Biplanes with Flag-Transitive Automorphism Groups of Almost Simple Type, with Alternating or Sporadic Socle.

Eugenia O'Reilly Regueiro

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, México D.F. 04510, Mexico.

Abstract

In this paper we prove that there cannot be a biplane admitting a primitive, flag-transitive automorphism group of almost simple type, with alternating or sporadic socle.

1 Introduction

A *biplane* is a $(v, k, 2)$-symmetric design, that is, an incidence structure of v points and v blocks such that every point is incident with exactly k blocks, and every pair of blocks is incident with exactly two points. Points and blocks are interchangeable in the previous definition, due to their dual role. A *nontrivial* biplane is one in which $1 < k < v - 1$. A *flag* of a biplane D is an ordered pair (p, B) where p is a point of D, B is a block of D, and they are incident. An automorphism group G of D is *flag-transitive* if it acts transitively on the flags of D.

The only values of k for which examples of biplanes are known are $k = 3$, 4, 5, 6, 9, 11, and 13. Due to arithmetical restrictions on the parameters, there are no examples with $k = 7$, 8, 10, or 12.

For $k = 3$, 4, and 5 the biplanes are unique up to isomorphism [4], for $k =$ 6 there are exactly three non-isomorphic biplanes [11], for $k = 9$ there are exactly four non-isomorphic biplanes [19], for $k = 11$ there are five known biplanes [2,8,10], and for $k = 13$ there are two known biplanes [1], namely a biplane and its dual.

Email address: eugenia@matem.unam.mx (Eugenia O'Reilly Regueiro).

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In [18] it is shown that if a biplane admits an imprimitive, flag-transitive automorphism group, then it has parameters (16,6,2). Among the three non-isomorphic biplanes with these parameters [11], one does not admit a flag-transitive automorphism group, and the other two admit flag-transitive automorphism groups which are imprimitive on points, (namely $2^4 S_4$, a subgroup of the full automorphism group $2^4 S_6$, acting primitively, and $(\mathbb{Z}_2 \times \mathbb{Z}_8) S_4$ [18]). Therefore, if any other biplane admits a flag-transitive automorphism group G, then G must be primitive. The O'Nan-Scott Theorem classifies primitive groups into five types [15]. It is shown in [18] that if a biplane admits a flag-transitive, primitive, automorphism group, its type can only be affine or almost simple. The affine case was analysed in [18]. Here we begin to analyse the almost simple case, namely when the socle of G is an alternating or a sporadic group, and prove that this is not possible.

We now state the main result of this paper:

Theorem 1 (Main Theorem) *If D is a biplane with a primitive, flag-transitive automorphism group G of almost simple type, then the socle of G cannot be alternating or sporadic.*

This, together with [18, Theorem 3], yields the following:

Corollary 1 *If D is a non-trivial biplane with a flag-transitive automorphism group G, then one of the following holds:*

(1) *D has parameters (16,6,2),*
(2) *$G \leq A\Gamma L_1(q)$, for some odd prime power q, or*
(3) *G is almost simple, and the socle X of G is either a classical or an exceptional group of Lie type.*

For the purpose of proving our Main Theorem, we will consider non-trivial biplanes that admit a primitive, flag-transitive automorphism group G of almost simple type, with alternating or sporadic socle. That is, if X is the socle of G (the product of all its minimal normal subgroups), then X is a simple (alternating or sporadic) group, and $X \trianglelefteq G \leq \mathrm{Aut}X$. We will also assume that $(v, k, \lambda) \neq (16, 6, 2)$.

2 Preliminary Results

In this section we state some preliminary results we will use in the proof of our Main Theorem.

Lemma 2 *If D is a $(v, k, 2)$-biplane, then $8v - 7$ is a square.*

PROOF. The result follows from [18, Lemma 4].

Corollary 3 *If D is a flag-transitive $(v, k, 2)$-biplane, then $2v < k^2$, and hence* $2|G| < |G_x|^3$.

PROOF. The equality $k(k-1) = 2(v-1)$, implies $k^2 = 2v - 2 + k$, so clearly $2v < k^2$. Since $v = |G : G_x|$, and $k \leq |G_x|$, the result follows.

From [6] we get the following:

Lemma 4 *If D is a biplane with a flag-transitive automorphism group G, then k divides $2d_i$ for every subdegree d_i of G.*

Lemma 5 *If G is a flag-transitive automorphism group of a biplane D, then k divides $2 \cdot \gcd(v-1, |G_x|)$.*

3 The Case in which X is an Alternating Group

In this section we suppose there is a non-trivial biplane D that has a primitive, flag-transitive almost simple automorphism group G with socle X, where X is an alternating group, and arrive at a contradiction. We follow the same procedure as in [7] for linear spaces.

Lemma 6 *The group X is not A_c.*

PROOF. We need only consider $c \geq 5$. Except for three cases (namely $c = 6$ and $G \cong M_{10},\ PGL_2(9)$, or $P\Gamma L_2(9)$), G is an alternating or a symmetric group. The three exceptions will be dealt with at the end of this section.

The point stabiliser G_x acts on the points of the biplane as well as on the set $\Omega_c = \{1, 2, \ldots, c\}$. The action of G_x on this set can be one of the following three:

(*1*) Not transitive.
(*2*) Transitive but not primitive.
(*3*) Primitive.

We analyse each of these actions separately.

3.1 Case (1)

Since G_x is a maximal subgroup of G, it is necessarily the full stabiliser of a proper subset S of Ω_c, of size $s \leq \frac{c}{2}$. The orbit of S under G consists of all the s-subsets of Ω_c, and G_x has only one fixed point in D and stabilises only one subset of Ω_c, hence we can identify the points of D with the s-subsets of Ω_c (we identify x with S).

Two points of the biplane are in the same G_x-orbit if and only if the corresponding s-subsets of Ω_c intersect S in the same number of points. Therefore G acting on the biplane has rank $s+1$, each orbit O_i corresponding to a possible size $i \in \{0, 1, \ldots, s\}$ of the intersection of an s-subset with S in Ω_c.

Now fix a block B in D incident with x. Since G is flag-transitive on D, B must meet every orbit O_i. Let $i < s$, and $y_i \in O_i \cap B$. Since D is a biplane, the pair $\{x, y_i\}$ is incident with exactly two blocks, B, and B_i. The group G_{xy_i} fixes the set of flags $\{(x, B), (x, B_i)\}$, and in its action on Ω_c stabilises the sets S and Y_i, as well as their complements S^c and Y_i^c. That is, G_{xy_i} is the full stabiliser in G of the four sets $S \cap Y_i$, $S \cap Y_i^c$, $S^c \cap Y_i$, and $S^c \cap Y_i^c$, so it acts as $S_{(s-i)}$ on $S^c \cap Y_i$, and at least as $A_{(c-2s+i)}$ on $S^c \cap Y_i^c$. Any element of G_{xy_i} either fixes the block B, or interchanges B and B_i, so the index of $G_{xy_i} \cap G_{xB}$ in G_{xy_i} is at most 2, and therefore $G_{xB} \cap G_{xy_i}$ acts at least as the alternating group on $S^c \cap Y_i$, and $S^c \cap Y_i^c$. Now G_{xB} contains such an intersection for each i, so G_{xB} is transitive on the s-subsets of S^c, that is, on O_0. This implies that the block B is incident with every point in the orbit, so every other block intersects this orbit in only one point, (since for every point y in O_0 the pair $\{x, y\}$ is incident with B and only one other block).

However, any pair of distinct points in O_0 must be incident with exactly two blocks, which is a contradiction.

3.2 Case (2)

Since G_x is maximal, then in its action on Ω_c it is the full stabiliser in G of some non-trivial partition P of Ω_c into t classes of size s, (with $s, t \geq 2$ and $st = c$), and since $G \cong A_c$ or S_c, G_x contains all the even permutations of Ω_c that preserve P. We now claim that P is the only non-trivial partition of Ω_c preserved by G_x.

To see this, suppose that G_x preserves two partitions P_1 and P_2 of Ω_c, with P_i having t_i classes each of size s_i, with $t_i, s_i \geq 2$, and $s_i t_i = c$. Denote by $C_{(i,a)}$ the class of the element a in the partition P_i, and suppose there is an element $b \in C_{(1,a)} \cap C_{(2,a)}$, with $b \neq a$. If $C_{(2,a)}$ is not contained in $C_{(1,a)}$, then there is an element $d \in C_{(2,a)} \setminus C_{(1,a)}$. The even 3-cycle (a, b, d) is in G and preserves P_2, but not P_1, a contradiction. So $C_{(2,a)} \subseteq C_{(1,a)}$, and similarly $C_{(1,a)} \subseteq C_{(2,a)}$. Therefore either $C_{(1,a)} = C_{(2,a)}$, or $C_{(1,a)} \cap C_{(2,a)} = \{a\}$.

Now suppose the latter, and suppose also that $s_1 \geq 3$. Take $b \in C_{(2,a)} \setminus C_{(1,a)}$, and $d, e \in C_{(1,b)}$. Then the 3-cycle (b, d, e) preserves P_1, but since $C_{(2,b)} \cap C_{(1,b)} = \{b\}$, it does not preserve $C_{(2,b)}$. However it is an even permutation preserving P_1, so it is in G_x and must therefore preserve P_2. Since it fixes a, it must stabilise $C_{(2,a)}$, but $C_{(2,a)} = C_{(2,b)}$. Hence $s_1 = s_2 = 2$, and $t_1 = t_2 = \frac{c}{2}$.

If $t_i \geq 3$, then take $b \in C_{(2,a)} \setminus C_{(1,a)}$ and $d \notin C_{(1,a)} \cup C_{(2,a)}$. That is, in P_2 we have $C_{(2,a)} = C_{(2,b)} = \{a, b\}$, and since $t_1 \geq 3$, we are considering three disjoint classes of size two in P_1: $C_{(1,a)}$, $C_{(1,b)}$, and $C_{(1,d)}$. Now consider the even permutation that has a transposition interchanging the two elements of $C_{(1,b)}$, the two elements of $C_{(1,d)}$, and fixes all the remaining points of Ω_c. Since it fixes a, it must stabilise $C_{(2,a)}$, but this is a contradiction because b is not fixed. We conclude that $s_i = t_i = 2$, so $c = 4$, contradicting our initial hypothesis.

Since G acts transitively on all the partitions of Ω_c into t classes of size s, we may identify the points of the biplane D with the partitions of Ω_c into t classes of size s.

We fix a point x of the biplane, that is, a partition X of Ω_c into t classes $C_0, C_1, \ldots, C_{t-1}$ each of size s. We say that a partition Y of Ω_c is *j-cyclic* (with respect to X) if X and Y have $t - j$ common classes, and if, numbering the other j classes $C_0, \ldots, C_{j-1}$, for each C_i $(i = 0, \ldots, j-1)$ there is a point c_i of C_i such that the j classes of Y which differ from those of X are $(C_i - \{c_i\}) \cup \{c_{i+1}\}$, with the subscripts computed modulo j. We define the *cycle* of Y to be the cycle $(C_0, \ldots, C_{j-1})$. As X is supposed to be fixed, if $s \geq 3$ then the points $c_0, \ldots, c_{j-1}$ are uniquely determined by Y, and are called the *special points* of Y. For every $j = 2, \ldots, t$, the set of j-cyclic partitions (with respect to X) is an orbit O_j of G_x.

Now fix a block B incident with x. Since we can identify the points of the biplane D with the partitions of Ω_c into t classes of size s, for simplicity we will refer to the partitions whose corresponding points of the biplane are incident with B simply as the partitions incident with B.

For every $j = 2, \ldots, t$, the block B is incident with at least one j-cyclic partition Y_j, (since G is flag-transitive), and there is an even permutation of the elements of Ω_c that preserves X and Y_j, stabilising each of their $t - j$ common classes and acting as $\mathbb{Z}_j$ on the remaining j classes of X. Therefore G_{xB} acts as S_t on the t classes of X. As a consequence, for any two classes C_0 and C_1 of X, the block B is incident with at least one 2-cyclic partition with cycle (C_0, C_1).

Now we claim that $s \geq 3$. Suppose to the contrary that the classes of X have size 2. Then there are only two 2-cyclic partitions with cycle (C_0, C_1), so B is incident with at least half of the points of the biplane corresponding to the 2-cyclic partitions, which implies that there are at most two blocks incident with x, a contradiction. Therefore $s \geq 3$.

Now we claim that any two 2-cyclic partitions incident with B have a common special point. Suppose to the contrary that for two points y and z incident with the block B, the corresponding 2-cyclic partitions Y and Z have cycle (C_0, C_1), the special points c_0 and c_1 of Y being both distinct from the special points of Z. There is an even permutation of Ω_c that stabilises the partitions X and Z, and maps $\{c_0, c_1\}$ onto any other disjoint pair $\{c'_0, c'_1\}$ (where $c'_i \in C_i$). Therefore, the number m of 2-cyclic partitions with cycle (C_0, C_1) incident with B satisfies $m \geq s^2 - 2s + 1$. However, the flag-transitivity of G and the fact that G_{xB} acts as S_t on the t classes of X imply that m divides the total number s^2 of 2-cyclic partitions with cycle (C_0, C_1), so $m = s^2$ since $s \geq 3$. Therefore the block B is incident with the whole orbit O_2 of G_x consisting of all 2-cyclic partitions, which implies that B is the only block incident with x, and this is a contradiction. Therefore any two 2-cyclic partitions incident with B have a common special point.

If $t \geq 3$, then since G_{xB} acts as S_t on the t classes of X, and since any two 2-cyclic partitions incident with B have a common special point, $t = 3$ and only one point c_i in each class C_i is a special point of some 2-cyclic partition incident with B. However there is an even permutation of Ω_c that preserves each of the classes C_0, C_1, C_2, fixing c_0 and c_1 but mapping c_2 onto any other point of C_2, preserving x and B but not $\{c_0, c_1, c_2\}$, a contradiction. Therefore $t = 2$.

It follows that B is incident with only one partition, say Y, with special points $\{c_0, c_1\}$. If the size of C_0 and C_1 is greater than 3, then B is incident with some partition Z different from Y and X, and there is an even permutation of Ω_c which leaves X and Z invariant, but does not preserve $\{c_0, c_1\}$, a contradiction. Therefore $s = 3$.

Hence $c = 6$, and since the points of D can be identified with the partitions of Ω_6 into 2 classes of size 3, $v = 10$. However, there is no biplane with 10 points, a contradiction.

3.3 Case (3)

Here first of all we mention that if $G \cong S_c$ then $G_x \ncong A_c$, since $[G : G_x] = v > 2$. If the number k of blocks incident with a point is a power of 2, then $v = [G : G_x]$ and $(v, k) \leq 2$ imply that the group G_x contains a subgroup acting transitively on 2 or 4 points of Ω_c, and fixing all other points, so by a theorem of Marggraf [21, Th.13.5], $c \leq 8$. Now v divides $|G|$, so v must be a divisor of $|S_c|$ for $5 \leq c \leq 8$. The only possibilities such that $v > 2$ and $8v - 7$ is a square are $v = 4$, 16, and 56. Since we had assumed the biplane to be non-trivial and to have parameters different to (16,6,2), we immediately rule out $v = 4$ or 16, and $v = 56$ forces $k = 11$, a contradiction.

If k is not a power of 2, then let p be an odd prime divisor of k, so p divides $|G_x|$. Since $v = [G : G_x]$ and $(k, v) \leq 2$, G_x contains a Sylow p-subgroup of G, and so G_x acting on Ω_c contains an even permutation with exactly one cycle of length p and $c - p$ fixed points. By a result of Jordan [21, Th. 13.9], the primitivity of G_x on Ω_c yields $c - p \leq 2$, that is $c - 2 \leq p \leq c$. This implies that p^2 does not divide $|G|$, so p^2 does not divide k. Therefore either k is a prime, namely $c - 2$, $c - 1$, or c, or the product of two twin primes, namely $c(c - 2)$. On the other hand, $k^2 > v$, and a result of Bochert [21, Th. 14.2], implies that $v \geq \frac{\lfloor \frac{c+2}{2} \rfloor !}{2}$. From this and the previous conditions on k, the possibilities are $c = 13 (k = 11 \cdot 13)$, 8, 7, 6, or 5.

If $c = 13$, then $k = 143$, so $k(k - 1) = 2(v - 1)$ forces $v = 10154$. But if v is even, then $k - 2 = 141$ must be a square (by a theorem of Schützenberger [20]), however 141 is not a square, which is a contradiction.

As we have seen earlier in this proof, for $5 \leq c \leq 8$ the only possibility is the (56,11,2) biplane, which cannot happen given the above conditions on k.

We now consider the case $c = 6$, and $G \cong M_{10}$, $PGL_2(9)$, or $P\Gamma L_2(9)$. Checking the divisors of $2^2|A_6|$, the only possibilities for v such that $8v - 7$ is a square are $v = 4$ and 16, which have been already ruled out.

This completes the proof of Lemma 6, and hence X is not an alternating group.

4 The Case in which X is a Sporadic Group

Here we consider X to be a sporadic group.

Lemma 7 *If D is a non-trivial biplane with a flag-transitive, primitive, almost simple automorphism group G, then* $\mathrm{Soc}(G) = X$ *is not a sporadic group.*

PROOF. The way we proceed is as follows: We assume that the automorphism group G of D is almost simple, such that $X \trianglelefteq G \leq \mathrm{Aut}X$ with X a sporadic group. Then $G = X$, or $G = \mathrm{Aut}X$, since for all sporadic groups X either $\mathrm{Aut}X = X$ or $\mathrm{Aut}X = 2.X$. We know that $v = [G : G_x]$, and G_x is a maximal subgroup of G. The lists of maximal subgroups of X and $\mathrm{Aut}X$ appear in [5,13,14,16]. (They are complete except for the 2-local subgroups of the Monster group). For each sporadic group (and its automorphism group), we rule out the maximal subgroups the order of which is too small to satisfy $2|G| < |G_x|^3$. In the remaining cases, for those $v > 2$, we check if $8v - 7$ is a square, or if $2\,(|G_x|)^2_{v'} > v$ (by $|G_x|_{v'}$ we mean the part of $|G_x|$ coprime to v). If this does happen, we check the remaining arithmetic conditions ($k - 2$ is a square if v is even, $k(k - 1) = 2(v - 1)$).

To illustrate this procedure, suppose $X = J_1$. Then $G = J_1$, since $|\mathrm{Out}J_1| = 1$. The maximal subgroups H of J_1, with their orders and indices are as follows:

$L_2(11)$, of order 660, $v = 266$,
$2^3.7.3$, of order 168, $v = 1045$,
$2 \times A_5$, of order 120, $v = 1463$,
$19 : 6$, of order 114, $v = 1540$,
$11 : 10$, of order 110, $v = 1596$,
$D_6 \times D_{10}$, of order 60, $v = 2926$, and
$7 : 6$, of order 42.

In the last case, the order of the group is too small to satisfy $|G_x|^3 > 2|G|$, and in all the remaining cases $8v - 7$ is not a square.

Proceeding in the same manner with the other sporadic groups, the only cases in which these arithmetic conditions are met are the following:

(*1*) $G = M_{23}$, $G_x = 2^4 : (A_5 \times 3) : 2$, $(v, k) = (1771, 60)$.
(*2*) $G = M_{24}$, $G_x = 2^6 : (3 \cdot S_6)$, $(v, k) = (1771, 60)$.

In the first case the subdegrees of M_{23} on $2^4 : (A_5 \times 3) : 2$ are 1, 60, 480, 160, 90, and 20 (calculated with GAP [9], my sincere thanks to A.A. Ivanov and D. Pasechnik), but 30 does not divide 20, contradicting the fact that k must divide twice every subdegree.

In the second case, the subdegrees are 1,90, 240, and 1440 [12, pp.126], however M_{24} has only one conjugacy class of subgroups of index 1771 [5], so if x is a point and B is a block G_x is conjugate to G_B, so G_x fixes a block, say, B_0. But x cannot be incident with B_0 since the flag-transitivity of G implies that G_x is transitive on the k blocks incident with x. Hence x and B_0 are not incident, and so some of the points incident B_0 form a G_x-orbit, which is a contradiction since the smallest non-trivial G_x-orbit has size 90, and B_0 is incident with 60 points.

This completes the proof of Lemma 7, and hence X is not a sporadic group.

The proof of our Main Theorem is now complete.

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