Chiral polyhedra and projective lines

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ABSTRACT

In this paper we give a new proof that that there are no chiral polyhedra whose automorphism group is isomorphic to $\mathrm{PSL}(2,q)$ *or* $\mathrm{PGL}(2,q)$*. Our proof uses the sharp 3-transitive action of* $\mathrm{PGL}(2,q)$ *on the projective line* $\mathrm{PG}(1,q)$*.*

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1 Introduction

Abstract polytopes, which generalise convex polytopes, were introduced in the 1970's as 'incidence polytopes'. In (McMullen-Schulte, 2002) McMullen and Schulte developed in great detail the theory of abstract regular polytopes. Regular polytopes have been much more studied, but in recent years there have been strong developments on chiral polytopes. In particular, people got interested in studying whether regularity or chirality is more prevalent. For some of them, the approach has been to determine whether a given group G is the full automorphism group of chiral and/or regular polytopes. Another relevant question is to determine the maximal rank such a polytope can have.

In this paper we focus on projective linear groups and chiral polytopes of rank 3, also called chiral polyhedra. We assume G is either $\mathrm{PGL}(2,q)$ or $\mathrm{PSL}(2,q)$. We prove, using the sharp 3-transitive action of $\mathrm{PGL}(2,q)$ on a projective line $\mathrm{PG}(1,q)$, that G is not the full automorphism group of a chiral polyhedron. Polyhedra are regular maps. Regular maps are a special

class of hypermaps, namely those with a linear Coxeter diagram. In this context, Sah enumerated in (Sah, 1969) orientably regular hypermaps with automorphism groups isomorphic to $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$. Singerman used a result of Macbeath (Macbeath, 1969) to prove in (Singerman, 1974, Theorem 3) that every regular map for $\mathrm{PSL}(2,q)$ is reflexible. Later, Conder, Potočnik and Širáň in (Conder et al., 2008) extended this result to reflexible hypermaps in both orientable and non-orientable surfaces. In (Conder et al., 2008, Section 6), they obtained that every regular map for $\mathrm{PGL}(2,q)$ is also reflexible. Their methods are algebraic involving matrix groups acting on hypermaps, whereas we give a more geometric proof in the context of abstract polytopes (finite abstract polytopes of rank 3 are hypermaps, but not necessarily vice-versa). The methods in our proof allow us to consider the more general case in which G is an almost simple group with socle $\mathrm{PSL}(2,q)$, and they are used in ongoing work involving such groups and chiral polytopes of higher rank, thus extending the main theorem in this paper in terms both of the groups involved and of the ranks of the polytopes, towards solving two conjectures stated below.
In Section 2 we give general definitions and background on abstract polytopes. In Section 3 we give some known experimental results and state conjectures based on this information, and in Section 4 we give our proof for the non-existence of chiral polyhedra with full automorphism group isomorphic to $\mathrm{PGL}(2,q)$ or $\mathrm{PSL}(2,q)$.

2 Abstract polytopes

For detailed definitions and results on abstract polytopes (regular and chiral), we refer the reader to (McMullen-Schulte, 2002) and (Schulte-Weiss, 1994). Here we only consider polytopes of rank 3, hence we give our definitions in the rank three case.
An abstract polyhedron $(\mathcal{P},\leq)$ (or *polyhedron* for short) is a partially ordered set whose elements are called faces. It has a strictly monotone rank function $\mathrm{rank}:\mathcal{P}\to\{-1,0,1,2,3\}$, a unique face F_{-1} corresponding to the empty set and a unique face F_3 corresponding to $\mathcal{P}$. A *flag* of $(\mathcal{P},\leq)$ is a maximal chain (which always has 5 faces, one of each rank). Two flags F and F' of $(\mathcal{P},\leq)$ are *adjacent* provided they have 4 common faces. A polyhedron $(\mathcal{P},\leq)$ is *strongly connected*, meaning that given any two flags F and F' of $(\mathcal{P},\leq)$, there exists a sequence of flags $F=:F_0,F_1,\dots F_n:=F'$ such that F_i and F_{i+1} are adjacent ($i=0,\dots,n-1$). A polyhedron $(\mathcal{P},\leq)$ satisfies the *diamond condition*, meaning that for any two faces $F\leq G$ of $(\mathcal{P},\leq)$ with $\mathrm{rank}(F)=\mathrm{rank}(G)-2$, there are exactly two elements H and H' of rank $\mathrm{rank}(F)+1$ such that $F\leq H,H'\leq G$. We will usually write $\mathcal{P}$ instead of $(\mathcal{P},\leq)$. Since we require $\mathcal{P}$ to be strongly flag connected and satisfy the diamond condition, any automorphism of $\mathcal{P}$ is determined by its action on a given flag. If $\mathcal{P}$ is regular (that is, its automorphism group is transitive on the set of flags), then fixing a *base flag* Φ yields a set $\{\rho_0,\rho_1,\rho_2\}$ of involutions which generate $\Gamma(\mathcal{P}):=\mathrm{Aut}(\mathcal{P})$, where for each $i=0,1,2$, the involution ρ_i maps Φ to its unique i-adjacent flag, that is the unique flag distinct from Φ whose j-elements with $j\neq i$ are the same as those of Φ. The *rotation subgroup* $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$ is the subgroup consisting of words of even length of ρ_0,ρ_1,ρ_2, namely $\Gamma^+(\mathcal{P})=\langle\rho_0\rho_1,\rho_1\rho_2\rangle$. Obviously, $\Gamma^+(\mathcal{P})$ has index at most 2 in $\Gamma(\mathcal{P})$. Let $\sigma_i:=\rho_{i-1}\rho_i$ ($i=1,2$). We have $(\sigma_1\sigma_2)^2=1_{\Gamma(\mathcal{P})}$. If $\Gamma^+(\mathcal{P})$ has index 2 in $\Gamma(\mathcal{P})$ then we say that

G	Rank 3	Rank 4		G	Rank 3	Rank 4
$PSL_2(4)$	0	0		$PGL_2(4)$	0	0
$PSL_2(5)$	0	0		$PGL_2(5)$	0	6
$PSL_2(7)$	0	0		$PGL_2(7)$	0	10
$PSL_2(8)$	0	2		$PGL_2(8)$	0	2
$PSL_2(9)$	0	0		$PGL_2(9)$	0	2
$PSL_2(11)$	0	0		$PGL_2(11)$	0	24
$PSL_2(13)$	0	6		$PGL_2(13)$	0	14
$PSL_2(16)$	0	2		$PGL_2(16)$	0	2
$PSL_2(17)$	0	10		$PGL_2(17)$	0	8
$PSL_2(19)$	0	4		$PGL_2(19)$	0	28
$PSL_2(23)$	0	0		$PGL_2(23)$	0	10
$PSL_2(25)$	0	2		$PGL_2(25)$	0	2
$PSL_2(27)$	0	0		$PGL_2(27)$	0	4
$PSL_2(29)$	0	10		$PGL_2(29)$	0	26
$PSL_2(31)$	0	6		$PGL_2(31)$	0	46
$PSL_2(32)$	0	6		$PGL_2(32)$	0	6

Table 1: Number of abstract chiral polytopes for $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$

$\mathcal{P}$ is *directly regular*. In this case there is an involution in $\mathrm{Aut}(\mathcal{P})$ inverting σ_1 and σ_2.
A polyhedron $\mathcal{P}$ is *chiral* if its automorphism group has two orbits on the set of flags and adjacent flags lie in different orbits. In this case there is a generating set $\{\sigma_1, \sigma_2\}$ of $\Gamma(\mathcal{P})$ satisfying the same conditions as those of the rotation subgroup of a directly regular polytope, but there is no involution in $\mathrm{Aut}(\Gamma(\mathcal{P}))$ inverting σ_1 and σ_2.

3 Experimental results and conjectures

Table 1 is extracted from (Hartley et al., 2012). It lists the number of abstract chiral polytopes associated to the groups $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ for $4 \leq q \leq 32$. It suggests the following conjectures:

Conjecture 3.1. No abstract chiral polyhedron has $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ as its automorphism group.

Conjecture 3.2. The maximal rank of an abstract chiral polytope whose automorphism group is either $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ is 4.

Conjecture 3.3. For each $q \geq 5$, there exists at least one abstract chiral polytope of rank four having $\mathrm{PGL}(2,q)$ as automorphism group.

Conjecture 3.1 is known to be true already as we mentioned in the introduction (see (Conder et al., 2008)). However, the proof given by Conder et al. uses matrix groups and linear algebra. Hence, it is hard to use their techniques to study groups $\mathrm{PSL}(2,q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}(2,q))$. We give in Section 4 a proof based on the sharply 3-transitive action of $\mathrm{PGL}(2,q)$ on a projective line $\mathrm{PG}(1,q)$.

4 The rank three case

Given a group G and two generators σ_1 and σ_2 of G, the group G is the full automorphism group of an abstract chiral polyhedron of type $\{o(\sigma_1), o(\sigma_2)\}$ provided that the following four conditions hold.

(C1) $\langle \sigma_1, \sigma_2 \rangle = G$

(C2) $(\sigma_1\sigma_2)^2 = 1_G$

(C3) $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = 1_G$

(C4) There does not exist $\alpha \in \mathrm{Aut}(G) : \alpha((\sigma_1, \sigma_2)) = (\sigma_1^{-1}, \sigma_2^{-1})$

All chiral polyhedra having G as full automorphism group are obtained by looking at pairs of elements $\{\sigma_1, \sigma_2\}$ that satisfy these four conditions. The main theorem of this article is as follows.

Theorem 4.1. *No abstract chiral polyhedron has full automorphism group isomorphic to either a* $\mathrm{PSL}(2, q)$ *or* $\mathrm{PGL}(2, q)$ *group.*

We first recall some basic results related to the projective linear groups $\mathrm{PGL}(2, q)$ and $\mathrm{PSL}(2, q)$, that will be used to prove Theorem 4.1. The proof of these results is left to the interested reader.

Lemma 4.2. *Let* $G \cong \mathrm{PGL}(2, q)$ *be a permutation group acting on the* $q+1$ *points of a projective line* $\mathrm{PG}(1, q)$*, with* $q = p^n$*,* p *a prime and* n *a positive integer.*

1. *If* $g \in G$ *is such that* $o(g) \mid q$*, then* $o(g) = p$*. Moreover, all elements of order* p *in* $\mathrm{PGL}(2, q)$ *are conjugate and inside the unique subgroup of* G *isomorphic to* $\mathrm{PSL}(2, q)$*.*
2. *For* $g \in G$ *with* $g^2 \neq 1_G$*,*
   - *(a)* $o(g) \mid q$ *if and only if* $|\mathrm{fix}(g)| = 1$*;*
   - *(b)* $o(g) \mid q - 1$ *if and only if* $|\mathrm{fix}(g)| = 2$*;*
   - *(c)* $o(g) \mid q + 1$ *if and only if* $|\mathrm{fix}(g)| = 0$*;*
3. *If* $\alpha \in \mathrm{PGL}(2, q)$ *swaps two distinct points of* $\mathrm{PG}(1, q)$ *then* α *is an involution.*

For the case where σ_i fixes exactly one point, we can go further by counting the number of distinct involutions which invert σ_i. These involutions act as candidates for counter examples to chirality condition (C4).

Lemma 4.3. *If* $\sigma_1 \in \mathrm{PGL}(2, q)$ *with* $o(\sigma_1) = p$*, then there are exactly* q *involutions of* $\mathrm{PGL}(2, q)$ *which invert* σ_1 *by conjugation.*

Proof. Let $\sigma_1 \in \mathrm{PGL}(2, q)$, such that $o(\sigma_1) \mid q$. Then by Lemma 4.2 (2)(a), σ_1 fixes one point, say P_1. By Lemma 4.2 (1), σ_1 has order p and all elements of order p are conjugate in $\mathrm{PGL}(2, q)$. So we can fix σ_1 to be the mapping $\sigma_1 : \mathrm{PG}(1, q) \to \mathrm{PG}(1, q) : x \mapsto x + 1$. It has inverse $\sigma_1^{-1} : x \mapsto x - 1$. We now define $I := \{\alpha \in \mathrm{PGL}(2, q) \mid \sigma_1^{\alpha} = \sigma_1^{-1}, o(\alpha) = 2\}$. Any α which inverts σ_1 must also fix P_1, so $\alpha \in \mathrm{PGL}(2, q)_{P_1} \cong \mathrm{AGL}(1, q)$. The form of α is then

$\alpha : x \mapsto ax+b$, and since we want α to be an involution, $\alpha^2(x) = a(ax+b)+b = a^2x+ab+b = x$. Equating coefficients, we see that $a = \pm 1$, and $ab + b = 0$. If $a = 1$ then $b = 0$, so α is the identity, and thus not an involution. However, if $a = -1$ then any $b \in \mathrm{GF}(q)$ gives an involution $\alpha : x \mapsto -x+b$. We require α to invert σ_1 by conjugation, so $\sigma_1^\alpha = -(-x+b+1)+b = \sigma_1^{-1} = x-1$, which holds for any $b \in \mathrm{GF}(q)$. Since there are q choices for b, $|I| = q$. i.e. there are q distinct $\alpha \in \mathrm{PGL}(2, q)$ which invert σ_1 by conjugation. □

We will show that for any σ_1 and σ_2 in G that satisfy the chirality conditions (C1), (C2) and (C3) above, we can always find an $\alpha \in \mathrm{PGL}(2, q)$ which makes condition (C4) fail. Each of σ_1 and σ_2 can fix between 0 and 2 points. Hence, in terms of fixed points, there are 9 possible ordered pairs $[\sigma_1, \sigma_2]$. We also know that the order in which we choose σ_1 and σ_2 is irrelevant. One can easily see that $\langle\sigma_1, \sigma_2\rangle = \langle\sigma_2, \sigma_1\rangle$, $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \langle\sigma_2\rangle \cap \langle\sigma_1\rangle$ and $\alpha([\sigma_1, \sigma_2]) = [\sigma_1^{-1}, \sigma_2^{-1}]$ if and only if $\alpha([\sigma_2, \sigma_1]) = [\sigma_2^{-1}, \sigma_1^{-1}]$. So it remains to show that condition (C2) is symmetric. Indeed, assuming $(\sigma_1\sigma_2)^2 = 1_G$ and letting $(\sigma_2\sigma_1)^2 = x$, then by expansion and left multiplication we can see that $\sigma_1(\sigma_2\sigma_1\sigma_2\sigma_1) = \sigma_1 x$, but since $(\sigma_1\sigma_2)^2 = 1_G$, it follows that $\sigma_1 = \sigma_1 x$, and hence $x = 1_G$. This leaves 6 cases. However we shall prove the cases where o$(\sigma_i) \mid q \pm 1$ simultaneously. So it suffices to consider the following 3 main cases.

1. $o(\sigma_1) \mid q \pm 1$ and o$(\sigma_2) \mid q \pm 1$
2. $o(\sigma_1) \mid q \pm 1$ and o$(\sigma_2) \mid q$
3. $o(\sigma_1) \mid q$ and o$(\sigma_2) \mid q$

We will first consider the case where both σ_1 and σ_2 fix either 0 or 2 points, that is $o(\sigma_1) \mid q \pm 1$ and $o(\sigma_2) \mid q \pm 1$.

Lemma 4.4. *Let G be isomorphic to either $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, with $q \geq 4$. Let $\sigma_1, \sigma_2 \in G$ be elements that satisfy conditions (C1), (C2) and (C3). If $o(\sigma_1) \mid q \pm 1$ and $o(\sigma_2) \mid q \pm 1$, then there exists $\alpha \in \mathrm{PGL}(2, q)$ such that $\sigma_1^\alpha = \sigma_1^{-1}$ and $\sigma_2^\alpha = \sigma_2^{-1}$ and hence condition (C4) fails.*

Proof. Note that $(q^2 - 1) = (q + 1)(q - 1)$, so necessarily $o(\sigma_1)$ and $o(\sigma_2)$ both divide $(q^2 - 1)$. Thus, we may view σ_1, σ_2 as elements of $\mathrm{PGL}(2, q^2)$ acting on $\mathrm{PG}(1, q^2)$ and G as the stabiliser of a subline $\mathrm{PG}(1, q)$ of $\mathrm{PG}(1, q^2)$. Both σ_1 and σ_2 then fix 2 points of $\mathrm{PG}(1, q^2)$ by Lemma 4.2 (2)(b). Since G is not a subgroup of the stabiliser of a point in $\mathrm{PGL}(2, q^2)$, we can assume that $\mathrm{fix}(\sigma_1) \cap \mathrm{fix}(\sigma_2) = \emptyset$.
Define $A := N_{\mathrm{PGL}(2,q^2)}(\langle\sigma_1\rangle)$ and $B := N_{\mathrm{PGL}(2,q^2)}(\langle\sigma_2\rangle)$. Then $A \cong B \cong D_{2(q^2-1)}$. As dihedral groups, A and B have cyclic subgroups of index 2, say A' and B', each of which fixes the same 2 points as σ_1 and σ_2 respectively. The intersection $A' \cap B'$ therefore fixes 4 points, which forces it to be the identity as $\mathrm{PGL}(2, q^2)$ is sharply 3-transitive on $\mathrm{PG}(1, q^2)$. If $H := A \cap B$, then H must consist solely of involutions since $A' \cap B' = \{1_G\}$. Let $\mathrm{fix}(\sigma_1) := \{s, t\}$ and $\mathrm{fix}(\sigma_2) := \{u, v\}$. By the 3-transitivity of $\mathrm{PGL}(2, q^2)$, we can choose an $\alpha \in \mathrm{PGL}(2, q^2)$ such that $\alpha(s) = t, \alpha(t) = s$ and $\alpha(u) = v$. By Lemma 4.2 (3), α must then be an involution, and so $\alpha(v) = u$. Moreover, $\alpha \neq 1_G$ must be in H as it interchanges s and t and it also interchanges u and v. Therefore $H \geq C_2$. Also, $\alpha \in \mathrm{PGL}(2, q^2)$ must then invert both σ_1, σ_2 as it is a non-central involution of

both A and B, so it remains to show that α is in the unique subgroup $\mathrm{PGL}(2,q)$ containing G in $\mathrm{PGL}(2,q^2)$.
We know $\alpha \in \mathrm{PGL}(2,q^2)$ is such that $\sigma_1^\alpha = \sigma_1^{-1}$ and $\sigma_2^\alpha = \sigma_2^{-1}$, so $\alpha \in N_{\mathrm{PGL}(2,q^2)}(\langle\sigma_1,\sigma_2\rangle^\alpha)$. However, since $\langle\sigma_1,\sigma_2\rangle^\alpha = \langle\sigma_1^\alpha,\sigma_2^\alpha\rangle = \langle\sigma_1^{-1},\sigma_2^{-1}\rangle = \langle\sigma_1,\sigma_2\rangle$, we have that $\alpha \in N_{\mathrm{PGL}(2,q^2)}(\langle\sigma_1,\sigma_2\rangle) =: N$. By looking at the subgroup structure of $\mathrm{PGL}(2,q^2)$ (see (Cameron et al., 2006) for instance), one readily sees that N must be isomorphic to $\mathrm{PGL}(2,q)$. Since $\alpha \in N$, it is clear that α is in the unique $\mathrm{PGL}(2,q)$ that contains $\langle\sigma_1,\sigma_2\rangle$. Hence α contradicts condition (C4). □

We now prove the case where σ_1 and σ_2 both fix exactly 1 point of $\mathrm{PG}(1,q)$.

Lemma 4.5. *Let G be either $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, with $q \geq 4$. Let $\sigma_1,\sigma_2 \in G$ be elements that satisfy conditions (C1), (C2) and (C3). If $o(\sigma_1) \mid q$ and $o(\sigma_2) \mid q$, then there exists $\alpha \in \mathrm{PGL}(2,q)$ such that $\sigma_1^\alpha = \sigma_1^{-1}$ and $\sigma_2^\alpha = \sigma_2^{-1}$ and hence condition (C4) fails.*

Proof. By Lemma 4.2 (1), we know that σ_i has order p, and thus it fixes exactly one point P_i ($i = 1,2$). Also $P_1 \neq P_2$ in order to have $(C1)$. By Lemma 4.3, there are q distinct involutions $\alpha \in \mathrm{PGL}(2,q)$ with $\sigma_1^\alpha = \sigma_1^{-1}$.
We can assume that q is odd, for if it were even then $o(\sigma_1) = 2$. Then $G = \langle\sigma_1,\sigma_2\rangle = \langle\sigma_1,\sigma_1\sigma_2\rangle$, the latter being a group generated by two involutions. Hence G is a dihedral group, not an almost simple group of type $\mathrm{PSL}(2,q)$, and $q \leq 3$, a contradiction.
Define $J := \{\sigma_2 \in \mathrm{PGL}(2,q) \mid o(\sigma_2) = p, o(\sigma_1\sigma_2) = 2\}$. As $P_1 \neq P_2$, there are then $q+1-1 = q$ choices for the point P_2. Since σ_1 only fixes P_1, there exist $a, b \in \mathrm{PG}(1,q)$ with $\sigma_1(b) = P_2$ and $\sigma_1(P_2) = a$. There are $q-1$ elements of order p in G_{P_2}, but by the sharp 3-transitivity of $\mathrm{PGL}(2,q)$, only one of them sends a to b, a necessary condition for $\sigma_1\sigma_2$ to be an involution. There is then a unique $\sigma_2 \in J$ for each of the q choices of P_2, so we can conclude that $|J| = q$.
By Lemma 4.3, there are exactly q involutions which invert σ_1 in $\mathrm{PGL}(2,q)$. Take α that fixes P_1 and inverts σ_1 by conjugation, so as in the proof of Lemma 4.3,

$$\sigma_1 : \mathrm{PG}(1,q) \to \mathrm{PG}(1,q) : x \mapsto x+1$$

and

$$\alpha : \mathrm{PG}(1,q) \to \mathrm{PG}(1,q) : x \mapsto -x+b.$$

Now, α fixes another point as $\alpha(x) = -x+b = x$ if and only if $x = 2^{-1}b$. Note that the assumption that the characteristic is odd implies this point is unique. Hence, given $\sigma_1 \in \mathrm{PGL}(2,q)$ and $P_2 \in \mathrm{PG}(1,q)$, there is a unique $\alpha \in \mathrm{PGL}(2,q)$ which fixes both P_1, P_2 and inverts σ_1, and a unique $\sigma_2 \in \mathrm{PGL}(2,q)$ fixing P_2 in $\mathrm{PG}(1,q)$ with $(\sigma_1\sigma_2)^2 = 1_G$. We know that $\sigma_2^\alpha(P_2) = \sigma_2^{-1}(P_2) = P_2$ and $\sigma_2^\alpha(b) = \sigma_2^{-1}(b) = a$, and that σ_2^α is of order p. Hence we can conclude that σ_2^α is equal to σ_2^{-1}. Therefore, if σ_1, σ_2 satisfy chirality conditions (C1), (C2) and (C3), then there exists $\alpha \in \mathrm{Aut}(G)$ that inverts them both and condition (C4) is not satisfied. □

We now combine the techniques used in the first two cases to prove the final case where σ_1 fixes exactly 1 point, and σ_2 fixes either 0 or 2 points.

Lemma 4.6. *Let G be either $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, with $q \geq 4$. Let $\sigma_1,\sigma_2 \in G$ be elements that satisfy conditions (C1), (C2) and (C3). If $o(\sigma_1) \mid q$ and $o(\sigma_2) \mid q \pm 1$, then there exists $\alpha \in \mathrm{PGL}(2,q)$ such that $\sigma_1^\alpha = \sigma_1^{-1}$ and $\sigma_2^\alpha = \sigma_2^{-1}$ and hence condition (C4) fails.*

Proof. Let G be either $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, with $q \geq 4$, $\sigma_1 \in G$ be an element with order dividing q. By Lemma 4.2 (1), σ_1 has order p, so σ_1 fixes one point, say P_3. Let $\sigma_2 \in G$ be an element with order dividing $q \pm 1$, then as with Lemma 4.4, $o(\sigma_2) \mid q^2 - 1$, so we consider σ_1, σ_2 as elements of $\mathrm{PGL}(2,q^2)$, acting on the projective line $\mathrm{PG}(1,q^2)$. In this case, σ_2 fixes two points, P_1 and $P_2 \in PG(1,q^2)$, both distinct from P_3. Using the sharp 3-transitivity of $\mathrm{PGL}(2,q^2)$ on $\mathrm{PG}(1,q^2)$, we can choose $\alpha \in \mathrm{PGL}(2,q^2)$ such that $\alpha(P_3) = P_3$, $\alpha(P_1) = P_2$ and $\alpha(P_2) = P_1$. Note that since P_1 and P_2 are swapped by α, it must be an involution by Lemma 4.2 (3). Additionally, σ_1 is in the stabiliser of the point P_3, so it is in $\mathrm{PGL}(2,q^2)_{P_3} \cong \mathrm{AGL}(1,q^2)$. By the same argument used in the proof of Lemma 4.5, α necessarily inverts σ_1. Now, σ_1 only fixes P_3, so there exist $a, b \in \mathrm{PG}(1,q^2)$ with $\sigma_1(a) = P_2$ and $\sigma_1(P_2) = b$. We require $(\sigma_1\sigma_2)^2 = 1_G$, so σ_2 must map b to a. Since α is an inner automorphism it preserves the order of elements, specifically $(\sigma_1\sigma_2)^\alpha$ is an involution, hence $(\sigma_1\sigma_2)^\alpha(a) = P_2$. We know that $\sigma_1^\alpha(b) = P_2$ and $\sigma_1^\alpha(P_2) = a$ as α inverts σ_1. Moreover, $\sigma_2^\alpha(P_2) = P_2$. Hence $\sigma_2^\alpha(a) = \sigma_1(\sigma_2^\alpha)^{-1}\sigma_1(a) = \sigma_1(\sigma_2^\alpha)^{-1}(P_2) = \sigma_1(P_2) = b$. By definition of α, we also have $\sigma_2^\alpha(P_1) = P_1$ and $\sigma_2^\alpha(P_2) = P_2$. So by the sharp 3-transitivity of $\mathrm{PGL}(2,q^2)$, $\sigma_2^\alpha = \sigma_2^{-1}$. There is then an $\alpha \in \mathrm{PGL}(2,q^2)$ which inverts σ_1, σ_2, but by the same reasoning as in the proof of Lemma 4.4, α is also in the unique subgroup $\mathrm{PGL}(2,q)$ containing G. Therefore, if σ_1, σ_2 satisfy chirality conditions (C1), (C2) and (C3), condition (C4) is not satisfied. □

Proof of Theorem 4.1. Combining the 3 lemmas above, we get the proof for $q \geq 4$. The cases $\mathrm{PSL}(2,2) \cong S_3$, $\mathrm{PSL}(2,3) \cong A_4$, and $\mathrm{PGL}(2,3) \cong S_4$ are straightforward to check by hand. □

5 Future work

The techniques developed in this article have been pushed further by Leemans and Moerenhout in order to study almost simple groups with socle $\mathrm{PSL}(2,q)$ (Leemans-Moerenhout, In preparation). Conjectures 3.2 and 3.3 will be addressed in a joint paper of Leemans, Moerenhout and O'Reilly-Regueiro (Leemans et al., In preparation).

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