotropic subspaces of dimension n-1$\}$,
- the incidence is given by inclusion.

This is a partial linear space with lines with exactly three points since:

- every maximal totally isotropic subspace has dimension n, and
- every totally isotropic subspace of dimension $n-1$ is contained in exactly three maximal totally isotropic subspaces.

The dimension of the universal embedding of the dual polar space is the dimension of $U(\mathcal{G}_n)$. A. E. Brouwer produce an embedding of $\mathcal{G}_n$ of dimension exactly $g(n) = (2^n + 1)(2^{n-1} + 1)/3$. Thus

$$\dim U(\mathcal{G}_n) \geqslant g(n) .$$

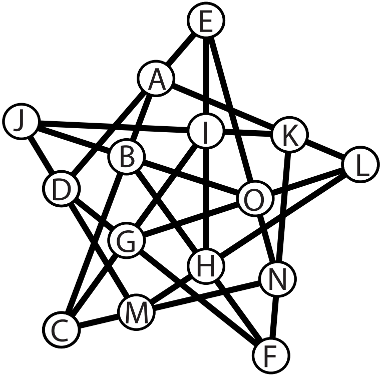
The Brouwer's conjecture state the equality $\dim U(\mathcal{G}_n) = g(n)$, so in order to prove this conjecture, it suffice to show that $U(\mathcal{G}_n)$ is spanned by $g(n)$ vectors. For $n = 1$, $\mathcal{P}_1 = \{10, 01, 11\}$ and $\mathcal{L}_1 = \{0\}$. Hence $\dim U(\mathcal{G}_1) = 2$.
For $n = 2$, $\mathcal{P}_2$ is

$$
\text{A}\begin{pmatrix} 0001 \\ 0010 \end{pmatrix}\text{B}\begin{pmatrix} 0001 \\ 1000 \end{pmatrix}\text{C}\begin{pmatrix} 0001 \\ 1010 \end{pmatrix}\text{D}\begin{pmatrix} 0010 \\ 0100 \end{pmatrix}\text{E}\begin{pmatrix} 0010 \\ 0101 \end{pmatrix}
$$

$$
\text{F}\begin{pmatrix} 0100 \\ 1000 \end{pmatrix}\text{G}\begin{pmatrix} 0100 \\ 1010 \end{pmatrix}\text{H}\begin{pmatrix} 0101 \\ 1000 \end{pmatrix}\text{I}\begin{pmatrix} 0101 \\ 1010 \end{pmatrix}\text{J}\begin{pmatrix} 0110 \\ 1001 \end{pmatrix}
$$

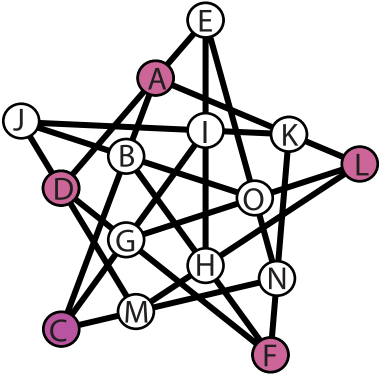
$$
\text{K}\begin{pmatrix} 0011 \\ 1100 \end{pmatrix}\text{L}\begin{pmatrix} 0011 \\ 1101 \end{pmatrix}\text{M}\begin{pmatrix} 0110 \\ 1011 \end{pmatrix}\text{N}\begin{pmatrix} 0111 \\ 1011 \end{pmatrix}\text{O}\begin{pmatrix} 0111 \\ 1001 \end{pmatrix}
$$

They form the *Cremona-Richmond* configuration



A-B-C	J-D-M	C-M-N
A-K-L	E-O-N	F-N-K
D-A-E	B-H-F	M-H-L
D-G-F	J-B-O	G-O-L
E-I-H	C-G-I	J-I-K

$\dim U(\mathcal{G}_2) = 5$



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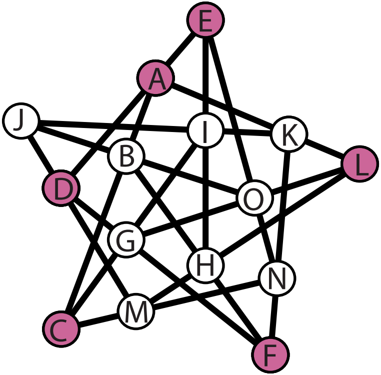

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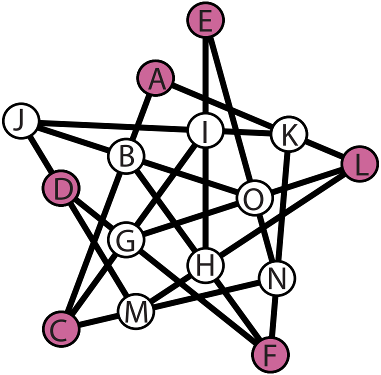

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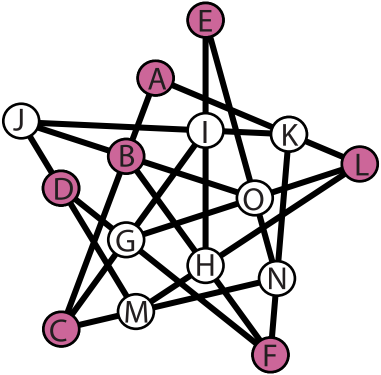

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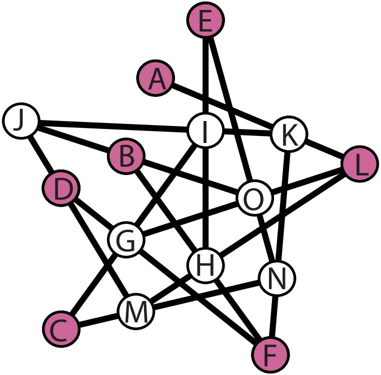

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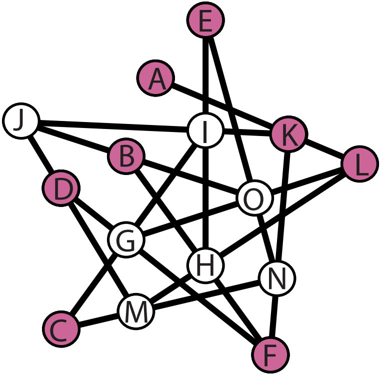

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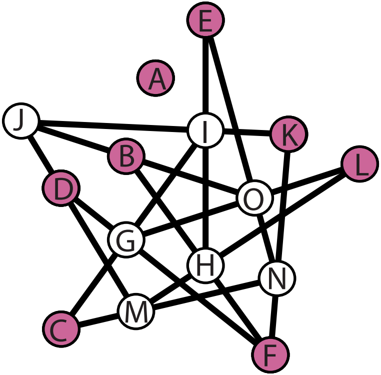

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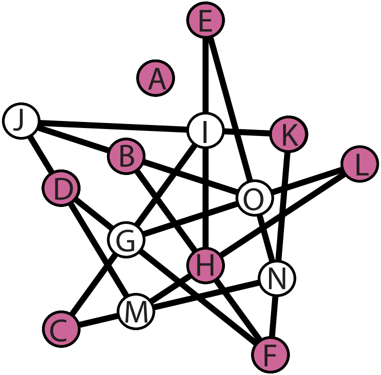

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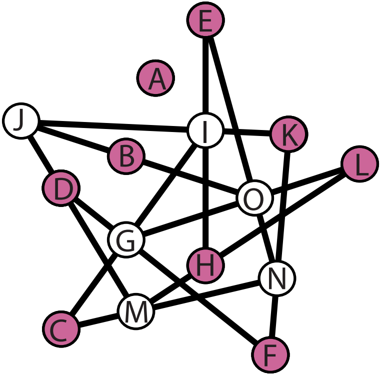

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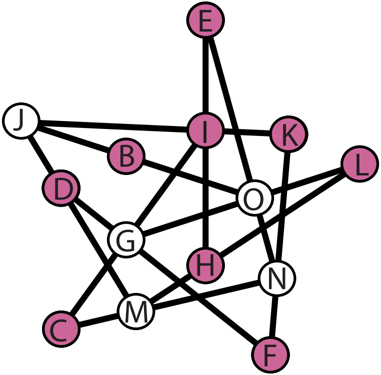

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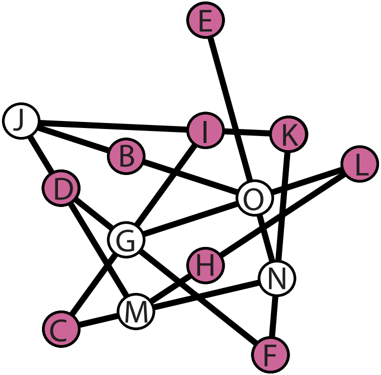

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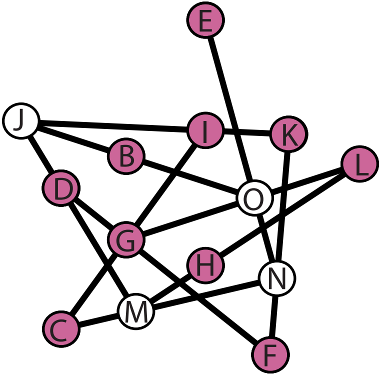

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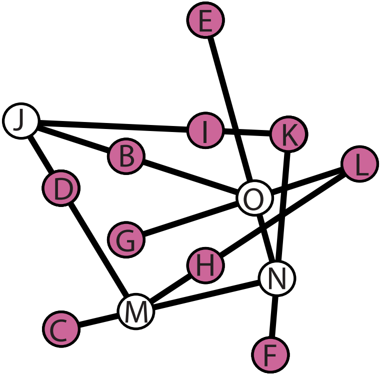

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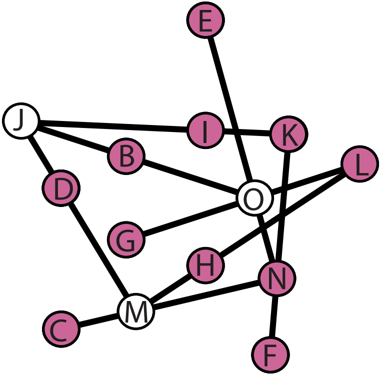

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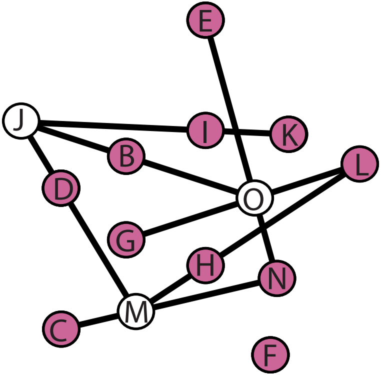

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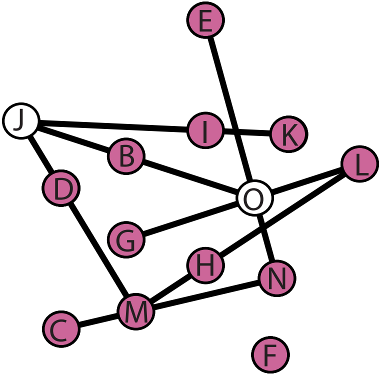

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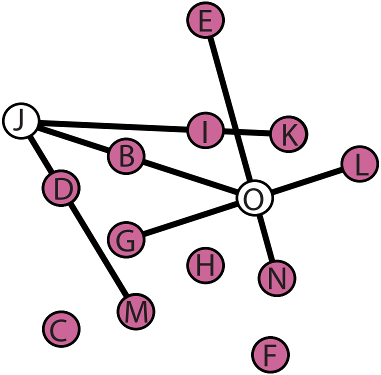

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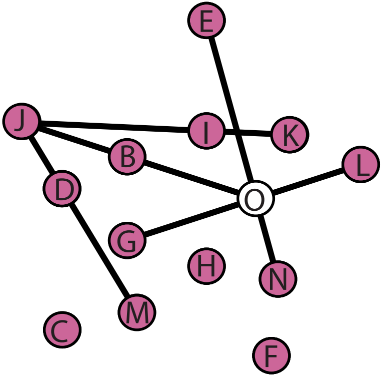

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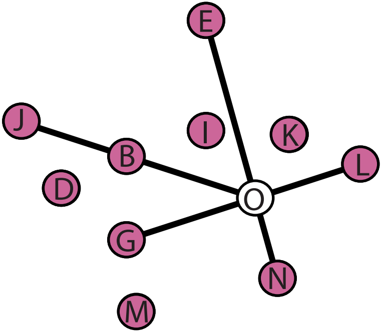

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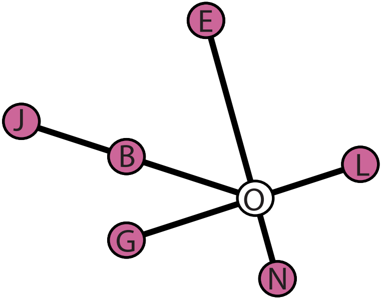

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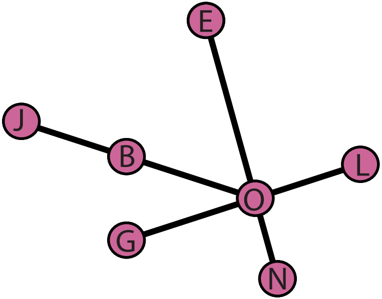

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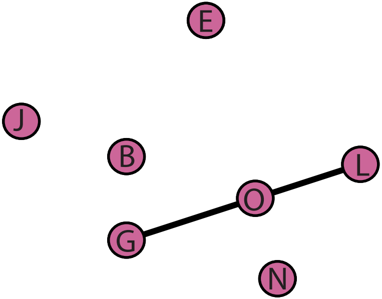

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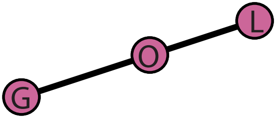

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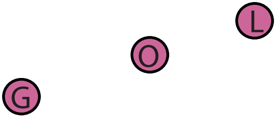
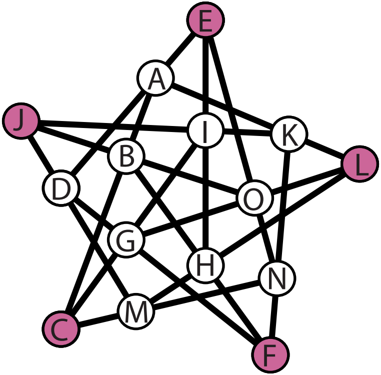


$\dim U(\mathcal{G}_2) = 5$

Counterexample



To find a set of generators, for $n \geq 3$, this could be a difficult problem. The number of maximal isotropic spaces is given by the formula

$$(2^n + 1)(2^{n-1} + 1) \cdots (2^2 + 1)(2 + 1) = 3, 15, 135, 2295, \cdots$$

Paul Li proposes a strategy:
Let Γ the collinearity graph of $\mathcal{G}_n$. Fix a vertex $x_0 \in \Gamma$ and let Γ_k $(0 \leqslant k \leqslant n)$ denote the set of vertices at distance k from x_0. Then $y \in \Gamma_k$ if and only if $\dim(y \cap x_0) = n - k$. The geometry of $\mathcal{G}_n$ has the following properties:

- every line of $\mathcal{G}_n$ contains two elements from Γ_k and one from Γ_{k-1}, for some $1 \leqslant k \leqslant n$;
- every point $y \in \Gamma_{k-1}$ is adjacent to at least a point in Γ_k, for $1 \leqslant k \leqslant n$.

Therefore we have a filtration

$$0 \leqslant \langle \Gamma_0^\theta \rangle \leqslant \cdots \leqslant \langle \Gamma_n^\theta \rangle = U(\mathcal{G}_n)\,,$$

We say that two elements $p, q \in \Gamma_k$ are connected if $p \cap x_0 = q \cap x_0$, where $x_0 \in \Gamma$ is the fixed point. Therefore, the connected components of Γ_k correspond to the $n-k$-subspaces of x_0.

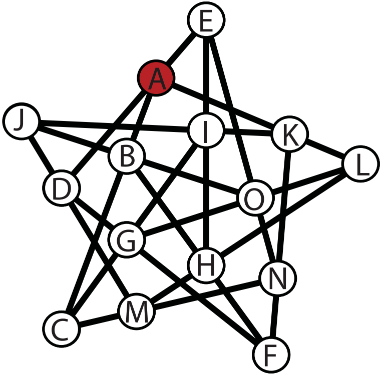
Since Γ_0, Γ_1, and Γ_n, respectively have 1, $2^n - 1$. and 1 connected components, we obtain

$$\dim U(\mathcal{G}_n) \leqslant 1 + (2^n - 1) + \sum_{k=1}^{n-2} \dim \frac{\langle \Gamma^{\theta}_{k+1} \rangle}{\langle \Gamma^{\theta}_{k} \rangle} + 1\,.$$

Theorem

For the universal embedding $U(\mathcal{G}_n)$. There exists a subset $\mathcal{N}^n$ of subspaces of x_0, such that for a subset $T^n \subset \mathcal{P}_n$ with the map $y \in T \longmapsto y \cap x_0$ a bijection restricted to $\mathcal{N}^n$. Therefore, the set T^n is a basis for $U(\mathcal{G}_n)$.

For example for $n = 2$ we take $x_0 = A$



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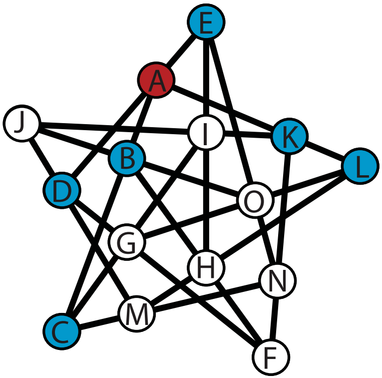

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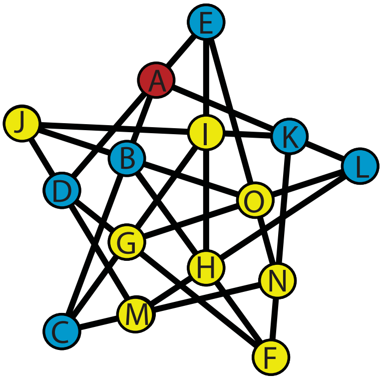

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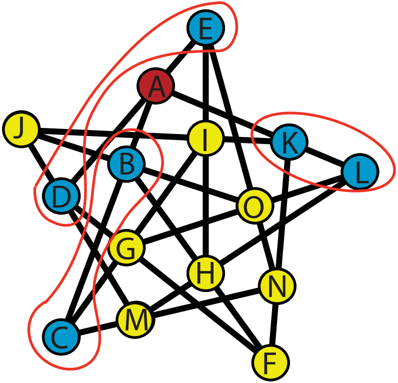

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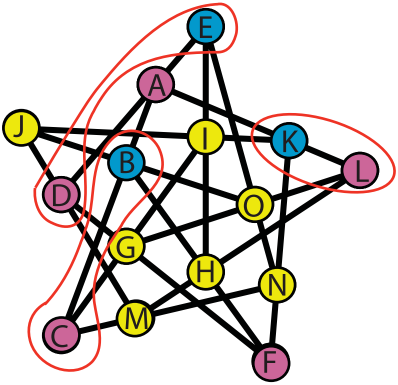


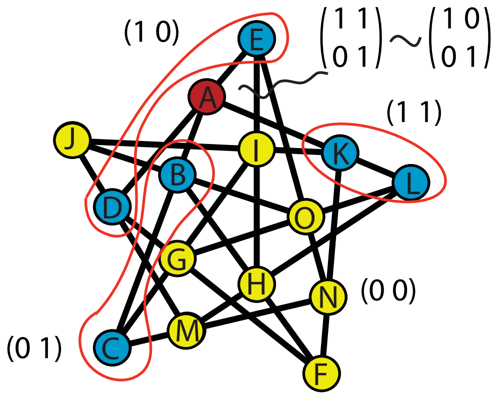
Properties of $\mathcal{N}^n$:

- the cardinality is exactly the number $g(n) = (2^n + 1)(2^{n-1} + 1)/3$;
- the set is participated in seven families which are denoted $\mathcal{N}^n(i)$ for $i = 1, 2, ..., 7$;
- there are bijections $E_j : \mathcal{N}^n(j) \longrightarrow \mathcal{N}^{n-1}$ for $j = 1, 2$;
- there are bijections $E_j : \mathcal{N}^n(j) \longrightarrow \mathcal{N}^{n-1}(1, 2)$ for $j = 3, 4, 5$;
- there is a bijection $E_6 : (\mathcal{N}^n(6) \cup \mathcal{N}^n(7)) - \langle 00 \cdots 011 \rangle \longrightarrow \mathcal{N}^{n-1}(1, 2)$;
- inductively we can define bijections with the languages where

$$
\begin{array}{ccc}
\mathcal{N}^n(i) & \xrightarrow{F_n} & L^n(i) \\
{\scriptstyle E_i}\downarrow & & \uparrow{\scriptstyle A_i} \\
\mathcal{N}^{n-1} & \xrightarrow[F_{n-1}]{} & L^{n-1}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{N}^n(j) & \xrightarrow{F_n} & L^n(j) \\
{\scriptstyle E_j}\downarrow & & \uparrow{\scriptstyle A_j} \\
\mathcal{N}^{n-1}(1,2) & \xrightarrow[F_{n-1}]{} & L^{n-1}(1,2)
\end{array}
$$

for $i = 1, 2$, $j = 3, 4, 5$ and similarly for the case 6 with $\langle 00 \cdots 011 \rangle \longmapsto 11 \cdots 122$.

Case 1	Case 2	Case 3, 4, 5, 6	Case 7
1**1** (0 0)	1**2** (0 1)	$\emptyset$	22 (1 1)
2**1** (1 0)	2**3** $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$		



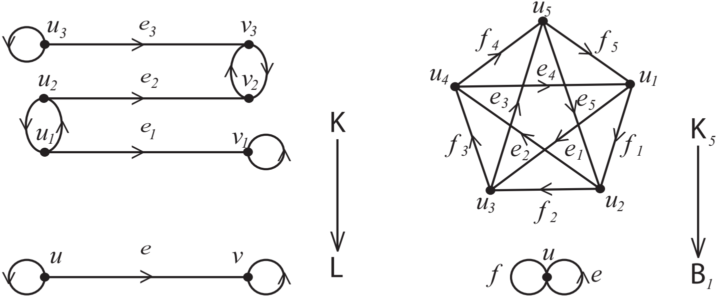
Case 1	Case 2	Case 3	Case 4	Case 5	Case 7
11**1** (000)	11**2** (001)	2**1**2 (101)	2**2**2 $\begin{pmatrix}110\\011\end{pmatrix}$	2**3**2 $\begin{pmatrix}101\\010\end{pmatrix}$	**1**22 (011)
12**1** (010)	12**3** $\begin{pmatrix}010\\001\end{pmatrix}$				2**33** $\begin{pmatrix}100\\011\end{pmatrix}$
21**1** (100)	21**3** $\begin{pmatrix}100\\001\end{pmatrix}$				
22**1** (110)	22**3** $\begin{pmatrix}110\\001\end{pmatrix}$				
23**1** $\begin{pmatrix}110\\010\end{pmatrix}$	23**4** $\begin{pmatrix}110\\010\\001\end{pmatrix}$				

Coverings of graphs

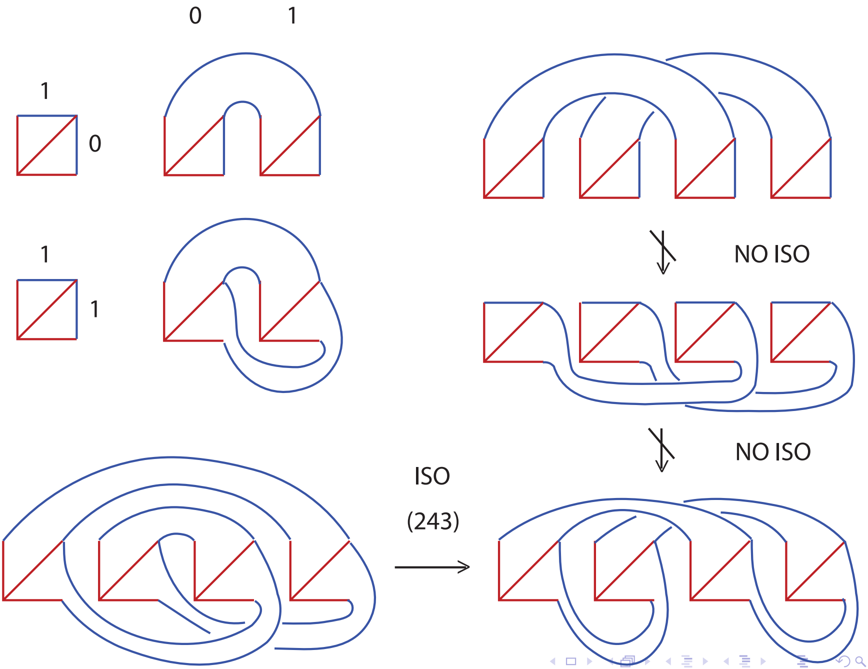
We take $L = (V(L), D(L))$ a (directed) graph where $V(L)$ are the vertices and $D(L)$ the edges.
A covering of L,

$$K \longrightarrow L$$

is a surjection $p : V(K) \longrightarrow V(L)$ such that
$p|_{N(\widetilde{v})} : N(\widetilde{v}) \longrightarrow N(v)$ is a bijection for all v and $\widetilde{v} \in p^{-1}(v)$.
Where $N(v)$ is the neighborhood of v, i.e. the set of vertices adjacent to v.



- K is an n-fold covering of L if the projection is n to 1;
- a covering $p : K \longrightarrow L$ is regular if there is a group G which acts free and transitively over K, i.e. there is monomorphism of groups $G \longrightarrow \mathrm{Aut}(K)$ and the quotient K/G is isomorphic to L;
- two covering graphs $p_1 : K_1 \longrightarrow L$ and $p_2 : K_2 \longrightarrow L$ are isomorphic if there is a graph isomorphism $\Phi : K_1 \longrightarrow K_2$ such that $p_1 = p_2\Phi$.
- every regular covering of a graph L can be constructed through a voltage map $\phi : D(L) \longrightarrow G$ with G a finite group called the voltage group (Gross and Tucker);
- A voltage map has a graph associated, called the voltage graph, which we denote by $L \times_\phi G$ and which vertex set is $V(L) \times G$ and an edge joints a vertex (u, g) to $(v, \phi(uv)g)$.


0
1
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NO ISO
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(243)

Theorem

Let G be an abelian group. Any two n-fold voltage coverings $L \times_\phi G$ and $L \times_\psi G$ are isomorphic if only if there exists a permutation f in S_n such that

$$\psi(uv) = f(v)\,(\phi(uv) + g) - f(u)(g)\,.$$

From here we suppose $G = \mathbb{F}_p^r$ the r-dimensional vector space over the field $\mathbb{F}_p$, where p is a prime number. Let $\mathrm{Isom}(L, \mathbb{F}_p^r)$ be the set of isomorphism regular graph covering with voltage group $\mathbb{F}_p^r$. Now we show the isomorphism

$$\mathrm{Isom}(L, \mathbb{F}_p^r) \cong (\mathbb{F}_p^r)^{\beta(L)} / \, \mathrm{Gl}_r(\mathbb{F}_p)\,.$$

where $\beta(L) = |D(L)| - |V(L)| + 1$ is the Betti number of L and $\mathrm{Gl}_r(\mathbb{F}_p)$ is the general linear group.

- ϕ a voltage map, fix vertex u_0, a spanning tree T. We replace by

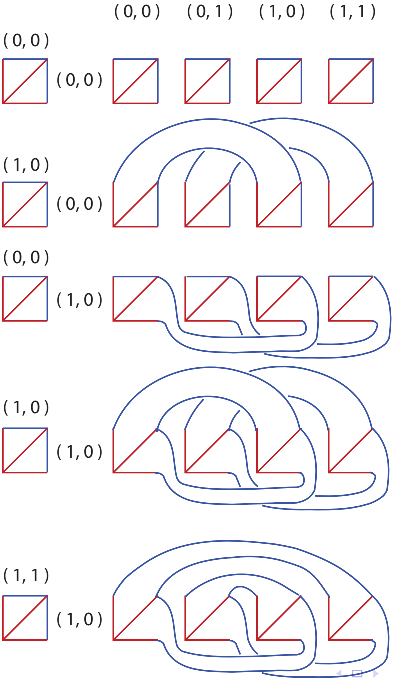
$$\phi(P_w) = \sum_{xy \in P_w} \phi(xy)\,, \qquad \phi_T(uv) = \phi(P_u) + \phi(uv) - \phi(P_v)\,,$$

$$f(v)(g) = g - \phi(P_v)\,, \qquad \phi_T(uv) = f(v)(g + \phi(uv)) - f(u)(g)\,.$$

P_u the unique path in T from u_0 to u. $\phi_T \equiv 0$ on T.
- ϕ_1, ϕ_2 voltage maps with isomorphic voltage graphs and $\phi_1 \equiv \phi_2 \equiv 0$ on T.

$$f(a\phi_1(u_1v_1) + b\phi_1(u_2v_2)) = a\phi_2(u_1v_1) + b\phi_2(u_2v_2) + f(0)\,,$$

for $a, b \in \mathbb{Z}$. Thus there exists $A \in \mathrm{Gl}(2, \mathbb{Z})$ with $f(g) = Ag + f(0)$ for $g \in \mathbb{F}_p^r$. Therefore, $\phi_2(uv) = A\phi_1(uv)$.
- The isomorphisms follows since the cotree T^* of the spanning tree has $\beta(L)$ edges.
- For $p = r = 2$ we have the initial sequen 2ce.
The number of isomorphism classes of regular coverings of a graph L with voltage group $\mathbb{F}_2^2$.



(0, 0) (0, 1) (1, 0) (1, 1)
(0, 0)
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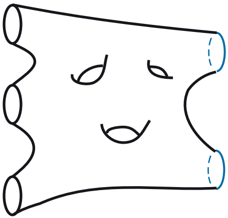
(1, 0)

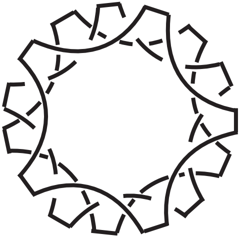
Cobordism category

A *category* is a collection of objects and morphisms. Example: the category of sets and functions, the category of topological spaces and continuous functions, the category of smooth manifolds and smooth maps, etc.

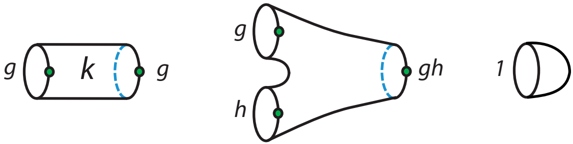
The *cobordism category*. Its objects are finite disjoint unions of circles. A *cobordism* between two objects Σ_1 and Σ_2 is an oriented surface M whose boundary is the disjoint union $\partial M = \Sigma_1 \sqcup \overline{\Sigma}_2$. We consider two cobordisms as equal if there exists a diffeomorphism between them which is the identity in the boundary. The equivalence classes of the cobordisms are the morphisms of the cobordism category.



Let G a finite abelian group of order $|G| = n$. The *G-cobordism category* has objects finite sequences $(g_1, \ldots, g_m)$ of elements in G. Each $g \in G$ defines an n-fold covering of the unit circle by taking the product $G \times [0,1]$ up to the identification $(h, 0) \sim (h + g, 1)$, for every $h \in G$. For $G = \mathbb{Z}_{15}$ and $g = 3$, we have a 15-fold covering of the circle whose total space is the disjoint union of three circles, see the following figure



For $(g_1, \ldots, g_m)$ and $(h_1, \ldots, h_l)$ objects, consider cobordisms between the total spaces of the n-fold coverings. Every such cobordism comes with a free action of the group G. We identify two G-cobordisms if there exists a diffeomorphism between them which commutes with the action and which is the identity in the boundary. The equivalence classes of the G-cobordisms obtained by this identification are the morphisms of the G-cobordism category. They are generated by elementary components



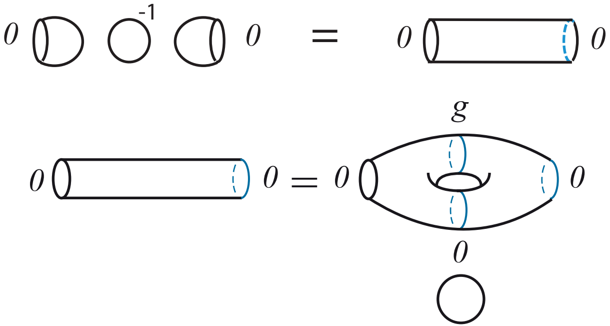
We denote by Cob^G the G-cobordism category.

For $\mathcal{C}$ a small category we denote by $\mathcal{C}[\mathcal{C}^{-1}]$ the groupoid obtained by formally adjoint the inverses of all arrows. For example $\mathbb{Z} = \mathbb{N}[\mathbb{N}^{-1}]$ and $\mathbb{Q} = \mathbb{Z}[\mathbb{Z}^{-1}]$. For $\mathcal{C}$, we associate a topological space called *the classifying space* $B\mathcal{C}$. For example $B\mathbb{Z} \simeq S^1 \simeq B\mathbb{N}$.

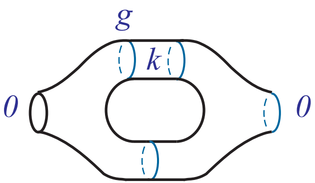
Theorem (Quillen)

The fundamental group $\pi_1(B\mathcal{C}, x)$ *is canonical isomorphic with* $\mathcal{C}[\mathcal{C}^{-1}]_x$ *for* x *and object of* $\mathcal{C}$.

There are two properties in the G-cobordism category which let us to find the fundamental group of its classifying space

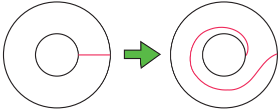


For $G = 0$ the assignment *genus* $-$ *spheres* gives the isomorphism $\pi_1(B\,\mathrm{Cob}^0) = \mathbb{Z}$, so the rank is 1. For an arbitrary finite abelian group, the group $\pi_1(B\,\mathrm{Cob}^G)$ is *generated* by elements of the form



up to the identification given by the diffeomorphism of the torus, which are generated essentially by two which give the identifications $(g, k) \sim (g, g + k)$ and $(g, k) \sim (k, -g)$. Thus the rank of the fundamental group of the classifying space of the G-cobordism category is given by the cardinality of the quotient of $G \times G$ under the action of $\mathrm{Sl}_2(\mathbb{F}_2)$.

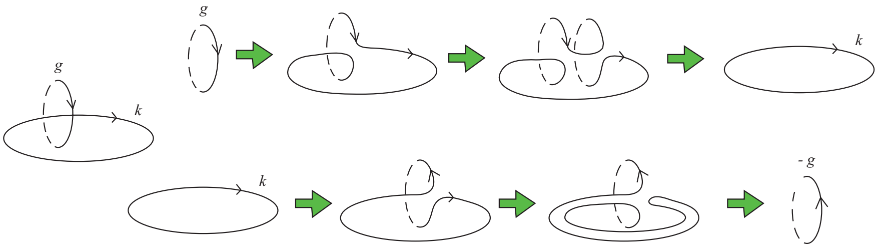
$(g, k) \sim (g, g + k)$



Dehn twist

$$(e^{i\theta}, t) \longmapsto (e^{i(\theta+2\pi t)}, t)$$

$(g, k) \sim (k, -g)$



Resume

- We establish a relation between languages and subvector spaces which give a set of representatives of the dual polar space by bijections
$$F_n : \mathcal{N}^n \longrightarrow L^n$$
where both sets have cardinality $g(n) = (2^n + 1)(2^{n-1} + 1)/3$.
- We prove that the number of isomorphism of regular coverings of a graph with Betti number n with voltage group $\mathbb{F}_p^r$ is given by
$$k(r, p, n) := |\operatorname{Isom}(L, \mathbb{F}_p^r)| = \left|(\mathbb{F}_p^r)^n / \operatorname{Gl}_r(\mathbb{F}_p)\right|$$
and the rank of the $\mathbb{F}_p^n$-cobordism category is the cardinality of the quotient
$$m(p, n) := \left|(\mathbb{F}_p^2)^n / \operatorname{Sl}_2(\mathbb{F}_p)\right|$$
Thus
$$k(2, p, n) \leq m(p, n)\,.$$

Languages

Fourcoverings

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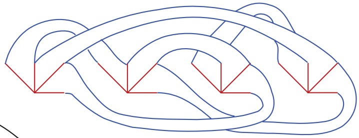
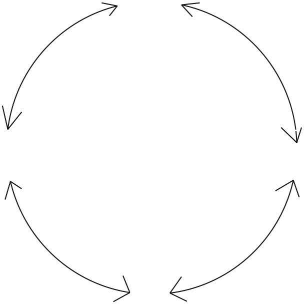
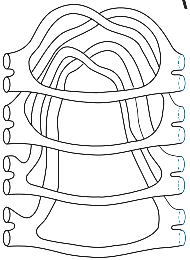


$$\begin{pmatrix} 10 \\ 11 \\ 01 \end{pmatrix}$$





$$\begin{pmatrix} 110 \\ 010 \\ 001 \end{pmatrix}$$

Dual polar space

Cobordism

Languages

Fourcoverings

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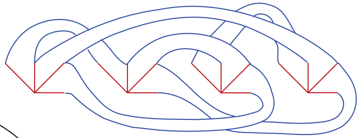
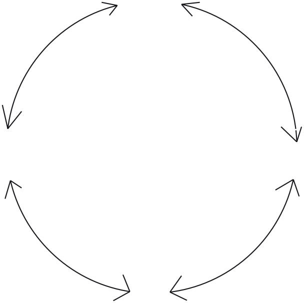
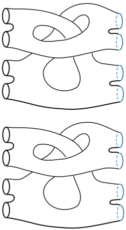


$$\begin{pmatrix} 10 \\ 11 \\ 01 \end{pmatrix}$$

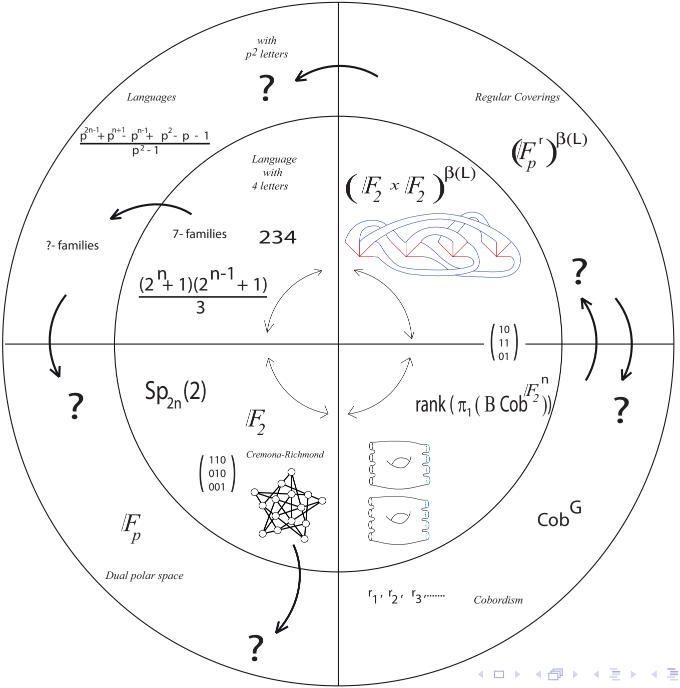


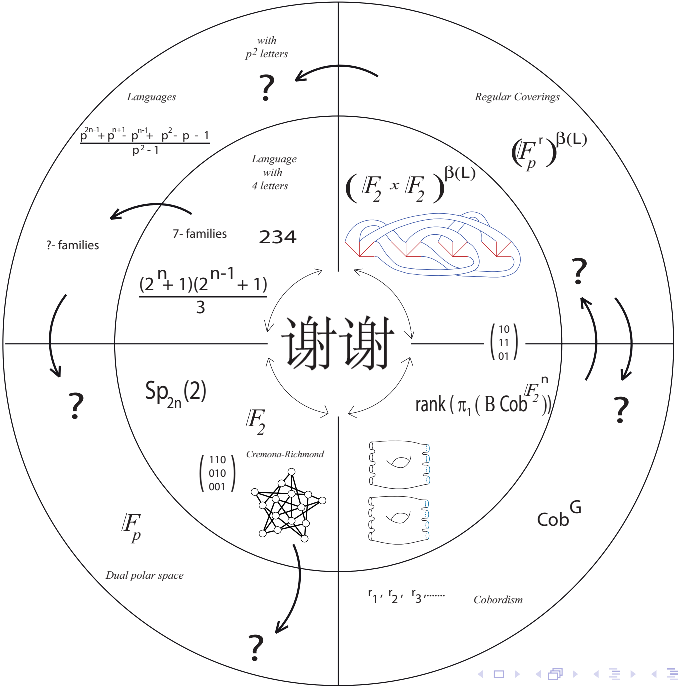


$$\begin{pmatrix} 110 \\ 010 \\ 001 \end{pmatrix}$$

Dual polar space

Cobordism


with
p^2 letters
?
Languages
Regular Coverings
$\frac{p^{2n-1}+p^{n+1}-p^{n-1}+p^2-p-1}{p^2-1}$
$(\mathbb{F}_p^r)^{\beta(L)}$
Language
with
4 letters
$(\mathbb{F}_2 \times \mathbb{F}_2)^{\beta(L)}$
7- families
234
?- families
$\frac{(2^n+1)(2^{n-1}+1)}{3}$
?
$\begin{pmatrix} 10 \\ 11 \\ 01 \end{pmatrix}$
$Sp_{2n}(2)$
rank (π_1 (B Cob $^{\mathbb{F}_2^n}$))
?
?
$\mathbb{F}_2$
Cremona-Richmond
$\begin{pmatrix} 110 \\ 010 \\ 001 \end{pmatrix}$
Cob^G
$\mathbb{F}_p$
Dual polar space
$r_1, r_2, r_3, \ldots$
Cobordism
?



with
p2 letters
?
Languages
Regular Coverings
p2n-1 + pn+1 - pn-1 + p2 - p - 1
p2 - 1
Language
with
4 letters
β(L)
7- families
234
?- families
(2n + 1)(2n-1 + 1)
3
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谢谢
10
11
01
Sp2n(2)
?
?
rank (π1 (B Cob))
F2
Cremona-Richmond
110
010
001
CobG
Fp
Dual polar space
r1, r2, r3,.......
Cobordism
?