On Primitivity and Reduction for Flag-Transitive Symmetric Designs

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Abstract

We present some results on flag-transitive symmetric designs. First we see what conditions are necessary for a symmetric design to admit an imprimitive, flag-transitive automorphism group. Then we move on to study the possibilities for a primitive, flag-transitive automorphism group, and prove that for $\lambda \leq 3$, the group must be affine or almost simple, and finally we analyse the case in which a biplane admits a primitive, flag-transitive automorphism group of affine type.

1 Introduction

If D is a (v, k, λ)-symmetric design, then D', the *complement* of D is a $(v, v-k, v-2k+\lambda)$-symmetric design whose set of points is the same as the set of points of D, and whose blocks are the complements of the blocks of D, that is, incidence is replaced by non-incidence and vice-versa. The *order* of D (and of its complement) is $n = k - \lambda$. A *flag* of D is an ordered pair (p, B) where p is a point of D, B is a block of D, and they are incident. Hence if G is an automorphism group of D, then G is *flag-transitive* if it acts transitively on the flags of D.

Flag-transitivity is just one of many conditions that can be imposed on the automorphism group G of a symmetric design D. In the case $\lambda = 1$, in which symmetric designs are *projective planes*, Kantor [23] proved that either D is Desarguesian and $G \rhd PSL(3, n)$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, and $n^2 + n + 1$ is a prime.

Here we will consider non-trivial symmetric designs with $\lambda > 1$. There is a considerable difference between these two cases in that for $\lambda = 1$ there are

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infinitely many symmetric designs, whereas for any given $\lambda > 1$ there are finitely many known examples, and indeed it is conjectured that for any given $\lambda > 1$ only finitely many exist. In the case $\lambda = 2$, the designs are known as *biplanes*.

The only values of k for which examples of non-trivial biplanes are known are $k = 4, 5, 6, 9, 11$, and 13 [6, p. 76]. Due to arithmetical restrictions on the parameters, there are no examples with $k = 7, 8, 10$, or 12. We will give a brief summary of these examples at the end of this section.

In [6, p. 76] we find that the values of k for which $(v, k, 3)$-symmetric designs are known with $4n - 1 < v < n^2 + n + 1$ (where $n = k - \lambda$) are $k = 6, 9, 10$, 12, and 15. By the above inequality, this list does not include, for example, the parameters (15,7,3) for which five non-isomorphic designs exist [6, p. 11].

In the case of flag-transitivity, in [13] it is shown that there are no projective planes admitting a flag-transitive imprimitive automorphism group. Davies [7] proved that for any given λ, there are finitely many (v, k, λ)-designs (not necessarily symmetric) admitting an imprimitive, flag-transitive automorphism group, by showing that if the automorphism group is imprimitive, then k is bounded, thus allowing only a finite number of possibilities, (in the case $\lambda = 1$, the group must be primitive [7,8,21]). Although it is shown in [7] that k is bounded, an explicit bound is not given. Here we calculate for symmetric designs a bound for k in terms of λ and give some conditions that the parameters must satisfy. Similar results appear in [8, p. 79], and [21], where conditions are given for a design to admit a flag-transitive, imprimitive automorphism group. We analyse the case of biplanes, showing that the only admissible parameters for a biplane with a flag-transitive, imprimitive automorphism group are (16,6,2), and indeed, there are two such examples. The reduction to primitive groups allows us to use the O'Nan-Scott Theorem, which classifies primitive groups into five types, and so we may analyse each case separately. We prove a reduction theorem (Theorem 2) for small values of λ.

1.1 Results

Here we state the results that we will prove in this paper. Firstly, we give some necessary conditions for a symmetric design to have a flag-transitive, imprimitive, automorphism group:

Theorem 1 *If D is a (v, k, λ)-symmetric design admitting a flag-transitive, imprimitive automorphism group G, then either:*

(1) $(v, k, \lambda) = (\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$, *or*

(*2*) $k \leq \lambda(\lambda - 2)$.

Corollary 1 *If G is a flag-transitive automorphism group of a (v, k, λ)-symmetric design D with $\lambda \leq 4$, then either G is primitive, or D has parameters (16,6,2), (45,12,3), (15,8,4), or (96,20,4).*

Thus for $\lambda \leq 3$, if $k \neq 6$ or 12, flag-transitivity implies primitivity. We have the following reduction theorem for primitive groups:

Theorem 2 *If D is a (v, k, λ)-symmetric design admitting a flag-transitive, primitive autopmorphism group G with $\lambda \leq 3$, then G is of affine, or almost simple type.*

Focusing on biplanes, we analyse the case in which the automorphism group is flag-transitive, primitive, of affine type, and together with Theorem 2, we get the following:

Theorem 3 *If D is a non-trivial biplane with a primitive, flag-transitive automorphism group G, then one of the following holds:*

(*1*) *D has parameters (16,6,2).*
(*2*) *$G \leq A\Gamma L_1(q)$, for some odd prime power q.*
(*3*) *G is of almost simple type.*

The known flag-transitive (37,9,2) biplane (constructed from a difference set on the non-zero squares of $\mathbb{Z}_{37}$ with automorphism group $\mathbb{Z}_{37}.\mathbb{Z}_9$), is an example of the one-dimensional affine case. The one-dimensionanl affine case for projective planes has been treated extensively by Ho (see for example [14–17].

The almost simple case is treated in [31–33]. It is shown that the only biplanes admitting a flag-transitive, primitive automorphism group of almost simple type have parameters (7,4,2) (this is the complement of the Fano plane), or (11,5,2) (this is the unique Hadamard design of order 3). Here it is worth mentioning that as a consequence if a non-trivial biplane has a flag-transitive automorphism group and v is even, then $v = 16$.

We now list the parameters of the five known non-trivial flag-transitive biplanes, with their full automorphism groups and point stabilisers:

(*1*) (7,4,2), $PSL_2(7)$, S_4.
(*2*) (11,5,2), $PSL_2(11)$, A_5.
(*3*) (16,6,2), 2^4S_6, S_6.
(*4*) (16,6,2), $(\mathbb{Z}_2 \times \mathbb{Z}_8)\,(S_4.2)$, $(S_4.2)$.
(*5*) (37,9,2), $\mathbb{Z}_{37} \cdot \mathbb{Z}_9$, $\mathbb{Z}_9$.

1.2 Examples

Here we give a brief summary of the known examples of biplanes, as well as some symmetric designs which admit imprimitive, flag-transitive automorphism groups.

1.2.1 Biplanes

For a more detailed description of these examples, see [18, Section 3.6].

For $k = 4$ we have the unique (7,4,2) biplane, on the set of points $P = \mathbb{Z}_7$. Take $B_0 = \{3, 5, 6, 7\}$ a difference set, so the set of block is $B = \{B_0 + i,\ i = 1, \ldots, 7\}$. This is the complement of the Fano Plane, and the full automorphism group is $PSL_2(7)$, which is flag-transitive (in fact it is 2-transitive). The point stabiliser is S_4.

For $k = 5$ we have the unique (11,5,2) biplane, the set of points is $P = \mathbb{Z}_{11}$, and $B_0 = \{1, 3, 4, 5, 9\}$ (the set of squares modulo 11) is a Paley Difference Set. The set of blocks is $B = \{B_0 + i,\ i = 1, \ldots, 11\}$. This is a Hadamard design, and the full automorphism group is $PSL_2(11)$, also flag-transitive (and 2-transitive). The point stabiliser is A_5.

For $k = 6$ there are exactly three non-isomorphic biplanes [19]. The first one arises from a difference set in $\mathbb{Z}_2^4$: Take the set of points $P = \mathbb{Z}_2^4$, and the set of blocks $B = \{B_0 + p : p \in P\}$, where $B_0 = \{\overline{0}, e_1, e_2, e_3, e_4, \sum_{i=1}^{4} e_i\}$, and e_i is the vector with 1 in the i-th place, and 0 elsewhere, so $\{e_1, \ldots, e_4\}$ is the canonical basis for $\mathbb{Z}_2^4$. The automorphism group is $2^4S_6 < 2^4GL_4(2)$. Since the stabiliser $G_{\overline{0}} = S_6$ is transitive on the six blocks incident with $\overline{0}$, and the group of translations 2^4 acts regularly on the points of P, G is flag-transitive.

Next we have a biplane arising from a difference set in $\mathbb{Z}_2 \times \mathbb{Z}_8$, and the stabiliser of order 48 acts as the full group of symmetries of the cube, hence is a central extension of the symmetric group S_4 by a group of order 2. The group $\mathbb{Z}_2 \times \mathbb{Z}_8$ acts regularly, and so the full automorphism group is flag-transitive.

The last (16,6,2) biplane can be seen as a difference set in $Q \times \mathbb{Z}_2$, where Q is the quaternion group. The stabiliser of a block has two orbits on the points of the block, of size 2 and 4, therefore the full automorphism group is not flag-transitive. The order of the stabiliser is 24, and it acts as the inverse image of A_4 in the central extension of S_4 by a cyclic group of order 2 described in the previous case. The order of the automorphism group is $16 \cdot 24$.

For $k = 9$ there are exactly four non-isomorphic biplanes [35], only one of which has a flag-transitive automorphism group.

The first one was first found by Hussain [20], the (Hussain) graphs are given by the elements of order 3 in $PSL_2(8)$, and the full automorphism group is $P\Gamma L_2(8)$. The second is the dual of the first, and so has the same automorphism group.

The third biplane can be constructed from the difference set of nine quartic residues modulo 37. The automorphism group is $\mathbb{Z}_{37} \cdot \mathbb{Z}_9$, and it is flag-transitive, with the stabiliser of a point $G_x \cong \mathbb{Z}_9$.

The last of these has an automorphism group of order 54, which fixes a unique point.

For $k = 11$ there are five known biplanes [12,3,9], (see also [24]), none of which has a flag-transitive automorphism group. The first was found by Hall, Lane, and Wales [12] in terms of a rank-3 permutation group, and its associated strongly regular graph. The group of automorphisms is a subgroup of index 3 of $\mathrm{Aut}\,(PSL_3(4))$, represented on the 56 cosets of A_6, which is the full stabiliser of a block. However if the automorphism group G is flag-transitive, then k divides twice the order of G_x, but in this case $G_x \cong A_6$, and 11 does not divide 720. Hence the group is not flag-transitive.

The next was found by a computer search by Assmus, Mezzaroba and Salwach [3]. The automorphism group has order 288.

The next two were found by Denniston [9]. His constructions are based on $GF(9)$, and two other symbols A and B. As the points he takes the 55 unordered pairs of these symbols, and a further point (which he denotes - -), and assumes that addition and multiplication in $GF(9)$ (taking as its elements $a + bi,\ a, b \in GF(3)$) carry over to a biplane. Multiplication can be done by two methods, either fixing or interchanging A and B. Their automorphism groups have orders 144 and 64 respectively.

The last one of these was constructed by Janko, assuming that a group of order 6 acts on the biplane. The full automorphism group is of order 24.

Finally for $k = 13$ there are two known examples. One was constructed by Aschbacher [2] in 1970, and the other is its dual. If we consider the elements of $GF(11)$ and two further elements A and B, we can take the unordered pairs of these elements to be the points of the biplane, plus one other point X. Addition and multiplication in $GF(11)$ fix A and B, but multiplication by a primitive root exchanges X and AB. The full group of automorphisms is $G = \langle x, y, z; x^2 = y^5 = z^{11}, x^y = x^4, x^z = x^{-1}, yz = zy\rangle$ which is of order 110, and is the only possible group of automorphisms for a biplane with $k = 13$ that has at least $v = 79$ points. Here k does not divide twice the order of the group so the group cannot be flag-transitive.

1.2.2 Imprimitive Designs

We give some examples of symmetric designs with flag-transitive, imprimitive automorphism groups, whose parameters are according to Corollary 1.

There are exactly three non-isomorphic (16,6,2) biplanes [19], of which exactly two admit flag-transitive automorphism groups, and these are $2^4 S_6$, and $(\mathbb{Z}_2 \times \mathbb{Z}_8)\,(S_4.2)$. Now, both of these are affine groups contained in $AGL_4(2)$, where S_6 and $S_4.2$ are the point stabilisers in $GL_4(2)$. The group S_4 is contained in both of these stabilisers, and is transitive on the six cosets of V_4, so it is transitive on the six blocks containing the fixed point. Therefore the subgroups $2^4 S_4$ and $(\mathbb{Z}_2 \times \mathbb{Z}_8)\,(S_4)$ are still flag-transitive on the respective biplanes. However S_4 fixes a subspace of dimension 2 in 2^4, so it is not irreducible, and therefore these subgroups are imprimitive.

There are at least 3752 non-isomorphic (45,12,3) symmetric designs [6, p. 16], [29], and at least 1136 have a trivial automorphism group, which is not flag-transitive. It seems unlikely that there is a flag-transitive example, however conducting a thorough search is beyond the scope of this paper.

There is an example of a (15,8,4) symmetric design with a flag-transitive imprimitive group:

Take $P = \{1, ..., 15\}$ to be the set of points, and $B_1 = \{1, 2, 3, 4, 8, 11, 12, 14\}$ to be a block. Now take the set of blocks to be $B = \{B_1 + i,\ i \in \mathbb{Z}_{15}\}$. This construction gives a (15,8,4) symmetric design (the complement of a (15,7,3) symmetric design, which arises from the difference set $\{1, 2, 3, 5, 6, 9, 11\}$ [1, p. 68]).

The permutations $\alpha = (2, 5)(4, 14)(6, 11)(7, 15)(8, 13)(10, 12)$,
$\beta = (2, 8)(3, 7)(5, 10)(6, 11)(9, 14)(12, 13)$, and
$\gamma = (2, 5)(3, 9)(4, 13)(7, 10)(8, 14)(12, 15)$ all fix the point 1. The group H generated by α, β, and γ is transitive on the eight blocks incident with 1, and preserves the partition of P into the sets $\{1, 6, 11\}$, $\{2, 7, 12\}$, $\{3, 8, 13\}$, $\{4, 9, 14\}$, and $\{5, 10, 15\}$. This group has order 24 and is isomorphic to S_4 (calculated with GAP [11]).

The group $\mathbb{Z}_{15}$ of translations acts regularly on the points, (and the blocks) and note it preserves the same partition of the points. Hence the group $G = \mathbb{Z}_{15} H$ which is isomorphic to $3S_5$ (again, with GAP [11]), acts imprimitively, and flag-transitively on the design.

There is also an example of a flag-transitive, imprimitive (96,20,4) design.

A *finite generalised quadrangle* with parameters (s, t), $(s, t \geq 1)$, [6, p. 357] is an incidence structure (P, L, I) with set of points P and set of lines L such that every point is incident with $t + 1$ lines (and two distinct points are incident with at most one line), every line is incident with $1+s$ points (and two distinct lines are incident with at most one point), and if x is a point and j is a line not incident with x, then there is a unique pair $(y, m) \in P \times L$ such that $xImIyIj$.

Take the generalised quadrangle with parameters (5,3), and construct the design as follows: The points are the same as in the quadrangle, and the blocks are the points different from x that are collinear with x for every point x. There are 96 points (and blocks), and it is a (96,20,4) symmetric design. The automorphism group is $2^4 3 S_6$ which is imprimitive, and the point stabiliser is A_6 which has a transitive action on 20 points [11], and so is transitive on the 20 blocks through the fixed point. Therefore the automorphism group is flag-transitive.

2 Primitivity

In this section we will prove Theorem 1. We begin by stating some arithmetic conditions on the parameters of the design.

Lemma 1 *[4, Chapter II, Proposition 3.11] If D is a (v, k, λ)-symmetric design with $n = k - \lambda$, then $4n - 1 \leq v \leq n^2 + n + 1$.*

The upper bound for v is achieved if and only if D or D' is a projective plane, and the lower bound is achieved if and only if D or D' has parameters $v = 4n - 1$, $k = 2n - 1$, and $\lambda = n - 1$. (This is a Hadamard design).

We also have that if D is a (v, k, λ)-symmetric design with a flag-transitive automorphism group G, then k divides λd_i, for every subdegree d_i of G, [7]. Combining this with the well known result that $k(k - 1) = \lambda(v - 1)$, we get the following:

Corollary 2 *If G is a flag-transitive automorphism group of a (v, k, λ)-symmetric design D, then k divides $\lambda \cdot \gcd(v - 1, |G_x|)$, for every point stabiliser G_x.*

Finally, we have the following:

Lemma 3 *If D is a (v, k, λ)-symmetric design, then $4\lambda(v-1)+1$ is a square.*

PROOF. Solving the quadratic equation $k(k-1) = \lambda(v-1)$ for k, the result follows from the the fact that k must be an integer.

Now we proceed to prove Theorem 1:

PROOF. Suppose a (v, k, λ)-symmetric design D admits a flag-transitive automorphism group G which is imprimitive. Then the set of points is partitioned into n non-trivial blocks of imprimitivity Δ_j, $j = 1, \ldots, n$, each of size c. So $v = cn$, with $c, n > 1$.

Now, since G is flag-transitive, each block of D and each block of imprimitivity that intersect non trivially, do so in a constant number of points, say d, since G permutes these intersections transitively. Hence d divides k, and so $k = ds$, where s is the number of blocks of imprimitivity which intersect each block of D, and $d, s > 1$.

Now fix a point x, and count all the flags (p, B_i) such that both p and x are in the same block of imprimitivity, (say Δ), and also both p and x are incident with B_i. Since each block of imprimitivity has constant size c, there are $c - 1$ such points p, and each of them is, together with x, incident with exactly λ blocks. On the other hand, there are exactly k blocks through x, and each of them intersects Δ in d points, of which $d-1$ are not x. Therefore $\lambda(c-1) = k(d-1)$.

Hence, we have the following equations:

$$
\begin{aligned}
v &= cn && (1)\\
k &= ds && (2)\\
\lambda(v-1) &= k(k-1) && (3)\\
\lambda(c-1) &= k(d-1) && (4)
\end{aligned}
$$

with $c, n, d, s > 1$. From Equation 4, we get $c = \frac{k(d-1)+\lambda}{\lambda}$ and $\lambda n(c-1) = kn(d-1)$, and from Equations 1 and 3 we obtain

$$v = cn = \frac{k(k-1)+\lambda}{\lambda}.$$

Subtracting the previous two equations we get $\lambda(n-1) = k(k-1-n(d-1))$. [1]

Let $x = k - 1 - n(d-1)$. Then x is a positive integer, and $\lambda(n-1) = kx$, hence $n = \frac{kx+\lambda}{\lambda}$.

[1] We should mention here that up to now our proof is similar to the proofs in [7], [8, p. 80], and [22, Theorems 4.7 and 4.8]. Our variables (c, n, d) correspond to the variables (t, s, μ) in [7], $(w, \mathrm{p}, \mathrm{k}^*)$ in [8], and (c, n, t) in [21]. Additionally, our equation (4) corresponds to equation (1) in [21], and equation (2) in [21] also appears in this proof. Some of these equations appear too in [7], however they are not numbered.

Combining this with the previous two equations we get that

$$cn = \frac{k(k-1)+\lambda}{\lambda} = \frac{(k(d-1)+\lambda)(kx+\lambda)}{\lambda^2},$$

and solving for k we get that

$$k = \frac{\lambda(x+d)}{\lambda - x(d-1)}.$$

Therefore $\lambda > x(d-1)$, which is a positive integer, so we have the following possibilities:

- $x(d-1) < x+d < \lambda$,
- $x(d-1) < \lambda \leq x+d$, or
- $x+d \leq x(d-1) < \lambda$.

Suppose $x(d-1) < x+d$. Then $x=1$, or $x=2$ and $d \leq 3$, or $d=2$.

First consider $x+d < \lambda$, so $\lambda \geq 4$. Also, $k < \frac{\lambda^2}{2}$, so since $\lambda \geq 4$ we have $k \leq \lambda(\lambda-2)$, satisfying condition (*2*) of the theorem. Now consider $x(d-1) < \lambda \leq x+d$.

First assume $x=1$. Then either $\lambda = d$, or $\lambda = d+1$. If $\lambda = d+1$ then $k = \frac{(d+1)^2}{2}$, but then d does not divide k, which is a contradiction. If $\lambda = d$ then $k = \lambda(\lambda+1)$, and since $k(k-1) = \lambda(v-1)$, then $v = \lambda^2(\lambda+2)$, which corresponds to conclusion (*1*) of the theorem.

Now assume $x=2$. If $d=2$ then $2 < \lambda \leq 4$, and $k = \frac{4\lambda}{\lambda-2}$. If $\lambda=4$ then $k=8$, and $v=15$, which satisfies conclusion (*2*) of the theorem. These parameters correspond to the complement of the $(15,7,3)$ Hadamard design. If $\lambda=3$ then $k=12$, and $v=45$, satisfying again (*1*).

If $d=3$ then $\lambda=5$, so $k=25$, but then d does not divide k, which is a contradiction.

Finally, assume $x \geq 3$ and $d=2$. Then either $\lambda = x+1$, or $\lambda = x+2$. If $\lambda = x+2$ then $k = \frac{\lambda^2}{2}$, and $v = \frac{\lambda^3}{4} - \lambda^2 + 1$, satisfying (*2*). If $\lambda = x+1$, then again $k = \lambda(\lambda+1)$ and $v = \lambda^2(\lambda+2)$, once more, (*1*).

Next suppose $x+d = x(d-1)$. Then $d \neq 2$, and $x \neq 1$. From $x+d = xd-x$, we obtain $2x = d(x-1)$ and $d = x(d-2)$, so either $x=2$ and $d=4$, or $x=3=d$. In either case, $k = \frac{\lambda(x+d)}{\lambda-x(d-1)}$ forces $k = \frac{6\lambda}{\lambda-6}$. If $\lambda \geq 12$, then $v \leq k \leq \lambda$, which is a contradiction. If $\lambda = 11$, then $k = \frac{66}{5} \notin \mathbb{Z}$, another contradiction. If $\lambda = 10$, then we get the parameters (22,15,10), but here $v=22$ is even, and $k-\lambda=5$ is not a square, contradicting Schutzenberger's Theorem [37]. For $7 \leq \lambda \leq 9$, we have the following parameters for D: (247,42,7), (70,24,8), and (35,18,9). The two latter correspond to conclusion (*2*) of the theorem. Now suppose there is a (247,42,7)-symmetric design with a flag-transitive, imprimitive automorphism group. Then $v = cn = 13 \cdot 19$. Also, we know $d = 3$ or 4, but $k = 2 \cdot 3 \cdot 7$ so $d = 3 = x$ and so each block intersects 14 "blocks" (of imprimitivity) in three points each. Now recall $x = k - 1 - n(d-1)$, so $n = 19$. There are eight transitive groups on 19 points [11]. Five of these have order less than $247 = v$, and are therefore ruled out. Of the remaining three, one has order 342 which is not divisible by 247, so it is also ruled out. The remaining two are A_{19} and S_{19}. These groups produce at least one block per 14 "blocks", and this forces more than v blocks altogether.

Finally, suppose $x + d < x(d-1) < \lambda$, then $k \leq \lambda(\lambda - 2)$, and this completes the proof of Theorem 1.

Now we prove Corollary 1:

PROOF. By Theorem 1, if $\lambda = 2$ or 3, then we have conclusion (*1*), which forces $v = 16$ and $k = 6$ in the first case, and $v = 45$ and $k = 12$ in the second.

If $\lambda = 4$ then either conclusion (*1*) holds, forcing $v = 96$ and $k = 20$, or conclusion (*2*) holds, forcing $k \leq 8$. For the design to be non-trivial, $k > 5$. The equation $k(k-1) = \lambda(v-1)$ forces $k(k-1)$ to be divisible by 4, so k cannot be 6 nor 7. If $k = 8$ then $v = 15$.

3 Reduction

In this section we will prove Theorem 2. Here we will investigate the case in which D admits a flag-transitive primitive group. The O'Nan-Scott Theorem classifies primitive groups into the following five types [27]:

(*1*) Affine.
(*2*) Almost simple.
(*3*) Simple diagonal.
(*4*) Product.
(*5*) Twisted wreath.

First suppose G has a product action on the set of points P. Then there is a group H acting primitively on Γ (with $|\Gamma| \geq 5$) of almost simple or diagonal type, where:

$$P = \Gamma^l, \text{ and } G \leq H^l \rtimes S_l = H \text{ wr } S_l,$$

and $l \geq 2$. We have the following lemmas:

Lemma 4 *If G is a primitive group acting flag-transitively on a (v,k,λ)-symmetric design D, with a product action on P, (the set of points of D), as defined above, then $v = |\Gamma|^l \leq \lambda l^2(|\Gamma|-1)^2$, and $l = 2$ forces $\lambda > 4$.*

PROOF. Take $x \in P$. If $x = (\gamma_1, \dots, \gamma_l)$, define for $1 \leq j \leq l$ the *cartesian line of the j^{th} parallel class* through x to be the set:

$$\mathcal{G}_{x,j} = \{(\gamma_1, \dots, \gamma_{j-1}, \gamma, \gamma_{j+1}, \dots, \gamma_l) \mid \gamma \in \Gamma\},$$

that is,

$$\mathcal{G}_{x,j} = \{\gamma_1\} \times \dots \times \{\gamma_{j-1}\} \times \Gamma \times \{\gamma_{j+1}\} \times \dots \times \{\gamma_l\}.$$

(So there are l cartesian lines through x).

Denote $|\Gamma| = m$.

Since G is primitive, G_x is transitive on the l cartesian lines through x. Denote by Δ the union of those lines (excluding x). Then Δ is a union of orbits of G_x, and so every block through x intersects it in the same number of points. Hence k divides $\lambda l(m-1)$. Also, $k^2 > \lambda(m^l - 1)$, so $(m^l - 1) < \lambda l^2(m-1)^2$.

Hence $v = m^l \leq \lambda l^2(m-1)^2$.

Suppose $l = 2$. Then the fact that k divides $2\lambda(m-1)$, implies that $k = \frac{2\lambda(m-1)}{r}$, with $1 \leq r < 2\lambda$.

First assume $r = 1$. Then since $k(k-1) = \lambda(m^2-1)$, we have

$$4\lambda^2(m-1)^2 - 2\lambda(m-1) = \lambda m^2 - \lambda.$$

Solving the quadratic equation for m, we get that

$$m = \frac{4\lambda + 1 \pm 4}{4\lambda - 1} \geq 5.$$

This implies that $2\lambda \leq 1$, which is a contradiction.

Now assume $r = 2$. Then $k = \lambda(m-1)$. By the same procedure,

$$(\lambda - 1)m^2 - (2\lambda + 1)m + (\lambda + 2) = 0,$$

and solving for m forces either $m = 1$, or $m = \frac{\lambda+2}{\lambda-1}$. In both cases $m < 5$, which is a contradiction.

Hence $r \geq 3$, and so $k \leq \frac{2\lambda(m-1)}{3}$. Then, in the same manner as above, we have the following:

$$9\lambda(m^2-1) \leq 4\lambda^2(m-1)^2 - 6\lambda(m-1),$$

so

$$0 \geq (9-4\lambda)m^2 + 2(4\lambda+3)m - (4\lambda-3),$$

and since $m \geq 5$ then $\lambda > 4$.

Lemma 5 *If D is a (v,k,λ)-symmetric design with $\lambda \leq 3$ admitting a flag-transitive, primitive automorphism group G, then G does not have a non-trivial product action or twisted wreath action on the points of D.*

PROOF. The case $\lambda = 1$ was done in [5], so assume first that $\lambda = 2$, and suppose G has a non-trivial product action. Since $m^l \leq 2l^2(m-1)^2$, and $m \geq 5$, by the previous lemma $l = 3$. Then $m < 18$, and k divides $2(3(m-1), m^3-1)$, so k divides

$$2(m-1)(3, 1+m+m^2).$$

Now $(3, 1+m+m^2) = 3$ only when $m \equiv 1 \pmod 3$, that is, when $m = 7, 10, 13,$ or 16. In the first three of these cases $8v-7$ is not a square, contradicting Lemma 3. If $m = 16$ then v is even, but $k-\lambda = 89$ is not a square, contradicting a theorem by Schutzenberger [37]. Therefore $k = 2(m-1)$, and so

$$2m-3 = m+1,$$

which implies that $m = 4$, a contradiction.

Now assume that $\lambda = 3$. Then $m^l \leq 3l^2(m-1)^2$, implies $l < 5$. If $l = 4$ then $m = 5$ or 6, but then in both cases $12v-11$ is not a square, contradicting Lemma 3. Therefore $l = 3$. Now k divides $(9(m-1), 3(m^3-1))$. If $m \equiv 1 \pmod 3$ then k divides $9(m-1)$, and $k^2 > 3(m^3-1)$, so $m = 7$, 10, 13, 16, 19, 22, or 25. We check that the only value of m for which $12m^3-11$ is a square is $m = 25$. So $v = 5^6$, which forces $k = 217$, but then k does not divide $9(m-1) = 216$, which is a contradiction. If $m \equiv 0$ or $2 \pmod 3$ then k divides $3(m-1)$, so $m^3 \leq 3(m-1)^2$, which is a contradiction.

Groups with a twisted wreath action are contained in twisted wreath groups H wr S_l with a product action and H of diagonal type. Here we have also considered subgroups of G, thereby also ruling out groups with a twisted wreath action.

Now suppose G is of simple diagonal type. Then

$$\text{Soc }(G) = N = T^m,\ m \geq 2$$

for some non-abelian simple group T, where $T \cong N_\alpha \lhd G_\alpha \leq \text{ Aut } T \times S_m$.

Here $v = |T|^{m-1} = |N_\alpha|^{m-1}$.

We have the following lemma:

Lemma 6 *If D is a (v, k, λ)-symmetric design with $\lambda \leq 3$ which admits a flag-transitive, primitive automorphism group G, then G is not of simple diagonal type.*

PROOF. The fact that G is flag-transitive implies that G_x is transitive on the k blocks through x, so $N_x \lhd G_x$ implies that the orbits of N_x on the set of k blocks through x all have the same size, say, l. Therefore l divides k, so it divides $\lambda(v-1)$, and also divides $|T|$, that is, l divides $(|T|, \lambda(|T|^{m-1}-1) \leq \lambda$. If $\lambda < 4$ then $l = 1$ as $T \cong N_\alpha$ is simple, and so N_x fixes all the k blocks through x.

We assume $\lambda > 1$, since $\lambda = 1$ was dealt with in [5].

Choose $t \in N_x$ of odd order. Then $o(t) \geq 3$. There is a point y which is not fixed by t. The pair $\{x, y\}$ is incident with exactly λ blocks. Since y is in each of these blocks, the t-orbit of y (which is of size at least three) must also be incident with each of these blocks (together with x) as these blocks are fixed by N_x. This contradicts the fact that every pair of blocks is incident with exactly λ points.

So now we proceed to prove Theorem 2:

PROOF. By Lemmas 5 and 6, G does not have a product or twisted wreath action on the points of D, and is not of simple diagonal type. Hence G must be of affine or almost simple type.

4 Affine Case

Finally, in this section we will prove Theorem 3. For this purpose we consider biplanes which have a flag-transitive automorphism group G of affine type, that is, the points of the biplane can be identified with the vectors in a vector space $V = V_d(p)$ of dimension d over the field $\mathbb{F}_p$, (with p prime), so that $G = TG_x \leq AGL_d(p) = AGL(V)$, where $T \cong (\mathbb{Z}_p)^d$ is the translation group, and G_x (the stabiliser of the point x) is an irreducible subgroup of $GL_d(p)$, by Corollary 1, unless the parameters are (16,6,2).

Now, for each divisor n of d, there is a natural irreducible action of the group $\Gamma L_n\left(p^{\frac{d}{n}}\right)$ on V. Choose n to be the minimal divisor of d such that $G_x \leq \Gamma L_n(p^{\frac{d}{n}})$ in this action, and write $q = p^{\frac{d}{n}}$. Hence $G_x \leq \Gamma L_n(q)$, and $v = p^d = q^n$.

The following result restricts the possibilities for biplanes where v is a power of 2:

Theorem 7 *If D is a non-trivial $(2^b, k, 2)$-biplane, then b=4.*

PROOF. This follows from a result in [28].

We also have the following proposition, provided by Cameron (private communication):

Proposition 8 *Let G be an affine automorphism group of a biplane. Suppose that $G = TH$, where T is the translation group of $V(d,p)$ (acting regularly on the points of the biplane) and $H \leq GL(d,p)$, and p is odd. Then $|G|$ is odd.*

PROOF. We have $v = p^d$, so

$$p^d = 1 + \frac{k(k-1)}{2}.$$

Suppose that $|G|$ is even. Then H contains an involution t. The fixed points of t form an e-dimensional subspace of V for some e, so t fixes p^e points. Also, $G_x = H$ permutes the k blocks incident with x. Suppose t has m transpositions and $k - 2m$ fixed blocks. Then, since the points different from x correspond bijectively to pairs of blocks incident with x, we see that t has $1 + m + \frac{(k-2m)(k-2m-1)}{2}$ fixed points. Thus

$$p^e = 1 + m + \frac{(k-2m)(k-2m-1)}{2}.$$

Subtracting the two displayed equations gives

$$p^d - p^e = 2m(k - m - 1).$$

Note that since $m \leq \frac{k}{2}$, the number of fixed points is at least $\frac{k+1}{2}$, with equality only if $k - 2m = 1$. So $p^e \geq \frac{k+1}{2}$.

It cannot happen that $p \mid m$ and $p \mid k - m - 1$, for then $p \mid k - 1$ and $p^d = 1 + \frac{k(k-1)}{2} \equiv 1 \pmod{p}$. Hence either $p^e \mid m$ or $p^e \mid k - m - 1$. The former is impossible since $m \leq \frac{k}{2}$ and $p^e \geq \frac{k+1}{2}$. We conclude that $p^e \mid k-m-1$, so that indeed

$$p^e = k - m - 1.$$

Now $k - m - 1 = p^d - 2m(k-m-1)$, so

$$(2m+1)(k-m-1) = p^d,$$

so $2m + 1 = p^{d-e}$.

If $m = 0$, then p divides $k-1$ and $p^d = 1+\frac{k(k-1)}{2} \equiv 1 \pmod{p}$, a contradiction. If $m \geq 1$, then p divides $2(k-m-1) + (2m+1) = 2k-1$, so p^2 divides $(2k-1)^2 = 8p^d - 7$, also a contradiction. This completes the proof.

For the proof of Theorem 3, by Theorem 7 we need only consider $p > 2$. Since the case $G \leq A\Gamma L_1(q)$ is a conclusion of Theorem 3, we may also assume $G \not\leq A\Gamma L_1(q)$.

We will assume $G \leq AGL_d(p)$ to be a flag-transitive automorphism group of odd order of a biplane D. Note that the odd order of G implies k is odd. Also, $k \equiv 1 \bmod 4$, since $k(k-1) = 2(v-1) \equiv 0 \bmod 4$.

We now proceed with the proof of Theorem 3:

PROOF. Since $|G|$ is odd, the Feit-Thomson Theorem [10] implies that G is solvable, so all the complements of the regular normal subgroup of G (which we can identify with $V_d(p)$) are conjugate. This implies that every point stabiliser also stabilises a block, that is, $G_x = G_B$.

The point x and the block B cannot be incident, since the flag-transitivity of G implies that G_x is transitive on the k blocks incident with x, and G_B is transitive on the k points incident with B. So $G_x = G_B$ has at least one orbit of size k.

Take a non-trivial element $t \in G_x$, of order s (of course s is odd), and count the number of blocks incident with x which are fixed by t, and those incident with x which are moved by t. Say t moves m blocks incident with x, and fixes $k - m$ of these blocks.

There is a one-to-one correspondence between the points different from x, and the unordered pairs of blocks incident with x, since for any point $p \neq x$ the pair $\{x, p\}$ is incident with exactly two blocks. Therefore t fixes at least $\binom{k-m}{2}$ points different from x. If in addition to these points t fixed another point different from x, it would correspond to an unordered pair of blocks incident with x, however this is not possible, t has odd order so it can only fix pairs of blocks that are fixed individually.

So t fixes $1+\frac{(k-m)(k-m-1)}{2}$ points, and hence it moves $v-1-\frac{(k-m)(k-m-1)}{2}$ points. Now $v = \frac{k(k-1)}{2} + 1$, so a small calculation shows that t moves $\frac{2mk-(m)^2-m}{2}$ points, and this is $\binom{m}{2} = \frac{m(m-1)}{2}$. This forces $\frac{m(2k-m-1)}{2} = \frac{m(m-1)}{2}$, that is, $k = m$.

This means that any non-trivial element of G_x fixes only x and only B, that is, only the identity fixes two points. So $G_{xy} = 1$, and since $k = [G_x : G_{xy}]$, we conclude that $|G_x| = k$, and G_x fixes x and has $\frac{k-1}{2}$ orbits each of size k.

Now since $G_{xy} = 1$, G_x is a Frobenius group, so by [34, 18.2], $G_x = \langle a, b \rangle$, with certain conditions including $a^l = b^m = 1$, $a^{-1}ba = b^r$, and $(r-1, m) = (l, m) = 1$. Also G_x is metacyclic, with $\langle b \rangle$ a maximal abelian subgroup, that is, $C_{G_x}(b) = \langle b \rangle$.

Now G_x is irreducible in $V_d(p)$, ($G_x < GL_d(p)$), so by Schur's Lemma [38, p. 159], $C_{GL_d(p)}(b) = GL_n(p^{\frac{d}{n}})$ for some divisor n of d.

Since any non-identity power of a does not centralise b, a has to be a field automorphism of $GL_n(p^{\frac{d}{n}})$. We also have that $b \in Z(GL_n(p^{\frac{d}{n}}))$, so $b \in GL_1(p^d)$.

That is, $G_x = \langle a, b \rangle < \Gamma L_1(p^d)$, which is a contradiction.

This completes the proof of Theorem 3

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