CLASSIFICATION OF SYMMETRIC TABAČJN GRAPHS

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Abstract

A *bicirculant* is a graph admitting an automorphism whose cyclic decomposition consists of two cycles of equal length. In this paper we introduce the *Tabačjn graphs*, a family of pentavalent bicirculants which are a natural generalization of generalized Petersen graphs obtained from them by adding two additional perfect matchings between the two orbits of a semiregular automorphism.

The main result is the classification of symmetric Tabačjn graphs. In particular, it is shown that the only such graphs are the complete graph K_6, the complete bipartite graph minus a perfect matching $K_{6,6} - 6K_2$ and the icosahedron graph.

Keywords: pentavalent graph, bicirculant, symmetric, s-arc

1 Introductory remarks

A graph is said to be *symmetric*, also called *arc-transitive*, if its automorphism group acts transitively on the set of arcs of the graph. A non-identity automorphism of a graph is *semiregular*, in particular, (k, n)-*semiregular*, if it has k cycles of equal length n in its cycle decomposition. A graph admitting a $(2, n)$-semiregular automorphism is said to be a *bicirculant*.

We may think of the classical result by Frucht, Graver and Watkins [3] in which they have classified all symmetric generalized Petersen graphs as the main step in the classification of cubic connected symmetric bicirculants. The classification, which was completed much later by Marušič and Pisanski [16, 18], states that a connected cubic symmetric graph is a bicirculant if and only if it is isomorphic to one of the following graphs: the complete graph K_4, the complete bipartite graph $K_{3,3}$, the seven symmetric generalized Petersen graphs GP$(4, 1)$, GP$(5, 2)$, GP$(8, 3)$, GP$(10, 2)$, GP$(10, 3)$, GP$(12, 5)$, and GP$(24, 5)$ (see [3, 17]), the Heawood graph F014A, and a Cayley graph Cay$(D_{2n}, \{b, ba, ba^{r+1}\})$ on a dihedral group $D_{2n} = \langle a, b \,|\, a^n = b^2 = baba = 1\rangle$ of order $2n$ with respect to the generating set $\{b, ba, ba^{r+1}\}$, where $n \geq 11$ is odd and $r \in \mathbb{Z}_n^*$ is such that $r^2+r+1 \equiv 0 \pmod{n}$.

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The classification of connected tetravalent symmetric bicirculants was, in a sense, obtained in a similar way. The first step was done by Kovács, Kutnar and Marušič when they classified symmetric rose window graphs [10]. (The rose window graphs, introduced by Wilson in [21], are a natural generalization of the generalized Petersen graphs obtained from them by adding an additional perfect matching between the two orbits of a semiregular automorphism.) The classification was completed quite recently by Kovács, Kuzman, Malnič and Wilson [11, 12].

The aim of this paper is to initiate the research towards the classification of pentavalent symmetric bicirculants. In accordance with the line of research that led to classifications in the case of valencies 3 and 4 we first consider the pentavalent symmetric bicirculants obtained from rose window graphs by adding another perfect matching between the two orbits of a semiregular automorphism. In particular, given natural numbers $n \geq 3$ and $1 \leq a, b, r \leq n-1$, where $a \neq b$ and $r \neq n/2$, the *Tabačjn graph* $T(n; a, b; r)$ is a pentavalent graph with vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set

$$\{\{x_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\} \ \cup \ \{\{y_i, y_{i+r}\} \mid i \in \mathbb{Z}_n\} \cup$$

$$\{\{x_i, y_i\} \mid i \in \mathbb{Z}_n\} \ \cup \ \{\{x_i, y_{i+a}\} \mid i \in \mathbb{Z}_n\} \ \cup \ \{\{x_i, y_{i+b}\} \mid i \in \mathbb{Z}_n\}.$$

Three examples are shown in Figure 1. The edges from the last three of the above five sets will be called the 0-*spokes*, the a-*spokes* and the b-*spokes*, respectively. A rose window graph is thus obtained by removing all b-spokes from a Tabačjn graph while a generalized Petersen graph is obtained by removing all a- and b-spokes.

Observe that

$$\rho = (x_0 \ x_1 \ \ldots \ x_{n-1})(y_0 \ y_1 \ \ldots \ y_{n-1})$$

is a $(2, n)$-semiregular automorphism of $T(n; a, b; r)$. We will say that ρ gives the $(n; a, b; r)$-*tabačjn structure* to the graph. Of course, a Tabačjn graph does not determine the quadruple $(n; a, b; r)$ uniquely (see Proposition 3.1 for some isomorphisms between Tabačjn graphs).

The main result of this paper is the following classification of symmetric Tabačjn graphs which states that the only such graphs are the ones represented in Figure 1 (for the definition of s-transitivity see Section 2).

Theorem 1.1 *A Tabačjn graph is symmetric if and only if it is isomorphic to one of the graphs* $T(3; 1, 2; 1) \cong K_6$, $T(6; 2, 4; 1) \cong K_{6,6} - 6K_2$ *and* $T(6; 1, 5; 2)$*, which is isomorphic to the icosahedron graph. Moreover, the first two are* 2*-transitive while the third one is* 1*-transitive.*

The classification is obtained by first considering the so-called *core-free Tabačjn graphs*, that is Tabačjn graphs admitting a $(2, n)$-semiregular automorphism ρ giving a tabačjn structure, such that the subgroup $\langle \rho \rangle$ contains no nontrivial normal subgroup of the full automorphism group of the graph. A remarkable group-theoretic result of Herzog and Kaplan [8], which says that 'sufficiently large' cyclic subgroups are never core-free (see Proposition 3.3), combined together with a result, recently extracted by Guo and Feng [7] from the work of Weiss [19, 20], which gives the upper bound for the order of the automorphism group (see Proposition 2.1), enable us to prove that $T(3; 1, 2; 1) \cong K_6$ is the only core-free symmetric Tabačjn graph (see Theorem 3.5). As for non-core-free symmetric Tabačjn graphs, we use the fact that any such graph is a regular cyclic cover of a core-free symmetric Tabačjn graph (see Lemma 3.6). This then enables us to use graph covering techniques, a short review of which is given in Subsection 2.1, to prove that the graphs $T(6; 2, 4; 1) \cong K_{6,6} - 6K_2$ and $T(6; 1, 5; 2)$ are the only non-core-free symmetric Tabačjn graphs.

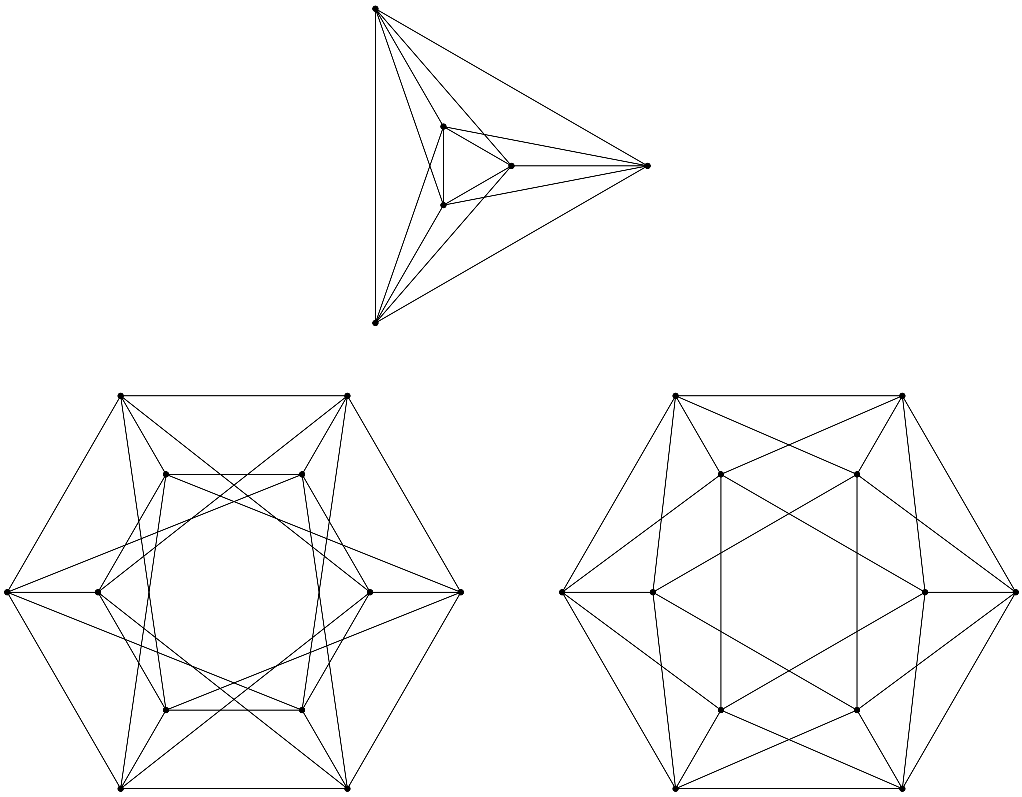

Figure 1: The Tabačjn graphs $T(3;1,2;1)$, $T(6;2,4;1)$ and $T(6;1,5;2)$, which are isomorphic to the complete graph K_6, the complete bipartite graph minus a perfect matching $K_{6,6} - 6K_2$, and the icosahedron, respectively.

2 Preliminaries

Throughout this paper graphs are simple, finite, undirected and connected. Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. A sequence of $k+1$ not necessarily distinct vertices of X such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct is called a *k-arc*. If $v \in V(X)$ then $N(v)$ denotes the set of neighbors of v. The *girth* of X is the length of a shortest cycle contained in X.

A subgroup $G \leq \mathrm{Aut}X$ is said to be *transitive on vertices*, *transitive on edges* and *transitive on arcs* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. In this case the graph X is said to be *G-vertex-transitive*, *G-edge-transitive* and *G-arc-transitive*, respectively. In case of $G = \mathrm{Aut}X$ the prefix G is omitted. An arc-transitive graph is also called *symmetric*. A subgroup $G \leq \mathrm{Aut}X$ is said to be *s-arc-transitive* if it acts transitively on the set of s-arcs of X, and it is said to be *s-regular* if it is s-arc-transitive and the stabilizer of an s-arc in G is trivial. A graph X is said to be *(G,s)-arc-transitive* or *(G,s)-regular* if G is transitive or regular on the set of s-arcs of X, respectively. A (G,s)-arc-transitive graph is said to be *(G,s)-transitive* if the graph is not $(G,s+1)$-arc-transitive. By Weiss [19, 20], for a pentavalent (G,s)-transitive graph, $s \geq 1$, the order of the vertex stabilizer G_v in G is a divisor of $2^{17} \cdot 3^2 \cdot 5$. In addition, the following result can be deduced from his work, as was recently observed by Guo and Feng [7].

Proposition 2.1 [7, Theorem 1.1.] *Let X be a connected pentavalent (G, s)-transitive graph for some $G \leq Aut(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 5$ and one of the following holds:*

(i) For $s = 1$, $G_v \cong \mathbb{Z}_5, D_{10}$ or D_{20};

(ii) For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5;

(iii) For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;

(iv) For $s = 4$, $G_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ or $\mathrm{A\Gamma L}(2, 4)$;

(v) For $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2, 4)$.

For a partition $\mathcal{W}$ of $V(X)$, we let $X_{\mathcal{W}}$ be the associated *quotient graph* of X relative to $\mathcal{W}$, that is, the graph with vertex set $\mathcal{W}$ and edge set induced naturally by the edge set $E(X)$. In the case when $\mathcal{W}$ corresponds to the set of orbits of a subgroup N of Aut$X$, the symbol $X_{\mathcal{W}}$ will be replaced by X_N.

2.1 Graph Covers

A *covering projection* of a graph $\widetilde{X}$ is a surjective mapping $p\colon \widetilde{X} \to X$ such that for each $\tilde{u} \in V(\widetilde{X})$ the set of arcs emanating from $\tilde{u}$ is mapped bijectively onto the set of arcs emanating from $u = p(\tilde{u})$. The graph $\widetilde{X}$ is called a *covering graph* of the *base graph* X. The set $\mathrm{fib}_u = p^{-1}(u)$ is the *fibre* of the vertex $u \in V(X)$. The subgroup K of all automorphisms of $\widetilde{X}$ which fix each of the fibres setwise is called the *group of covering transformations*. The graph $\tilde{X}$ is also called a K-*cover* of X. It is a simple observation that the group of covering transformations of a connected covering graph acts semiregularly on each of the fibres. In particular, if the group of covering transformations is regular on the fibres of $\tilde{X}$, we say that $\tilde{X}$ is a *regular K-cover*. We say that $\alpha \in \mathrm{Aut}(X)$ *lifts* to an automorphism of $\tilde{X}$ if there exists an automorphism $\tilde{\alpha} \in \mathrm{Aut}(\tilde{X})$, called a *lift* of α, such that $\tilde{\alpha}p = p\alpha$. If the covering graph $\widetilde{X}$ is connected then K is the lift of the trivial subgroup of $\mathrm{Aut}(X)$. Note that a subgroup $G \leq \mathrm{Aut}(\widetilde{X})$ projects if and only if the partition of $V(\tilde{X})$ into the orbits of K is G-invariant.

A combinatorial description of a K-cover was introduced through so-called voltages by Gross and Tucker [6] as follows. Let X be a graph and K be a finite group. A *voltage assignment* on X is a mapping $\zeta\colon A(X) \to K$ with the property that $\zeta(u, v) = \zeta(v, u)^{-1}$ for any arc $(u, v) \in A(X)$ (here, and in the rest of the paper, $\zeta(u, v)$ is written instead of $\zeta((u, v))$ for the sake of brevity). The voltage assignment ζ extends to walks in X in a natural way. In particular, for any walk $D = u_1u_2\cdots u_t$ of X we let $\zeta(D)$ denote the product voltage $\zeta(u_{t-1}, u_t)\cdots\zeta(u_2, u_3)\zeta(u_1, u_2)$ of D, that is, the ζ-voltage of D.

The values of ζ are called *voltages*, and K is the *voltage group*. The *voltage graph* $X \times_\zeta K$ derived from a voltage assignment $\zeta\colon A(X) \to K$ has vertex set $V(X) \times K$, and edges of the form $\{(u, g), (v, \zeta(x)g)\}$, where $x = (u, v) \in A(X)$. Clearly, $X \times_\zeta K$ is a covering graph of X with respect to the projection to the first coordinate. By letting K act on $V(X \times_\zeta K)$ as $(u, g)^{g'} = (u, gg')$, $(u, g) \in V(X \times_\zeta K)$, $g' \in K$, one obtains a semiregular group of automorphisms of $X \times_\zeta K$, showing that $X \times_\zeta K$ can in fact be viewed as a K-cover of X.

Given a spanning tree T of X, the voltage assignment $\zeta\colon A(X) \to K$ is said to be T-*reduced* if the voltages on the tree arcs are trivial, that is, if they equal the identity element in K. In [5] it is shown that every regular covering graph $\tilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ζ with respect to an arbitrary fixed spanning tree T of X.

The problem of whether an automorphism α of X lifts or not is expressed in terms of voltages as follows (see Proposition 2.2). Given $\alpha \in \mathrm{Aut}(X)$ and the set of fundamental closed walks $\mathcal{C}$ based at a fixed vertex $v \in V(X)$, we define $\bar{\alpha} = \{(\zeta(C), \zeta(C^\alpha)) \mid C \in \mathcal{C}\} \subseteq K \times K$. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X. Also, from the definition, it is clear that for a T-reduced voltage assignment ζ the derived graph $X \times_\zeta K$ is connected if and only if the voltages of the cotree arcs generate the voltage group K.

We conclude this section with four propositions dealing with lifting of automorphisms in graph covers. The first one may be deduced from [14, Theorem 4.2], the second one from [9] whereas the third one is taken from [2, Proposition 2.2], but it may also be deduced from [15, Corollaries 9.4, 9.7, 9.8].

Proposition 2.2 [14] *Let K be a finite group, and let $X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K. Then an automorphism α of X lifts if and only if $\bar{\alpha}$ is a function which extends to an automorphism α^* of K.*

For a connected regular cover $X \times_\zeta K$ of a graph X derived from a T-reduced voltage assignment ζ with an abelian voltage group K and an automorphism $\alpha \in \mathrm{Aut}(X)$ that lifts, $\bar{\alpha}$ will always denote the mapping from the set of voltages of the fundamental cycles on X to the voltage group K and α^* will denote the automorphism of K arising from $\bar{\alpha}$.

Two coverings $p_i\colon \widetilde{X}_i \to X$, $i \in \{1,2\}$, are said to be *isomorphic* if there exists a graph isomorphism $\phi\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\phi p_2 = p_1$.

Proposition 2.3 [9] *Let K be a finite group. Two connected regular covers $X \times_\zeta K$ and $X \times_\varphi K$, where ζ and φ are T-reduced, are isomorphic if and only if there exists an automorphism $\sigma \in Aut(K)$ such that $\zeta(u,v)^\sigma = \varphi(u,v)$ for any cotree arc (u,v) of X.*

Proposition 2.4 [2] *Let K be a finite group, and let $X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K, and let the lifts of $\alpha \in Aut(X)$ centralize K, considered as the group of covering transformations. Then for any closed walk W in X, there exists $k \in K$ such that $\zeta(W^\alpha) = k\zeta(W)k^{-1}$. In particular, if K is abelian, $\zeta(W^\alpha) = \zeta(W)$ for any closed walk W of X.*

Given a voltage assignment ζ on X and $\beta \in \mathrm{Aut}(X)$, we let ζ^β be the voltage assignment on X given by $\zeta^\beta(u,v) = \zeta(u^{\beta^{-1}}, v^{\beta^{-1}})$, $(u,v) \in A(X)$; and we let $\widetilde{\beta}$ be the permutation of $V(X) \times K$ acting as $(u,k)^{\widetilde{\beta}} = (u^\beta, k)$. Our last proposition is straightforward.

Proposition 2.5 *Let K be a finite group, and let $\widetilde{X} = X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K, and let $\beta \in Aut(X)$. Then the following hold.*

(i) $\widetilde{\beta}$ *is an isomorphism from* $\widetilde{X}$ *to* $X \times_{\zeta^\beta} K$.

(ii) *If* $\widetilde{\alpha}$ *is in* $Aut(\widetilde{X})$ *which projects to* α, *then* $\widetilde{\beta}^{-1}\widetilde{\alpha}\widetilde{\beta}$ *is in* $\mathrm{Aut}(X \times_{\zeta^\beta} K)$, *and it projects to* $\beta^{-1}\alpha\beta$.

(iii) *If* $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ *centralizes the group* K *of covering transformations, then also* $\widetilde{\beta}^{-1}\widetilde{\alpha}\widetilde{\beta}$ *centralizes* K.

3 Symmetric Tabačjn graphs

We first record some fairly obvious isomorphisms between Tabačjn graphs which will be used in the subsequent analysis of symmetric Tabačjn graphs. Recall that the vertices of a Tabačjn graph $T(n;a,b;r)$ are indexed by elements of the additive group $\mathbb{Z}_n$, and so all the computations regarding the elements a,b and r are to be performed modulo n.

Proposition 3.1 *Let $n \geq 3$ and let $1 \leq a,b,r \leq n-1$ be such that $a \neq b$ and $r \neq n/2$. Then*

$$T(n;a,b;r) \cong T(n;a,b;-r) \cong T(n;-a,-b;r) \cong T(n;-a,b-a;r) \cong T(n;-b,a-b;r).$$

Moreover, if $gcd(n,r)=1$, then also $T(n;a,b;r) \cong T(n;-ar^{-1},-br^{-1};r^{-1})$ holds.

The first step towards the proof of Theorem 1.1 is the following result stating that the Tabačjn graphs can be at most 2-arc-transitive.

Proposition 3.2 *There exists no 3-arc-transitive Tabačjn graph.*

Proof. Suppose to the contrary that for some $n \geq 3$ and $1 \leq a,b,r \leq n-1$, where $a \neq b$ and $r \neq n/2$, the Tabačjn graph $X = T(n;a,b;r)$ is 3-arc-transitive. We first show that then $\mathrm{girth}(X) = 6$. Notice that, in general, $g \leq 6$, since $x_0x_1y_1x_{1-a}x_{-a}y_0$ is always a 6-cycle.

Since a regular 3-arc-transitive graph of valence more than 2 cannot have girth 3, we must have that $\mathrm{girth}(X) \geq 4$. The difference between any of 0, a and b is thus at least 2, and so $n \geq 6$. Consequently the 3-arc

$$(x_0, x_1, x_2, x_3) \tag{1}$$

does not lie on a 4-cycle, and so $\mathrm{girth}(X) \geq 5$. Since any pentavalent graph of girth at least 5 is of order at least 26, it follows that $n \geq 13$. Moreover, none of a,b and $b-a$ can be contained in $\{\pm 1, \pm 2\}$. Now, suppose $\mathrm{girth}(X) = 5$. The 3-arc (1) then lies on a 5-cycle of X, and so $n \geq 13$ implies that one of a,b and $b-a$ is 3 or -3. Similarly, the 3-arc

$$(y_0, x_0, x_1, y_1) \tag{2}$$

lies on a 5-cycle of X, and so the fact that $b-a \neq \pm 1$ implies that $2r \pm 1 = 0$. In particular, n is odd and $gcd(n,r) = 1$. By Proposition 3.1 we can thus assume that $r = \frac{n-1}{2}$, and hence that $X \cong T(n;2a,2b;-2) \cong T(n;2a,2b;2)$. We can now repeat the above argument to show that one of $2a, 2b$ and $2(b-a)$ is equal to 3 or -3. But then X contains a 4-cycle, contradicting $\mathrm{girth}(X) > 4$. This proves that $\mathrm{girth}(X) > 5$, and so the fact that $x_0x_1y_1x_{1-a}x_{-a}y_0$ is a 6-cycle of X implies $\mathrm{girth}(X) = 6$. Observe that this implies that no two distinct 6-cycles can contain a common 4-arc. It is easy to see that each 5-valent graph of girth 6 is of order at least 42, and so $n \geq 21$ holds. Moreover, $\mathrm{girth}(X) = 6$ also gives various restrictions on the parameters a,b and r. In particular, we have that

$$\begin{gathered} n \geq 21, \quad r \notin \{\pm 1, \pm 2\}, \\ a, b, b-a \notin \{\pm 1, \pm 2, \pm 3\} \quad \text{and} \quad 0 \notin \{2a, 2b, 2(a-b), a+b, 2a-b, 2b-a\}. \end{gathered} \tag{3}$$

We now show that each 3-arc of X lies on precisely two 6-cycles. To prove that observe that the 3-arc (2) lies on the 6-cycles

$$x_0x_1y_1x_{1-a}x_{-a}y_0 \quad \text{and} \quad x_0x_1y_1x_{1-b}x_{-b}y_0. \tag{4}$$

Suppose now that there exists an additional 6-cycle C of X containing the 3-arc (2). Since no 4-arc of X is contained on more than one 6-cycle, none of the edges of C containing y_0 or y_1 can be an a- or b-spoke. The only possibility is thus $3r \pm 1 = 0$. This implies $gcd(r, n) = 1$, and so Proposition 3.1 implies that we can assume $r = 3$. But applying the same argument, we find that $9 \pm 1 = 0$ holds in $\mathbb{Z}_n$, which is impossible in view of (3). Thus the 3-arc (2), and hence any 3-arc of X, lies on exactly two 6-cycles. This fact has the following consequence. As X is of valence 5 it has $5 \cdot 4^2 \cdot 2n = 160n$ 3-arcs. Since every 6-cycle contains 12 3-arcs a simple counting argument shows that $12c = 2 \cdot 160n = 320n$, where c is the number of 6-cycles of X, and so

$$3 \mid n. \tag{5}$$

Consider again the 3-arc (1) and let C_1 and C_2 be the two 6-cycles containing it. Observe that $r \neq \pm 3$ (otherwise the 3-arc (1) would be contained on at least three different 6-cycles of X, namely, the ones using two 0-spokes, two a-spokes and two b-spokes, respectively). Now, as $n \geq 21$ none of the 6-cycles C_1 and C_2 contains both x_{-1} and x_4. We distinguish two cases depending on whether any of the vertices x_{-1} and x_4 is contained on one of C_1 and C_2 or not. Before doing this analysis observe that one of C_1 and C_2 contains x_{-1} or x_4 if and only if one of a, b and $a - b$ is equal to 4 or -4. Consequently, either C_1 and C_2 each contain five vertices from the set $\{x_i : i \in \mathbb{Z}_n\}$ or they both contain just the four vertices x_0, x_1, x_2 and x_3 from this set.

Case 1: Neither of x_{-1} and x_4 is contained in any of C_1 and C_2 (that is, $x_{-1}, x_4 \notin C_1 \cup C_2$).

As noted above this implies that both C_1 and C_2 contain two vertices from the set $\{y_i : i \in \mathbb{Z}_n\}$. Using an isomorphism from Proposition 3.1 and the fact that $r \neq \pm 3$, we can assume, without lost of generality, that $C_1 = x_0x_1x_2x_3y_{a+3}y_0$ and hence that

$$r = a + 3 \tag{6}$$

holds. Let u and v be the two vertices of C_2 from the set $\{y_i : i \in \mathbb{Z}_n\}$, where u is the neighbor of x_0 and v is the neighbor of x_3. Since no 4-arc is contained on more than one 6-cycle we have that $u \in \{y_a, y_b\}$ and $v \in \{y_3, y_{b+3}\}$. Assume first that $u = y_b$. Since u and v are adjacent, $r \neq \pm 3$ then implies that $v = y_3$, and so $3 \pm r = b$. If $3 - r = b$, then (6) implies $b = -a$, contradicting (3). Therefore, $3 + r = b$, and so $b - a = 6$. Assume now that $u = y_a$. If $v = y_3$, then for u and v to be adjacent $a \pm r = 3$ must hold. However, in view of (6), $a - r = 3$ implies that $6 = 0$ holds in $\mathbb{Z}_n$, while $a + r = 3$ implies $2a = 0$, both of which contradict (3). Thus $v = y_{b+3}$, and consequently $a \pm r = b + 3$ holds. Now, $a + r = b + 3$ implies $2a = b$, which contradicts (3). It follows that $a - r = b + 3$, and so $b + 6 = 0$.

We have thus shown that in the case of $u = y_b$ we have that $v = y_3$ and $b - a = 6$ holds, while in the case of $u = y_a$ we have that $v = y_{b+3}$ and $b + 6 = 0$ holds. In view of the isomorphism $T(n; a, b; r) \cong T(n; -a, b - a, -r)$ these two possibilities are equivalent. Without loss of generality we can thus assume that $u = y_a$ and $v = y_{b+3}$, and consequently that

$$b + 6 = 0. \tag{7}$$

We next consider the 3-arc

$$(x_0, x_1, y_1, y_{r+1}). \tag{8}$$

Let C_1' and C_2' be the two 6-cycles of X containing it. Recall that the two 6-cycles containing the 3-arc (2) are the ones given in (4). Since none of them contains y_{r+1} we have that none of C_1' and C_2' contains the vertex y_0. It follows that two of the vertices x_{n-1}, y_a and y_b must be contained on the 6-cycles C_1' and C_2', one on each. In a similar way we can show that none of C_1' and C_2' contains the vertex x_{r+1}, and hence that two of the vertices x_{r-a+1}, x_{r-b+1} and y_{2r+1} must be contained on the 6-cycles C_1' and C_2', one on each.

We first show that x_{r-a+1} is not contained on any of C_1' and C_2'. Suppose to the contrary that, say C_1', contains x_{r-a+1}. If the remaining vertex v of C_1' is x_{n-1}, then $n-2 = r-a+1$ must hold, and so $a-3 = r = a+3$, contradicting (3). If $v = y_b$, then x_{r-a+1} and y_b are connected by a 0-spoke, and so $r-a+1 = b$, which by (7) implies that $a-7 = r = a+3$ holds, again contradicting (3). Finally, if $v = y_a$, then x_{r-a+1} and y_a are connected by a 0-spoke or a b-spoke. In the former case $r = 2a-1$ holds, and so (6) implies $a = 4$, which cannot hold since neither of x_{-1} and x_4 is contained on any of C_1 and C_2. In the latter case $r-a+1+b = a$, and so (7) implies $r = 2a+5$, forcing $a = -2$, which contradicts (3). This proves that x_{r-a+1} is indeed not contained on any of C_1' and C_2'.

We can thus assume that C_1' contains $x_{r-b+1} = x_{r+7}$ and C_2' contains y_{2r+1}. If the remaining vertex v of C_1' is x_{n-1}, then $n-2 = r+7$ must hold, and so $r = -9$, which by (6) gives $a = -12$. But then $a = 2b$, contradicting (3). If $v = y_a$, then x_{r+7} and y_a are connected via a 0-spoke, and so $r = a-7$. In view of (6) this contradicts (3). It follows that $v = y_b = y_{n-6}$. The vertices x_{r+7} and y_{n-6} are connected via a 0-spoke or an a-spoke. We thus have that either $r = -13$ or $r = -a-13$ holds. In view of (6) this implies that either

$$2r = -26 \quad \text{or} \quad 2r = -10 \tag{9}$$

holds. We now consider the 6-cycle C_2' (recall that we already know that it contains y_{2r+1}). Since no 4-arc is contained on more than one 6-cycle the remaining vertex v of the 6-cycle C_2' is either x_{n-1} or y_a. We cannot have $v = y_a$ for otherwise $1+3r = a$ must hold, which in view of (6) implies $2r = -4$. Combining together (5), (3) and (9), we find that this is not possible. It thus follows that $v = x_{n-1}$, implying that the equation $-1+i = 1+2r$ has a solution for some $i \in \{0, a, b\}$. If $i = a$, then $2r = a-2 = r-5$, but then $x_0 y_{n-6} y_{n-1} x_{n-1}$ is a 4-cycle, contradicting $\mathrm{girth}(X) = 6$. For $i = 0$ and $i = b$ we get $2r = n-2$ and $2r = n-8$, respectively. The latter case is impossible in view of (3) and (9). It thus follows that $2r = n-2$ holds, and so (3) and (9) imply that $n = 24$. Moreover, $r = n-13 = 11$, $a = 8$ and $b = 18$. But then $x_0 x_1 y_{19} y_8$ is a 4-cycle of X, contradicting $\mathrm{girth}(X) = 6$. This completes the analysis of Case 1.

Case 2: At least one of x_{-1} and x_4 is contained in one of C_1 and C_2.

As noted above Proposition 3.1 enables us to assume $a = 4$. Observe that in this case the 6-cycles C_1 and C_2 are $x_0 x_1 x_2 x_3 x_4 y_4$ and $x_{-1} x_0 x_1 x_2 x_3 y_3$, and so there exists no 6-cycle of X containing (1) and two vertices from the set $\{y_i \;:\; i \in \mathbb{Z}_n\}$. The equation $3+i \pm r - j = 0$, where $i, j \in \{0, 4, b\}$, thus has no solution. In addition to (3) we therefore have that

$$r \notin \{\pm 3, \pm 7, \pm(b-1), \pm(b+3), \pm(b-3), \pm(b-7)\}. \tag{10}$$

Consider now again the 3-arc (8) and let C_1' and C_2' be the two 6-cycles of X containing it. As in Case 1 we find that two of the vertices x_{r-3}, x_{r+1-b} and y_{2r+1} must be contained on C_1' and C_2', one on each, and similarly, two of the vertices x_{n-1}, y_4 and y_b must be contained on C_1' and C_2', one on each. We again first show that x_{r-3} is not contained on C_1' or C_2'. Suppose to the contrary that, say C_1', contains x_{r-3}. If the remaining vertex v of C_1' is x_{n-1}, then $n-2 = r-3$ must hold, contradicting (3). If $v = y_4$, then x_{r-3} and y_4 are connected by a 0-spoke or a b-spoke. In the former case $r-3 = 4$ and in the latter case $r-3+b = 4$, each of which contradicts (10). Finally, if $v = y_b$, then x_{r-3} and y_b are connected by a 0-spoke, and so $r-3 = b$, contradicting (10). This proves that x_{r-3} is indeed not contained on any of C_1' and C_2'. With no loss of generality we can thus assume that C_1' contains the vertex x_{r+1-b}. Now, if the remaining vertex v of C_1' is x_{n-1}, then $n-2 = r+1-b$, contradicting (10). Similarly if $v = y_4$, then x_{r+1-b} and y_4 are connected via a 0-spoke, so that $r+1-b = 4$, which again contradicts (10). Thus $v = y_b$, and so either $r+1-b = b$ or $r+1-b+4 = b$ must hold. In other words, we have that either

$$r = 2b-1 \quad \text{or} \quad r = 2b-5 \tag{11}$$

holds. We can now repeat the whole argument for the 3-arc (x_0, x_1, y_1, y_{1-r}) to find that either $-r = 2b-1$ or $-r = 2b-5$ holds. Since $2r \neq 0$, we thus must have that $2b-1 = 5-2b$, that is $4b = 6$. Thus either $2r = 4b-2 = 4 = a$ or $-2r = 4 = a$, contradicting $\mathrm{girth}(X) = 6$. This finally proves that no 3-arc-transitive Tabačjn graph exists, as claimed. ■

Recall that the *core* of the subgroup K in a group G (denoted by $\mathrm{core}_G(K)$) is the largest normal subgroup of G contained in K. Let $\mathcal{ST}$ be the family of all symmetric Tabačjn graphs. A graph $X \in \mathcal{ST}$ of order $2n$ is said to be a *core-free Tabačjn graph* if there exists a $(2,n)$-semiregular automorphisms $\rho \in \mathrm{Aut}(X)$ giving rise to a tabačjn structure of X such that the cyclic subgroup $\langle\rho\rangle$ has trivial core in $\mathrm{Aut}(X)$. In other words, a graph $X \in \mathcal{ST}$ of order $2n$ is not core-free if each of its $(2,n)$-semiregular automorphisms ρ giving rise to a tabačjn structure of X are such that $\langle\rho\rangle$ has nontrivial core in $\mathrm{Aut}(X)$. To obtain the classification of core-free symmetric Tabačjn graphs (see Theorem 3.5) the following group-theoretic result will be used.

Proposition 3.3 [8, Theorem B] *If H is a cyclic subgroup of a finite group G with $|H| \geq \sqrt{|G|}$, then H contains a non-trivial normal subgroup of G, that is $\mathrm{core}_G(H)$ is nontrivial.*

In the proof of Theorem 3.5 we shall see that this proposition implies that core-free symmetric Tabačjn graphs are of order less than 480. Moreover, the 1-transitive ones are of order less than 80. We thus first study such symmetric Tabačjn graphs. The proof of the next lemma is computer assisted. Here we explain the necessary theoretic arguments and the algorithm to obtain the results. The corresponding code for the actual algorithm can be provided by the first author upon request.

Lemma 3.4 *The graphs $T(3;1,2;1) \cong K_6$ and $T(6;2,4;1) \cong K_{6,6} - 6K_2$ are the only two 2-arc-transitive Tabačjn graphs $T(n;a,b;r)$ for $n < 240$ and $T(6;1,5;2)$ is the only 1-transitive Tabačjn graph $T(n;a,b;r)$ for $n < 40$.*

Proof. We first consider symmetric Tabačjn graphs of order less than 80. In this case the number of possible quadruples $(n;a,b;r)$ is small enough that an exhaustive computer search checking all of them can be performed. Using a standard software package (MAGMA [1] or GAP) one can verify that the graphs $T(3;1,2;1)$, $T(6;2,4;1)$ and $T(6;1,5;2)$ are the only symmetric Tabačjn graphs of order less than 80. Since the first two are isomorphic to K_6 and $K_{6,6} - 6K_2$, respectively, they are clearly 2-arc-transitive. Moreover, the icosahedron graph $T(6;1,5;2)$ is of girth 3 and is thus 1-transitive. This proves the second part of the proposition.

We now determine the 2-arc-transitive Tabačjn graphs $X = T(n;a,b;r)$ for $n < 240$. (Observe that by the above paragraph we could restrict to the n such that $40 \leq n < 240$ but as this would not shorten the argument we decided not to.) Note first that if the girth of a 2-arc-transitive graph of valence 5 is 3, it must be the complete graph $K_6 \cong T(3;1,2;1)$. For the rest of the proof we can thus assume that $\mathrm{girth}(X)$ is at least 4, and consequently that $a, b, b-a \notin \{\pm 1\}$. This implies $n \geq 6$. Now, if $n = 6$, then $X \cong T(6; 2, 4; 1)$, which is indeed 2-arc-transitive. We now show there is no other 2-arc-transitive Tabačjn graph of girth 4.

Suppose to the contrary that $n > 6$ and that $X = T(n; a, b; r)$ is a 2-arc-transitive Tabačjn graph of girth 4. The 2-arc (x_0, x_1, x_2) must then lie on a 4-cycle of X, and so $n > 6$ implies that one of a, b and $b - a$ equals 2 or -2. By Proposition 3.1 we can assume $a = 2$. Likewise the 2-arc (x_0, y_0, x_{n-b}) is contained on a 4-cycle, say C. The remaining vertex v of C cannot be x_{-1}, for then $n - b = n - 2$ would hold, which cannot occur since $b \neq a$. Suppose $v = y_2$ or $v = x_1$ holds. Then $b = n - 2$, and so the 2-arc (x_0, x_1, x_2) is contained in $x_0x_1x_2y_2$ and $x_0x_1x_2y_0$ while in view of $n > 6$ the 2-arc (x_0, y_2, x_4) is contained on just one 4-cycle. This contradiction shows that $v = y_b$ must hold, and so either $2b = 0$ or $2b = 2$ holds. Observe that we have now also shown that each 2-arc lies on precisely one 4-cycle of X. Consequently $r \neq \pm 1$.

We next consider the 2-arc (x_0, y_2, x_{2-b}). Using similar arguments as above we find that $2b = 0$ cannot hold, and so $2b = 2$. Thus n is even and $b = \frac{n}{2} + 1$. Considering the 2-arc $(x_0, x_1, y_{n/2+2})$ we find that the remaining vertex of the unique 4-cycle containing it is y_0, and so we can assume that $r = \frac{n}{2} + 2$. Observe that this implies $n \geq 10$ (otherwise $x_0y_2y_0$ is a 3-cycle). Considering finally the 2-arc $(y_0, y_{n/2+2}, y_4)$ we find that the remaining vertex of the 4-cycle containing it must be of the form x_i (otherwise $n = 8$), and so one of $\pm a$, $\pm b$ and $\pm(b - a)$ is 4. As $n \geq 10$ the only possibilities are $-b = 4$ and $b - a = 4$ which both occur if and only if $n = 10$. But then the 2-arc (y_0, y_3, y_6) lies on two 4-cycles of X, a contradiction. This finally proves that for $n > 6$ we have that $\mathrm{girth}(X) > 4$.

To complete the proof we thus only need to consider quadruples $(n; a, b; r)$, where $6 < n < 240$, and a, b and r are such that the girth of the corresponding Tabačjn graph $T(n; a, b; r)$ is at least 5. In view of the isomorphisms from Proposition 3.1 it suffices to consider those a and b, $3 \leq a, b \leq n - 3$, for which a is the smallest of the elements $\pm a, \pm b, \pm(b - a)$ in $\mathbb{Z}_n$. We can thus assume that $3 \leq a < \frac{n}{3}$, $2a < b < n - a$ and $2b \notin \{0, a, 2a\}$. Moreover, we can assume that $2 \leq r < \frac{n}{2}$ is such that $3r \neq 0$ and $4r \neq 0$, and that none of $r, r + 1, r - 1$ and $2r$ is contained in the set $\{\pm a, \pm b, \pm(b - a)\}$. We run a Python code that checked which of such quadruples $(n; a, b; r)$ (there are 53 000 862 of them) have the property, that in the corresponding Tabačjn graph $T(n; a, b; r)$ the 2-arcs (x_0, x_1, x_2), (x_0, x_1, y_1), (x_0, x_1, y_{a+1}), (x_0, x_1, y_{b+1}), (x_0, y_a, x_a), (x_0, y_b, x_b) and (x_0, y_b, x_{b-a}) all lie on the same number of 6-cycles. The computations revealed there are 225 842 such quadruples. Of course this check does not guarantee that in such Tabačjn graphs every 2-arc lies on a constant number of 6-cycles, but some preliminary calculations indicated, that these seven 2-arcs were the ones that need to be tested in order to get rid of most of the possibilities. Analyzing the remaining quadruples $(n; a, b; r)$ which could thus potentially give rise to 2-arc-transitive Tabačjn graphs we found that for each of them n is even and $r = \frac{n}{2} - 1$ holds. Moreover, the above seven 2-arcs are all contained on three 6-cycles of the corresponding Tabačjn graph $T(n; a, b; r)$. However, the 2-arc

$$(x_0, x_1, y_1) \tag{12}$$

is clearly contained on the 6-cycles $x_0x_1y_1x_{1-a}x_{-a}y_0$, $x_0x_1y_1x_{1-b}x_{-b}y_0$ and $x_0x_1y_1y_{r+1}y_{n-1}x_{n-1}$ (recall that $r = \frac{n}{2} - 1$). Two of these 6-cycles contain the neighbor y_0 of the endvertex x_0 of (12). Thus, if $T(n; a, b; r)$ was indeed 2-arc-transitive, it would admit an automorphism reversing the 2-arc (12), and so there would have to exist at least two 6-cycles both containing the 2-arc (12) and some neighbor of its other endvertex y_1. As this does not hold, this shows that there exists no 2-arc-transitive Tabačjn graph $T(n; a, b; r)$ for $n > 6$. ■

We are now ready to classify all core-free symmetric Tabačjn graphs.

Theorem 3.5 *A graph $X \in \mathcal{ST}$ is core-free if and only if it is isomorphic to the complete graph $T(3;1,2;1) \cong K_6$.*

PROOF. Let $X = T(n;a,b;r)$ be a core-free symmetric Tabačjn graph and let $\rho \in \mathrm{Aut}(X)$ be a $(2,n)$-semiregular automorphism giving the $(n;a,b;r)$-tabačjn structure of X. Let $m = |\mathrm{Aut}(X)_{x_0}|$ be the order of the vertex stabilizer of x_0 in $\mathrm{Aut}(X)$. Since X is core-free, Proposition 3.3 implies that $n^2 < |\mathrm{Aut}(X)| = 2nm$, and consequently $n < 2m$. Now, by Proposition 3.2 we have that X is s-transitive for some $1 \leq s \leq 2$. Proposition 2.1 implies that in the case of $s = 2$ we have that $m \leq 120$, while in the case of $s = 1$ we have that $m \leq 20$. Therefore, if X is 2-transitive $n < 240$ and if X is 1-transitive $n < 40$ holds. By Lemma 3.4 X is one of $T(3;1,2;1)$, $T(6;2,4;1)$ and $T(6;1,5;2)$. It is clear that $T(3;1,2;1) \cong K_6$ is core-free (the only nontrivial normal subgroups of the symmetric group S_6 are A_6 and S_6) while for the other two graphs the following can be verified (using MAGMA or GAP, for instance). The automorphism group of $T(6;1,5;2)$ (which is of order 120 and is isomorphic to $A_5 \times \mathbb{Z}_2$) has only one conjugacy class of cyclic semiregular subgroups of order 6 none of which is core-free in the full automorphism group of the graph. This shows that $T(6;1,5;2)$ is not a core-free Tabačjn graph. The automorphism group of $T(6;2,4;1)$ (which is of order 1440 and is isomorphic to $S_6 \times \mathbb{Z}_2$) has 3 conjugacy classes of cyclic semiregular subgroups of order 6, but only one conjugacy class consists of subgroups generated by $(2,6)$-semiregular automorphisms giving rise to a tabačjn structure of the graph (one of the other conjugacy classes gives a bipartite presentation, that is, there are five perfect matchings between the two orbits of the semiregular automorphism, and the other gives a presentation in which there are only two matchings between the two orbits of the semiregular automorphism). It follows that also $T(6;2,4;1)$ is not a core-free Tabačjn graph. ∎

The following lemma is a straightforward generalization of [13, Theorem 9].

Lemma 3.6 *Let $X \in \mathcal{ST}$ with a $(2,n)$-semiregular automorphism $\rho \in Aut(X)$ giving the tabačjn structure, and let N be the core of $\langle\rho\rangle$ in $Aut(X)$. Then N is the kernel of the action of $Aut(X)$ on the set of orbits of N and $Aut(X)/N$ acts arc-transitively on X_N. Morever, $X_N \in \mathcal{ST}$ is a core-free Tabačjn graph of order $\frac{2n}{|N|}$.*

We are now ready to prove the main theorem of this paper.

Theorem 1.1 *A Tabačjn graph is symmetric if and only if it is isomorphic to one of the graphs $T(3;1,2;1) \cong K_6$, $T(6;2,4;1) \cong K_{6,6} - 6K_2$ and $T(6;1,5;2)$, which is isomorphic to the icosahedron graph. Moreover, the first two are 2-transitive while the third one is 1-transitive.*

PROOF. Let $X = T(n;a,b;r)$ be a symmetric Tabačjn graph and let $\rho \in \mathrm{Aut}(X)$ be a $(2,n)$-semiregular automorphism of X giving the $(n;a,b;r)$-tabačjn structure. If X is core-free then, by Theorem 3.5, X is isomorphic to $T(3;1,2;1) \cong K_6$.

Suppose now that X is not core-free. Then there exists a nontrivial subgroup N of $\langle\rho\rangle$ which is normal in $\mathrm{Aut}(X)$. By Lemma 3.6, the quotient graph X_N is a connected core-free symmetric Tabačjn graph, and hence, by Theorem 3.5, it is isomorphic to $X_N \cong T(3;1,2;1) \cong K_6$. In fact, since N is a cyclic group, X is isomorphic to a regular $\mathbb{Z}_m$-cover of this graph, where $|N| = m$. Note also that the natural action of ρ on the quotient graph X_N is a $(2, n/m)$-semiregular automorphism of X_N giving the $(3;1,2;1)$-tabačjn structure of X_N. (Below, all arithmetic operations are to be taken modulo m if at least one argument is from $\mathbb{Z}_m$ and the symbol mod m is always omitted.)

The graph $T(3;1,2;1)$ is illustrated in Figure 2. Let us choose the automorphisms

$$\alpha = (x_0\, x_1\, y_0\, y_2\, x_2)(y_1) \text{ and } \beta = (x_0\, x_1\, x_2)(y_0\, y_1\, y_2)$$

of $T(3;1,2;1)$, and let $G = \langle \alpha, \beta \rangle$. It can be checked directly (for instance using MAGMA) that every $(2,3)$-semiregular automorphism of $T(3;1,2;1)$ is conjugate to β, and that every arc-transitive subgroup of its automorphism group is conjugate to a subgroup containing the subgroup G. By Lemma 3.6 the natural action of $\mathrm{Aut}(X)/N$ on the quotient graph X_N is arc-transitive, and so Proposition 2.5 implies that we may assume, without loss of generality, that ρ projects to β (therefore, the lifts of β centralize the group N of covering transformations) and that G lifts to an arc-transitive subgroup of $\mathrm{Aut}(X)$.

Since X is a regular $\mathbb{Z}_m$ cover of K_6 it can be derived from K_6 through a suitable voltage assignment $\zeta\colon A(K_6) \to \mathbb{Z}_m$. To find the possible voltage assignments ζ fix the spanning tree T of K_6 consisting of the edges

$$\{y_0, y_1\}, \{y_0, y_2\}, \{x_0, y_0\}, \{x_0, x_1\}, \{x_0, x_2\}$$

(see also Figure 2).

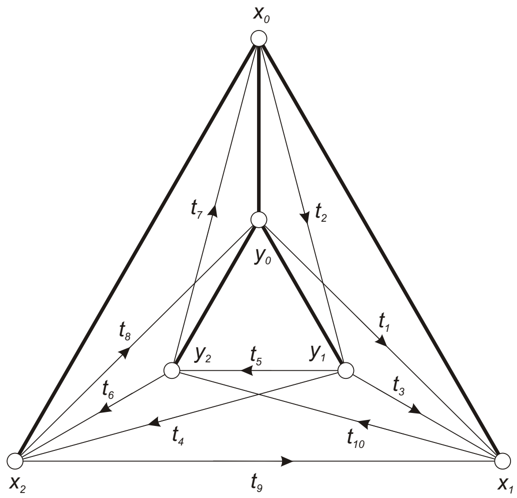

Figure 2: The voltage assignment ζ on $T(3;1,2;1) \cong K_6$. The spanning tree consists of undirected bold edges, all carrying trivial voltage.

The covering graph X is then completely determined by the voltages $t_1, t_2, \ldots, t_{10}$ of the ten arcs

$$(y_0, x_1),\ (x_0, y_1),\ (y_1, x_1),\ (y_1, x_2),\ (y_1, y_2),\ (y_2, x_2),\ (y_2, x_0),\ (x_2, y_0),\ (x_2, x_1) \text{ and } (x_1, y_2),$$

respectively, corresponding to the ten co-tree edges (see Figure 2). We denote the corresponding fundamental cycles by $C_1, C_2, \ldots, C_{10}$ (see Table 1). By Proposition 2.2 the relation $\bar{\alpha}$ extends to an automorphism α^* of $\mathbb{Z}_m$, and, by Proposition 2.4, the relation $\bar{\beta}$ extends to the identity automorphism of $\mathbb{Z}_m$. In Table 1 all fundamental cycles C_i and the voltages of their images C^α and C^β under the action of the automorphisms α and β are listed (for easier determination of the

		C	$\zeta(C)$	C^α	$\zeta(C^\alpha)$
1	C_1	(y_0, x_1, x_0, y_0)	t_1	(y_2, y_0, x_1, y_2)	$t_1 + t_{10}$
2	C_2	(y_0, x_0, y_1, y_0)	t_2	(y_2, x_1, y_1, y_2)	$-t_{10} - t_3 + t_5$
3	C_3	$(y_0, y_1, x_1, x_0, y_0)$	t_3	$(y_2, y_1, y_0, x_1, y_2)$	$-t_5 + t_1 + t_{10}$
4	C_4	$(y_0, y_1, x_2, x_0, y_0)$	t_4	$(y_2, y_1, x_0, x_1, y_2)$	$-t_5 - t_2 + t_{10}$
5	C_5	(y_0, y_1, y_2, y_0)	t_5	(y_2, y_1, x_2, y_2)	$-t_5 + t_4 - t_6$
6	C_6	$(y_0, y_2, x_2, x_0, y_0)$	t_6	$(y_2, x_2, x_0, x_1, y_2)$	$t_6 + t_{10}$
7	C_7	(y_0, y_2, x_0, y_0)	t_7	(y_2, x_2, x_1, y_2)	$t_6 + t_9 + t_{10}$
8	C_8	(y_0, x_0, x_2, y_0)	t_8	(y_2, x_1, x_0, y_2)	$-t_{10} - t_7$
9	C_9	(x_0, x_2, x_1, x_0)	t_9	(x_1, x_0, y_0, x_1)	t_1
10	C_{10}	$(y_0, x_0, x_1, y_2, y_0)$	t_{10}	$(y_2, x_1, y_0, x_2, y_2)$	$-t_{10} - t_1 - t_8 - t_6$
		C	$\zeta(C)$	C^β	$\zeta(C^\beta)$
11	C_1	(y_0, x_1, x_0, y_0)	t_1	(y_1, x_2, x_1, y_1)	$t_4 + t_9 - t_3$
12	C_2	(y_0, x_0, y_1, y_0)	t_2	(y_1, x_1, y_2, y_1)	$t_3 + t_{10} - t_5$
13	C_3	$(y_0, y_1, x_1, x_0, y_0)$	t_3	$(y_1, y_2, x_2, x_1, y_1)$	$t_5 + t_6 + t_9 - t_3$
14	C_4	$(y_0, y_1, x_2, x_0, y_0)$	t_4	$(y_1, y_2, x_0, x_1, y_1)$	$t_5 + t_7 - t_3$
15	C_5	(y_0, y_1, y_2, y_0)	t_5	(y_1, y_2, y_0, y_1)	t_5
16	C_6	$(y_0, y_2, x_2, x_0, y_0)$	t_6	$(y_1, y_0, x_0, x_1, y_1)$	$-t_3$
17	C_7	(y_0, y_2, x_0, y_0)	t_7	(y_1, y_0, x_1, y_1)	$t_1 - t_3$
18	C_8	(y_0, x_0, x_2, y_0)	t_8	(y_1, x_1, x_0, y_1)	$t_3 + t_2$
19	C_9	(x_0, x_2, x_1, x_0)	t_9	(x_1, x_0, x_2, x_1)	t_9
20	C_{10}	$(y_0, x_0, x_1, y_2, y_0)$	t_{10}	$(y_1, x_1, x_2, y_0, y_1)$	$t_3 - t_9 + t_8$

Table 1: Fundamental cycles and their images with corresponding voltages.

corresponding voltages of cycles the vertices of the cycle are comma separated and the starting vertex is repeated at the end).

Now, by assumption the $(2, n)$-semiregular automorphism ρ, giving the tabačjn structure of X, projects to β, and so the lift of at least one of the two orbits of β must induce an n-cycle. With no loss of generality we can assume it is the orbit $\{x_0, x_1, x_2\}$, and so $\langle t_9 \rangle = \mathbb{Z}_m$ holds. By Proposition 2.3 we can thus assume $t_9 = 1$. Since α^* is an automorphism of the cyclic group $\mathbb{Z}_m$ there exists some $s \in \mathbb{Z}_m^*$, coprime to m, such that $\alpha^*(i) = si$ for all $i \in \mathbb{Z}_m$. Since α is of order 5, we of course have that $s^5 = 1$. Moreover, by Rows 16 and 17 of Table 1 (for the rest of this proof, whenever we refer to a row we mean the corresponding row of Table 1) we have that $t_7 - t_6 = t_1$, and so Rows 6 and 7 imply that $\alpha^*(t_1) = \alpha^*(t_7 - t_6) = t_9 = 1$. Row 9 then forces $s^2 = (\alpha^*)^2(t_9) = \alpha^*(t_1) = 1$, and so $s = s^4 s = 1$ must hold, that is, also the automorphism α^* is the identity automorphism of $\mathbb{Z}_m$. Row 9 thus implies $t_1 = 1$, and so Row 11 forces $t_4 = t_3$. Then Rows 3 and 4 imply that $t_2 = -t_1 = -1$. On the other hand, Rows 2 and 12 imply $t_2 = -t_2$, and so $m = 2$. Using Table 1 it is now straightforward to show that the voltages t_i depend solely on whether $t_3 = 0$ or $t_3 = 1$ holds. In particular, if we let $t_3 = t$, then

$$t_1 = t_2 = t_9 = 1, \quad t_{10} = 0, \quad t_3 = t_4 = t_6 = t \quad \text{and} \quad t_5 = t_7 = t_8 = 1 - t.$$

It is now easy to see that in the case of $t = 0$ we get that $X \cong T(6; 2, 4; 1)$ and in the case of $t = 1$ we get that $X \cong T(6; 1, 5; 2)$. ■

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References

[1] W. Bosma, J. Cannon, and C. Playoust, The MAGMA Algebra System I: The User Language, *J. Symbolic Comput.* **24** (1997), 235–265.

[2] S. F. Du, J. H. Kwak and M. Y. Xu, On 2-arc-transitive covers of complete graphs with covering transformation group $\mathbb{Z}_p^3$, *J. Combin. Theory Ser. B* **93** (2005), 73–93.

[3] R. Frucht, J. E. Graver and M. E. Watkins, The groups of the generalized Petersen graphs, *Proc. Camb. Phil. Soc.* **70** (1971), 211–218.

[4] C. D. Godsil and G. F. Royle, "Algebraic Graph Theory", Springer-Verlag, New York, 2001.

[5] J. L. Gross and T. W. Tucker, Generating all graph coverings by permutation voltage assignment, *Discrete Math.* **18** (1977), 273–283.

[6] J. L. Gross and T. W. Tucker, "Topological Graph Theory", Wiley - Interscience, New York, 1987.

[7] S. T. Guo and Y. Q. Feng, A note on pentavalent s-transitive graphs, *Discrete Math.* **312** (2012), 2214–2216.

[8] M. Herzog and G. Kaplan, Large cyclic subgroups contain non-trivial normal subgroups, *J. Group Theory* **4** (2001), 247–253.

[9] S. Hong, J. H. Kwak and J. Lee, Regular graph coverings whose covering transformation groups have the isomorphism extension property, *Discrete Math* **168** (1996), 85–105.

[10] I. Kovács, K. Kutnar and D. Marušič, Classification of edge-transitive rose window graphs, *J. Graph Theory* **65** (2010), 216–231.

[11] I. Kovács, B. Kuzman and A. Malnič, On non-normal arc transitive 4-valent dihedrants, *Acta Math. Sinica, English Series* **26** (2010), 1485–1498.

[12] I. Kovács, B. Kuzman, A. Malnič and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, *J. Graph Theory* **69** (2012), 441–463.

[13] P. Lorimer, Vertex-transitive graphs: symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55–68.

[14] A. Malnič, Group actions, coverings and lifts of automorphisms, *Discrete Math.* **182** (1998), 203–218.

[15] A. Malnič, R. Nedela and M. Škoviera, Lifting graph automorphisms by voltage assignments, *Europ. J. Combin.* **21** (2000), 927–947.

[16] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croat. Chem. Acta* **73** (2000), 969–981.

[17] D. Marušič and T. Pisanski, The remarkable generalized Petersen graph $G(8,3)$, *Mathematica Slovaca* **50** (2000), 117–121.

[18] T. Pisanski, A classification of cubic bicirculants, *Discrete Math.* **307** (2007), 567–578.

[19] R. M. Weiss, Über symmetrische Graphen vom Grad fünf, *J. Combin. Theory Ser. B* **17** (1974), 59–64.

[20] R. M. Weiss, Über symmetrische graphen deren valenz eine primzahl ist, *Math. Z.* **136** (1974), 277–278.

[21] S. Wilson, Rose window graphs, *Ars Math. Contemp.* **1** (2008), 7–19.