Reduction for primitive flag-transitive $(v,k,4)$-symmetric designs.

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Abstract

It has been shown that if a (v,k,λ)-symmetric design with $\lambda \leq 3$ admits a flag-transitive automorphism group G which acts primitively on points, then G must be of affine or almost simple type. Here we extend the result to $\lambda = 4$.

1 Introduction

A $(v,k,\lambda)-$*symmetric design* is an incidence structure consisting of a set P of points, a set B of blocks (which can be thought of as subsets of P) and an incidence relation such that:

- $|P| = |B|$,
- Every block is incident with exactly k points, and
- Every pair of points is incident with exactly λ blocks.

An *automorphism* of a symmetric design is a permutation of the points which also permutes the blocks. A *flag* in a symmetric design is an incident point-block pair. An automorphism group G acting on a symmetric design D is *flag-transitive* if it acts transitively on the set of flags of D, and it is *point-primitive* if it leaves no non-trivial partition of the points invariant.

The O'Nan-Scott Theorem classifies primitive groups into the following five types (for a more detailed description, see [3]):

1. Affine
2. Almost simple
3. Simple diagonal
4. Product
5. Twisted wreath

In [1] it was shown that if a $(v,k,1)$-symmetric design admits a flag-transitive automorphism group G which acts point-primitively on points, then G can only be of affine or almost simple type. The case when $\lambda = 2$ or 3 was done in [2]. In this paper using the same procedures we go one step further proving the same result for λ=4, namely:

Theorem 1. *If D is a $(v,k,4)$-symmetric design admitting a flag-transitive point-primitive automorphism group G, then G is of affine or almost simple type.*

2 The Proof

We start this section with two lemmas that will be useful in our proof.

Lemma 1. *If D is a (v,k,λ)-symmetric design, then $\lambda(v-1) = k(k-1)$.*

Lemma 2. *If D is a (v,k,λ)-symmetric design, then $4\lambda(v-1)+1$ is a square.*

Proof. From the previous lemma, $k^2 - k - \lambda(v-1) = 0$, that is, $k = \frac{1+\sqrt{1+4\lambda(v-1)}}{2}$, and this must be an integer, hence the result. □

We will now assume that we have a (v,k,λ)-symmetric design which admits a flag-transitive point-primitive automorphism group G.

First suppose G has a product action on the set P of points. Then there is a group H acting primitively on Γ (with $|\Gamma| = m \geq 5$) of almost simple or diagonal type, where:

$P = \Gamma^l$, $G \leq H^l \rtimes S_l = H_{\mathrm{wr}} S_l$, and $l \geq 2$. By [2, Lemma 4], k divides $\lambda l(m-1)$, $v = m^l \leq \lambda l^2 (m-1)^2$, and $l > 2$ if $\lambda \leq 4$.

Lemma 3. *If D is a $(v,k,4)$-symmetric design admitting a flag-transitive, point-primitive automorphism group G, then G does not have a non-trivial product action nor a twisted wreath action on the points of D.*

Proof. Suppose G has a non-trivial product action. Since $m^l \leq \lambda l^2(m-1)^2$ and $m \geq 5$, we conclude $l \leq 4$. By [2, Lemma 4] $\lambda = 4$ implies $l > 2$, and so $l = 3$ or 4.

First assume $l = 4$. Then $m^4 \leq 64(m-1)^2$, which forces $m \leq 6$. We check that for none of these possibilities $16v - 15$ is a square, contradicting Lemma 2.

Now assume $l = 3$.

The inequality $m^3 \leq 36(m-1)^2$ forces $m < 36$. Since k divides $12(m-1)$ and $4(m^3-1)$, k divides $4(m-1)(3, 1+m+m^2)$. If $m \equiv 1 \pmod 3$ then $(3, 1+m+m^2) = 3$, otherwise $(3, 1+m+m^2) = 1$, and k divides $4(m-1)$, contradicting $k^2 > 4(m^3-1)$. Hence we need only consider $m = 7$, 10, 13, 16, 19, 22, 25, 28, 31, and 34. The only case in which $16v - 15$ is a square is $m = 34$. However k divides $12(m-1) = 2^2 \cdot 3^2 \cdot 11$, and $k(k-1) = 4(m^3-1) = 2^2 \cdot 3^2 \cdot 11 \cdot 397$, which is impossible.

Therefore G does not have a non-trivial product action on the points of D.

Since groups with a twisted wreath action are contained in twisted wreath groups $H \mathrm{wr} S_l$ with a product action and H of diagonal type, and here we have also considered subgroups of G, we have thereby also ruled out groups with a twisted wreath action. □

Lemma 4. *If D is a $(v,k,4)$-symmetric design admitting a flag-transitive, point-primitive automorphism group G, then G is not of simple diagonal type.*

Proof. If G is of simple diagonal type, then $\mathrm{Soc}(G) = N = T^m$, with $m \geq 2$, for some non-abelian simple group T, where $T \cong N_x \lhd G_x \leq \mathrm{Aut}\, T \times S_m$.

Here $v = |T|^{m-1} = |N_x|^{m-1}$.

The flag-transitivity of G implies G_x is transitive on the k blocks incident with x, and since $N_x \lhd G_x$ the orbits of N_x on these k blocks all have the same size, say s. Therefore s divides k, so it divides $\lambda(v-1)$, and it also divides $|N_x| = |T|$, so $s | (|T|, \lambda(|T|^{m-1} - 1))$, that is, s divides $\lambda = 4$.

Since N_x is non-abelian simple, its order is divisible by at least three primes, so there is an odd prime $p \geq 5$ that divides the order of N_x. Take $t \in N_x$ of order p, and a point y moved by t.

Since $s \leq 4$ t fixes the k blocks through x, and since it moves y, the 4 blocks incident with $\{x, y\}$ must necessarily contain the t-orbit of y, of size at least 5.

Therefore these 4 blocks intersect in at least 6 points, which is impossible in a symmetric design with $\lambda = 4$.

Therefore G is not of simple diagonal type. □

By the previous two lemmas, the proof of our main theorem is now complete.

References

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