Biplanes with Flag-Transitive Automorphism Groups of Almost Simple Type, with Alternating or Sporadic Socle.

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Abstract

In this paper we prove that there cannot be a biplane admitting a primitive, flag-transitive automorphism group of almost simple type, with alternating or sporadic socle.

1 Introduction

A biplane is a \((v, k, 2)\)-symmetric design, that is, an incidence structure of \(v\) points and \(v\) blocks such that every point is incident with exactly \(k\) blocks, and every pair of blocks is incident with exactly two points. Points and blocks are interchangeable in the previous definition, due to their dual role. A nontrivial biplane is one in which \(1 < k < v - 1\). A flag of a biplane \(D\) is an ordered pair \((p, B)\) where \(p\) is a point of \(D\), \(B\) is a block of \(D\), and they are incident. An automorphism group \(G\) of \(D\) is flag-transitive if it acts transitively on the flags of \(D\).

The only values of \(k\) for which examples of biplanes are known are \(k = 3, 4, 5, 6, 9, 11, \) and \(13\). Due to arithmetical restrictions on the parameters, there are no examples with \(k = 7, 8, 10, \) or \(12\).

For \(k = 3, 4, \) and \(5\) the biplanes are unique up to isomorphism [4], for \(k = 6\) there are exactly three non-isomorphic biplanes [11], for \(k = 9\) there are exactly four non-isomorphic biplanes [19], for \(k = 11\) there are five known biplanes [2,8,10], and for \(k = 13\) there are two known biplanes [1], namely a biplane and its dual.

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In [18] it is shown that if a biplane admits an imprimitive, flag-transitive automorphism group, then it has parameters (16,6,2). Among the three non-isomorphic biplanes with these parameters [11], one does not admit a flag-transitive automorphism group, and the other two admit flag-transitive automorphism groups which are imprimitive on points, (namely $2^4S_4$, a subgroup of the full automorphism group $2^4S_6$, acting primitively, and $(\mathbb{Z}_2 \times \mathbb{Z}_8)S_4$ [18]). Therefore, if any other biplane admits a flag-transitive automorphism group $G$, then $G$ must be primitive. The O’Nan-Scott Theorem classifies primitive groups into five types [15]. It is shown in [18] that if a biplane admits a flag-transitive, primitive, automorphism group, its type can only be affine or almost simple. The affine case was analysed in [18]. Here we begin to analyse the almost simple case, namely when the socle of $G$ is an alternating or a sporadic group, and prove that this is not possible.

We now state the main result of this paper:

**Theorem 1 (Main Theorem)** If $D$ is a biplane with a primitive, flag-transitive automorphism group $G$ of almost simple type, then the socle of $G$ cannot be alternating or sporadic.

This, together with [18, Theorem 3], yields the following:

**Corollary 1** If $D$ is a non-trivial biplane with a flag-transitive automorphism group $G$, then one of the following holds:

1. $D$ has parameters $(16,6,2)$,
2. $G \leq A\Gamma L_1(q)$, for some odd prime power $q$, or
3. $G$ is almost simple, and the socle $X$ of $G$ is either a classical or an exceptional group of Lie type.

For the purpose of proving our Main Theorem, we will consider non-trivial biplanes that admit a primitive, flag-transitive automorphism group $G$ of almost simple type, with alternating or sporadic socle. That is, if $X$ is the socle of $G$ (the product of all its minimal normal subgroups), then $X$ is a simple (alternating or sporadic) group, and $X \trianglelefteq G \leq \text{Aut}X$. We will also assume that $(v, k, \lambda) \neq (16, 6, 2)$.

## 2 Preliminary Results

In this section we state some preliminary results we will use in the proof of our Main Theorem.

**Lemma 2** If $D$ is a $(v, k, 2)$-biplane, then $8v - 7$ is a square.
PROOF. The result follows from [18, Lemma 4].

**Corollary 3** If $D$ is a flag-transitive $(v, k, 2)$-biplane, then $2v < k^2$, and hence $2|G| < |G_x|^2$.

**PROOF.** The equality $k(k - 1) = 2(v - 1)$, implies $k^2 = 2v - 2 + k$, so clearly $2v < k^2$. Since $v = |G : G_x|$, and $k < |G_x|$, the result follows.

From [6] we get the following:

**Lemma 4** If $D$ is a biplane with a flag-transitive automorphism group $G$, then $k$ divides $2d_i$ for every subdegree $d_i$ of $G$.

**Lemma 5** If $G$ is a flag-transitive automorphism group of a biplane $D$, then $k$ divides $2 \cdot \gcd (v - 1, |G_x|)$.

### 3 The Case in which $X$ is an Alternating Group

In this section we suppose there is a non-trivial biplane $D$ that has a primitive, flag-transitive almost simple automorphism group $G$ with socle $X$, where $X$ is an alternating group, and arrive at a contradiction. We follow the same procedure as in [7] for linear spaces.

**Lemma 6** The group $X$ is not $A_c$.

**PROOF.** We need only consider $c \geq 5$. Except for three cases (namely $c = 6$ and $G \cong M_{10}$, $PGL_2(9)$, or $PGL_2(9))$, $G$ is an alternating or a symmetric group. The three exceptions will be dealt with at the end of this section.

The point stabiliser $G_x$ acts on the points of the biplane as well as on the set $\Omega_c = \{1, 2, \ldots, c\}$. The action of $G_x$ on this set can be one of the following three:

1. Not transitive.
2. Transitive but not primitive.
3. Primitive.

We analyse each of these actions separately.
3.1 Case (1)

Since $G_x$ is a maximal subgroup of $G$, it is necessarily the full stabiliser of a proper subset $S$ of $\Omega_c$, of size $s \leq \frac{s}{2}$. The orbit of $S$ under $G$ consists of all the $s$-subsets of $\Omega_c$, and $G_x$ has only one fixed point in $D$ and stabilises only one subset of $\Omega_c$, hence we can identify the points of $D$ with the $s$-subsets of $\Omega_c$ (we identify $x$ with $S$).

Two points of the biplane are in the same $G_x$-orbit if and only if the corresponding $s$-subsets of $\Omega_c$ intersect $S$ in the same number of points. Therefore $G$ acting on the biplane has rank $s + 1$, each orbit $O_i$ corresponding to a possible size $i \in \{0, 1, \ldots, s\}$ of the intersection of an $s$-subset with $S$ in $\Omega_c$.

Now fix a block $B$ in $D$ incident with $x$. Since $G$ is flag-transitive on $D$, $B$ must meet every orbit $O_i$. Let $i < s$, and $y_i \in O_i \cap B$. Since $D$ is a biplane, the pair $\{x, y_i\}$ is incident with exactly two blocks, $B$ and $B_i$. The group $G_{xy_i}$ fixes the set of flags $\{(x, B), (x, B_i)\}$, and in its action on $\Omega_c$, stabilises the sets $S$ and $Y_i$, as well as their complements $S^c$ and $Y_i^c$. That is, $G_{xy_i}$ is the full stabiliser in $G$ of the four sets $S \cap Y_i$, $S \cap Y_i^c$, $S^c \cap Y_i$, and $S^c \cap Y_i^c$, so it acts as $S_{(s-i)}$ on $S^c \cap Y_i$, and at least as $A_{(s=2s+i)}$ on $S^c \cap Y_i^c$. Any element of $G_{xy_i}$ either fixes the block $B$, or interchanges $B$ and $B_i$, so the index of $G_{xy_i} \cap G_{xB}$ in $G_{xy_i}$ is at most 2, and therefore $G_{xB} \cap G_{xy_i}$ acts at least as the alternating group on $S^c \cap Y_i$, and $S^c \cap Y_i^c$. Now $G_{xB}$ contains such an intersection for each $i$, so $G_{xB}$ is transitive on the $s$-subsets of $S^c$, that is, on $O_0$. This implies that the block $B$ is incident with every point in the orbit, so every other block intersects this orbit in only one point, (since for every point $y$ in $O_0$ the pair $\{x, y\}$ is incident with $B$ and only one other block).

However, any pair of distinct points in $O_0$ must be incident with exactly two blocks, which is a contradiction.

3.2 Case (2)

Since $G_x$ is maximal, then in its action on $\Omega_c$ it is the full stabiliser in $G$ of some non-trivial partition $P$ of $\Omega_c$ into $t$ classes of size $s$, (with $s, t \geq 2$ and $st = c$), and since $G \cong A_c$ or $S_c$, $G_x$ contains all the even permutations of $\Omega_c$ that preserve $P$. We now claim that $P$ is the only non-trivial partition of $\Omega_c$ preserved by $G_x$.

To see this, suppose that $G_x$ preserves two partitions $P_1$ and $P_2$ of $\Omega_c$, with $P_i$ having $t_i$ classes each of size $s_i$, with $t_i, s_i \geq 2$, and $s_i t_i = c$. Denote by $C_{(i,a)}$ the class of the element $a$ in the partition $P_i$, and suppose there is an element $b \in C_{(1,a)} \cap C_{(2,a)}$, with $b \neq a$. If $C_{(2,a)}$ is not contained in $C_{(1,a)}$, then there is
Since \( G \) acts transitively on all the partitions of \( \Omega_c \) into \( t \) classes of size \( s \), we may identify the points of the biplane \( D \) with the partitions of \( \Omega_c \) into \( t \) classes of size \( s \).

We fix a point \( x \) of the biplane, that is, a partition \( X \) of \( \Omega_c \) into \( t \) classes \( C_0, C_1, \ldots, C_{t-1} \) each of size \( s \). We say that a partition \( Y \) of \( \Omega_c \) is \( j \)-cyclic (with respect to \( X \)) if \( X \) and \( Y \) have \( t-j \) common classes, and if, numbering the other \( j \) classes \( C_0, \ldots, C_{j-1} \), for each \( C_i \) (\( i=0,\ldots, j-1 \)) there is a point \( c_i \) of \( C_i \) such that the \( j \) classes of \( Y \) which differ from those of \( X \) are \( (C_i - \{c_i\}) \cup \{c_{i+1}\} \), with the subscripts computed modulo \( j \). We define the cycle of \( Y \) to be the cycle \( (C_0, \ldots, C_{j-1}) \). As \( X \) is supposed to be fixed, if \( s \geq 3 \) then the points \( c_0, \ldots, c_{j-1} \) are uniquely determined by \( Y \), and are called the special points of \( Y \). For every \( j=2, \ldots, t \), the set of \( j \)-cyclic partitions (with respect to \( X \)) is an orbit \( O_j \) of \( G_x \).

Now fix a block \( B \) incident with \( x \). Since we can identify the points of the biplane \( D \) with the partitions of \( \Omega_c \) into \( t \) classes of size \( s \), for simplicity we will refer to the partitions whose corresponding points of the biplane are incident with \( B \) simply as the partitions incident with \( B \).

For every \( j=2, \ldots, t \), the block \( B \) is incident with at least one \( j \)-cyclic partition \( Y_j \), (since \( G \) is flag-transitive), and there is an even permutation of the elements of \( \Omega_c \) that preserves \( X \) and \( Y_j \), stabilising each of their \( t-j \) common classes and acting as \( \mathbb{Z}_j \) on the remaining \( j \) classes of \( X \). Therefore \( G_{xB} \) acts as \( S_t \) on the \( t \) classes of \( X \). As a consequence, for any two classes \( C_0 \) and \( C_1 \) of \( X \), the block \( B \) is incident with at least one 2-cyclic partition with cycle \( (C_0, C_1) \).
Now we claim that \( s \geq 3 \). Suppose to the contrary that the classes of \( X \) have size 2. Then there are only two 2-cyclic partitions with cycle \((C_0, C_1)\), so \( B \) is incident with at least half of the points of the biplane corresponding to the 2-cyclic partitions, which implies that there are at most two blocks incident with \( x \), a contradiction. Therefore \( s \geq 3 \).

Now we claim that any two 2-cyclic partitions incident with \( B \) have a common special point. Suppose to the contrary that for two points \( y \) and \( z \) incident with the block \( B \), the corresponding 2-cyclic partitions \( Y \) and \( Z \) have cycle \((C_0, C_1)\), the special points \( c_0 \) and \( c_1 \) of \( Y \) being both distinct from the special points of \( Z \). There is an even permutation of \( \Omega_c \) that stabilises the partitions \( X \) and \( Z \), and maps \( \{c_0, c_1\} \) onto any other disjoint pair \( \{c'_0, c'_1\} \) (where \( c'_i \in C_i \)). Therefore, the number \( m \) of 2-cyclic partitions with cycle \((C_0, C_1)\) incident with \( B \) satisfies \( m \geq s^2 - 2s + 1 \). However, the flag-transitivity of \( G \) and the fact that \( G_{xB} \) acts as \( S_t \) on the \( t \) classes of \( X \) imply that \( m \) divides the total number \( s^2 \) of 2-cyclic partitions with cycle \((C_0, C_1)\), so \( m = s^2 \) since \( s \geq 3 \). Therefore the block \( B \) is incident with the whole orbit \( O_2 \) of \( G_x \) consisting of all 2-cyclic partitions, which implies that \( B \) is the only block incident with \( x \), and this is a contradiction. Therefore any two 2-cyclic partitions incident with \( B \) have a common special point.

If \( t \geq 3 \), then since \( G_{xB} \) acts as \( S_t \) on the \( t \) classes of \( X \), and since any two 2-cyclic partitions incident with \( B \) have a common special point, \( t = 3 \) and only one point \( c_i \) in each class \( C_i \) is a special point of some 2-cyclic partition incident with \( B \). However there is an even permutation of \( \Omega_c \) that preserves each of the classes \( C_0, C_1, C_2 \), fixing \( c_0 \) and \( c_1 \) but mapping \( c_2 \) onto any other point of \( C_2 \), preserving \( x \) and \( B \) but not \( \{c_0, c_1, c_2\} \), a contradiction. Therefore \( t = 2 \).

It follows that \( B \) is incident with only one partition, say \( Y \), with special points \( \{c_0, c_1\} \). If the size of \( C_0 \) and \( C_1 \) is greater than 3, then \( B \) is incident with some partition \( Z \) different from \( Y \) and \( X \), and there is an even permutation of \( \Omega_c \) which leaves \( X \) and \( Z \) invariant, but does not preserve \( \{c_0, c_1\} \), a contradiction. Therefore \( s = 3 \).

Hence \( c = 6 \), and since the points of \( D \) can be identified with the partitions of \( \Omega_6 \) into 2 classes of size 3, \( v = 10 \). However, there is no biplane with 10 points, a contradiction.

### 3.3 Case \((3)\)

Here first of all we mention that if \( G \cong S_c \) then \( G_x \not\cong A_c \), since \( [G : G_x] = v > 2 \). If the number \( k \) of blocks incident with a point is a power of 2, then \( v = [G : G_x] \) and \( (v, k) \leq 2 \) imply that the group \( G_x \) contains a subgroup
acting transitively on 2 or 4 points of Ωc, and fixing all other points, so by a theorem of Marggraf [21, Th.13.5], \( c \leq 8 \). Now \( v \) divides \( |G| \), so \( v \) must be a divisor of \( |S_c| \) for \( 5 \leq c \leq 8 \). The only possibilities such that \( v > 2 \) and \( 8v - 7 \) is a square are \( v = 4, 16, \) and \( 56 \). Since we had assumed the biplane to be non-trivial and to have parameters different to \((16,6,2)\), we immediately rule out \( v = 4 \) or \( 16 \), and \( v = 56 \) forces \( k = 11 \), a contradiction.

If \( k \) is not a power of 2, then let \( p \) be an odd prime divisor of \( k \), so \( p \) divides \( |G_x| \). Since \( v = [G : G_x] \) and \( (k, v) \leq 2 \), \( G_x \) contains a Sylow \( p \)-subgroup of \( G \), and so \( G_x \) acting on \( \Omega_c \) contains an even permutation with exactly one cycle of length \( p \) and \( c - p \) fixed points. By a result of Jordan [21, Th. 13.9], the primitivity of \( G_x \) on \( \Omega_c \) yields \( c - p \leq 2 \), that is \( c - 2 \leq p \leq c \). This implies that \( p^2 \) does not divide \( |G| \), so \( p^2 \) does not divide \( k \). Therefore either \( k \) is a prime, namely \( c - 2, \) \( c - 1, \) or \( c \), or the product of two twin primes, namely \( c(c - 2) \). On the other hand, \( k^2 > v \), and a result of Bochert [21, Th. 14.2], implies that \( v \geq \frac{c + 2}{2} \). From this and the previous conditions on \( k \), the possibilities are \( c = 13(k = 11 \cdot 13), 8, 7, 6, \) or \( 5 \).

If \( c = 13 \), then \( k = 143 \), so \( k(k - 1) = 2(v - 1) \) forces \( v = 10154 \). But if \( v \) is even, then \( k - 2 = 141 \) must be a square (by a theorem of Schützenberger [20]), however 141 is not a square, which is a contradiction.

As we have seen earlier in this proof, for \( 5 \leq c \leq 8 \) the only possibility is the \((56,11,2)\) biplane, which cannot happen given the above conditions on \( k \).

We now consider the case \( c = 6 \), and \( G \cong M_{10}, PGL_2(9), \) or \( PΓL_2(9) \). Checking the divisors of \( 2^2|A_6| \), the only possibilities for \( v \) such that \( 8v - 7 \) is a square are \( v = 4 \) and \( 16 \), which have been already ruled out.

This completes the proof of Lemma 6, and hence \( X \) is not an alternating group.

4 The Case in which \( X \) is a Sporadic Group

Here we consider \( X \) to be a sporadic group.

**Lemma 7** If \( D \) is a non-trivial biplane with a flag-transitive, primitive, almost simple automorphism group \( G \), then \( \text{Soc}(G) = X \) is not a sporadic group.

**Proof.** The way we proceed is as follows: We assume that the automorphism group \( G \) of \( D \) is almost simple, such that \( X \leq G \leq \text{Aut}X \) with \( X \) a
sporadic group. Then \( G = X \), or \( G = \text{Aut} X \), since for all sporadic groups \( X \) either \( \text{Aut} X = X \) or \( \text{Aut} X = 2 \cdot X \). We know that \( v = [G : G_x] \), and \( G_x \) is a maximal subgroup of \( G \). The lists of maximal subgroups of \( X \) and \( \text{Aut} X \) appear in [5,13,14,16]. (They are complete except for the 2-local subgroups of the Monster group). For each sporadic group (and its automorphism group), we rule out the maximal subgroups the order of which is too small to satisfy \( 2|G| < |G_x|^3 \). In the remaining cases, for those \( v > 2 \), we check if \( 8v - 7 \) is a square, or if \( 2 \left( |G_x| \right)^2 > v \) (by \( |G_x|_{v'} \) we mean the part of \( |G_x| \) coprime to \( v \)). If this does happen, we check the remaining arithmetic conditions \( (k - 2) \), is a square if \( v \) is even, \( k(k - 1) = 2(v - 1) \).

To illustrate this procedure, suppose \( X = J_1 \). Then \( G = J_1 \), since \( |\text{Out} J_1| = 1 \). The maximal subgroups \( H \) of \( J_1 \), with their orders and indices are as follows:

- \( L_2(11) \), of order 660, \( v = 266 \),
- \( 2^3.7.3 \), of order 168, \( v = 1045 \),
- \( 2 \times A_5 \), of order 120, \( v = 1463 \),
- \( 19 : 6 \), of order 114, \( v = 1540 \),
- \( 11 : 10 \), of order 110, \( v = 1596 \),
- \( D_6 \times D_{10} \), of order 60, \( v = 2926 \), and
- \( 7 : 6 \), of order 42.

In the last case, the order of the group is too small to satisfy \( |G_x|^3 > 2|G| \), and in all the remaining cases \( 8v - 7 \) is not a square.

Proceeding in the same manner with the other sporadic groups, the only cases in which these arithmetic conditions are met are the following:

1. \( G = M_{23} \), \( G_x = 2^4 : (A_5 \times 3) : 2 \), \( (v,k) = (1771,60) \).
2. \( G = M_{24} \), \( G_x = 2^6 : (3 \cdot S_6) \), \( (v,k) = (1771,60) \).

In the first case the subdegrees of \( M_{23} \) on \( 2^4 : (A_5 \times 3) : 2 \) are 1, 60, 480, 160, 90, and 20 (calculated with GAP [9], my sincere thanks to A.A. Ivanov and D. Pasechnik), but 30 does not divide 20, contradicting the fact that \( k \) must divide twice every subdegree.

In the second case, the subdegrees are 1,90, 240, and 1440 [12, pp.126], however \( M_{24} \) has only one conjugacy class of subgroups of index 1771 [5], so if \( x \) is a point and \( B \) is a block \( G_x \) is conjugate to \( G_B \), so \( G_x \) fixes a block, say, \( B_0 \). But \( x \) cannot be incident with \( B_0 \) since the flag-transitivity of \( G \) implies that \( G_x \) is transitive on the \( k \) blocks incident with \( x \). Hence \( x \) and \( B_0 \) are not incident, and so some of the points incident \( B_0 \) form a \( G_x \)-orbit, which is a contradiction since the smallest non-trivial \( G_x \)-orbit has size 90, and \( B_0 \) is incident with 60 points.
This completes the proof of Lemma 7, and hence $X$ is not a sporadic group.

The proof of our Main Theorem is now complete.

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