Homewor 1

Problem: Verify by substitutron that each given function is a solutron of the given differential equation:

$$x^2 y'' + 5x y' + 4y = 0$$

$y_1 = \frac{1}{x^2}$;

Answer: $y_1' = -2x^{-3}$, $y_1'' = 6x^{-4}$

$\Rightarrow x^2 y_1'' + 5x y' + 4y = x^2 6x^{-4} + 5x(-2x^{-3}) + 4x^{-2}$

$= 6x^{-2} - 10x^{-2} + 4x^{-2} = 0$. $\Rightarrow y_1$ is a solutron of the above ODE.

$y_2 = \frac{\ln x}{x^2}$

Answer: $y_2' = \frac{1}{x} x^{-2} + \ln x(-2x^{-3}) = x^{-3} - 2x^{-3}\ln x$

$y_2'' = -3x^{-4} + 6x^{-4}\ln x - 2x^{-3}\frac{1}{x} = -5x^{-4} + 6x^{-4}\ln x$

$\Rightarrow x^2 y_2'' + 5x y_2' + 4y_2 = x^2(-5x^{-4} + 6x^{-4}\ln x) + 5x(x^{-3} - 2x^{-3}\ln x)$
$+ 4x^{-2}\ln x$

$= \cancel{-5x^{-2}} + \cancel{6x^{-2}\ln x} + \cancel{5x^{-2}} \cancel{-10x^{-2}\ln x} + \cancel{4x^{-2}\ln x}$

$= 0 \Rightarrow y_2$ is a solution of the above ODE.

Problem: Use your knowledge of derivatives to find one solution of

$$x^3 y' + y^2 = -x^4$$

Answer: Notice that the degree of a power function $y = ax^\alpha$ decreases by one when we derive it. Given the structure of the ODE, we could try to find solutions of that form:

$y = ax^\alpha$ Substituting this above we get:

$\Rightarrow y' = a\alpha x^{\alpha-1}$

$x^3 (a \alpha x^{\alpha-1}) + a^2 x^{2\alpha} = -x^4$

$\Rightarrow \alpha = 2$

$\Rightarrow x^3 (2ax) + a^2 x^4 = -x^4 \Rightarrow (2a + a^2) x^4 = -x^4$

$\Rightarrow a^2 + 2a + 1 = 0 \quad a = \frac{-2 \pm \sqrt{4 - 4(+1)}}{2} = -1$

$\Rightarrow y = -x^2$ is one solution.

Problem: Consider the ordinary differential equation

$$y' = 3 - 2y.$$

Draw a direction field and determine the behavior of the solution as $t \to \infty$

Answer: For the slope field, see the attached document.

$y = 1.5$ is an equilibrium solution

For any other solution $y(t) \to 1.5$ as $t \to \infty$.

If $y(0) = y_0 < 1.5 \Rightarrow y$ increases as t increases

If $y(0) = y_0 > 1.5 \Rightarrow y$ decreases as t increase)

Hw 1.2

Section 1.1 Problem 22 A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Answer: Let's start with the following definitions.

$m(t)$ = amount of water in the raindrop (in kg).

S = Surface area $= 4\pi r^2$

r = radius of raindrop

V = volume $= \frac{4\pi}{3} r^3$

t = time (seconds)

The rate at which the raindrop evaporates is $\frac{dm}{dt}$

Since it evaporates at a rate proportional to its surface area, then $\frac{dm}{dt} = -kS$, k = constant of proportionality

m is proportional to the volume

$\Rightarrow \frac{dV}{dt} = -k_2 S$, k_2 another constant

k_2 has units of distance/time (m/s).

$S = 4\pi r^2$, $r = \left(\frac{3}{4\pi} V\right)^{1/3}$

$\Rightarrow S = 4\pi \left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}$

$\Rightarrow \frac{dV}{dt} = -k_3 V^{2/3}$ describes the volume of the raindrop as a function of time, where k_3 is a constant.

Section 1.1 Problem 23: Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temp. of the object itself and the temp. of its sorroundings. Suppose that the ambient temperature ~~of the~~ is $70^{\circ}F$ and that the rate is $0.05\ min^{-1}$. Write a differential eqn. for the temp. of the object at any time.

Answer:

Definitions:

T = Temp. of object (in $^{\circ}F$)

t = time in minutes.

$$\Rightarrow \frac{dT}{dt} = -0.05\,(T-70)$$

Note: The minus sign is because

If $T > 70 \Rightarrow$ ~~This~~ The temp. of object is higher than its sorroundings and it must decrease when the heat transfers to the ambient.

Problem 1 Section 1.2 Solve ~~the~~ each of the following initial value problems and plot solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.

(a) $\frac{dy}{dt} = -y+5,\ y(0)=y_0$

(b) $\frac{dy}{dt} = -2y+5,\ y(0)=y_0$

(c) $\frac{dy}{dt} = -2y+10,\ y(0)=y_0$

In class we derived the general solution of $\frac{dy}{dt} = -ay + b$

which is $y = \frac{b}{a} + Ce^{-at}$

Since $y(0) = y_0 \Rightarrow y_0 = \frac{b}{a} + C \Rightarrow C = y_0 - \frac{b}{a}$

$\Rightarrow y(t) = \frac{b}{a} + (y_0 - \frac{b}{a})e^{-at}$

(a) $y(t) = 5 + (y_0 - 5)e^{-t}$

(b) $y(t) = \frac{5}{2} + (y_0 - \frac{5}{2})e^{-2t}$

(c) $y(t) = 5 + (y_0 - 5)e^{-2t}$

See the attached plots of solutions and the corresponding slope fields.

All solutions converge to the equilibrium solution, but at possibly different rates.

Problem 9 Section 1.2 The falling object in Example 2 satisfies the initial value problem:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v(0) = 0.$$

(a) Find the time that must elapse for the object to reach 98% of ~~the~~ its limiting velocity.

Answer: $\frac{dv}{dt} = 9.8 - \frac{v}{5} = (9.8 \times 5 - v)/5$

$5\frac{dv}{dt} = 9.8 \times 5 - v \Rightarrow \frac{5\,dv}{9.8 \times 5 - v} = dt \Rightarrow -5\ln|9.8 \cdot 5 - v| = t + C$

$\Rightarrow 9.8 \times 5 - v = C_2 e^{-t/5}$

$v = 9.8\times5 - C_2 e^{-t/5}$ $\qquad v(0) = 9.8\times5 - C_2 = 0$

$\Rightarrow v = 9.8\times5(1-e^{-t/5})$

$v(t) \longrightarrow 9.8\times5$ as $t \to +\infty$

The limiting velocity is 9.8×5

Look for t such that:

$v(t) = 0.98(9.8\times5) = 9.8\times5(1-e^{-t/5})$

$\Rightarrow 1-e^{-t/5} = 0.98 \Rightarrow e^{-t/5} = 0.02 \Rightarrow -t/5 = \ln(0.02)$

$t = -5\ln(0.02) = 19.56$ s.

(b) How far does the object fall in the time found in part (a)?

Answer:

Let $x(t)$ = displacement (downward) of the object at time t.

$\Rightarrow \frac{dx}{dt} = 9.8\times5(1-e^{-t/5})$

$\Rightarrow x(t) = 9.8\times5(t+5e^{-t/5})+c$ $\qquad$ Since $x(0)=0$

$\Rightarrow c = -9.8\times5^2$

~~$\Rightarrow x(19.56s) = 9.8$~~

$\Rightarrow x(t) = 9.8\times5(t+5(e^{-t/5}-1))$

$x(19.56s) = 9.8\times5(19.56+5(0.02-1)) = 9.8\times5(19.56-5\times0.98)$

$= 718.34$ meters, assuming it was initially high enough.

Hw 1.4

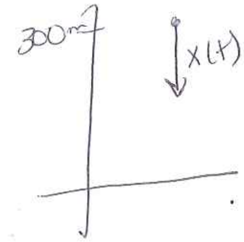
Problem 10 section 1.2

Modify Example 2 so that the falling object experiences no air resistance.

(a) Write down the modified initial value problem.

Answer: $\frac{dv}{dt} = g$

(b) Determine how long it takes for the object to reach ground.



Answer: In Example 2, $v(0)=0$,

$\Rightarrow v(t) = gt \Rightarrow x(t) = g\,t^2/2$

Because $x(0)=0$

~~The ob~~

$\Rightarrow$ Look for t such that $x(t) = 300\,m$

$\Rightarrow 9.8\,t^2 = 300 \times 2 \Rightarrow$, $t = 7.82\,s$

(c) Determine the velocity at time of impact:

Answer: $v = 9.8 \times 7.82 = 76.68$

Problem 19 section 1.2: Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of non toxic dye that leaves the swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a rate of 200 gal/min.

(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at time t

Answer: Consider an small time interval $[t, t+\Delta t]$.
t in minutes

Amount of water in Δt ~~seconds~~ minutes: $= 200\,\text{gal}/\text{min}\ \Delta t = 200\,\Delta t$

Amount of dye in that water: $200\,\Delta t \dfrac{q(t)}{60{,}000\text{ gal}}$

$$\Rightarrow q(t+\Delta t) \approx q(t) - \frac{200}{60{,}000}\, q(t)\, \Delta t$$

$$\Rightarrow \frac{q(t+\Delta t) - q(t)}{\Delta t} = -\frac{1}{300}\, q(t)$$

$$\Rightarrow \begin{cases} \dfrac{dq}{dt} = -\dfrac{1}{300}\, q(t) \\ q(0) = 5\text{ kg.} \end{cases}$$

(b) Solve the problem in part a

Answer:

$$q(t) = q_0\, e^{-t/300} = 5\text{ kg}\, e^{-t/300}$$

(c) You have invited several dozen friends to a pool party that is scheduled to begin 4 h. You have also determined that the effect of the dye is imperceptible ~~to the~~ if its concentration is less thant 0.02 g/gal

Is your filtering system capable of reducing the dye concentration to this level within 4 h?

Hw1.5

$$q(4h) = q(240\,min) = 5\,kg\, e^{-240/300} = 2.25\,kg$$

This corresponds to a concentration of

$$\frac{2.25\,kg}{60000\,gal} = 0.0374\ g/gal > 0.02\ g/gal$$

So: no.

(d) Find the time T at which the concentration of dye first reaches the value of $0.02\,g/gal$

Answer:

$$\frac{q(T)}{6000\,gal} = 0.02\ g/gal \Rightarrow q(T) = 0.02\,g \times 60000 = 1.2\,kg$$

$$\Rightarrow 5\,kg\, e^{-t/300} = 1.2\,kg$$

$$\Rightarrow \quad -t/300 = \ln(1.2/5)$$

$$t = -300 \ln(1.2/5) = 428.14\ min = 7.14\ hours.$$

(e) Find the flow rate that is sufficient to achieve the concentration $0.02\,g/gal$ within 4h.

Answer:

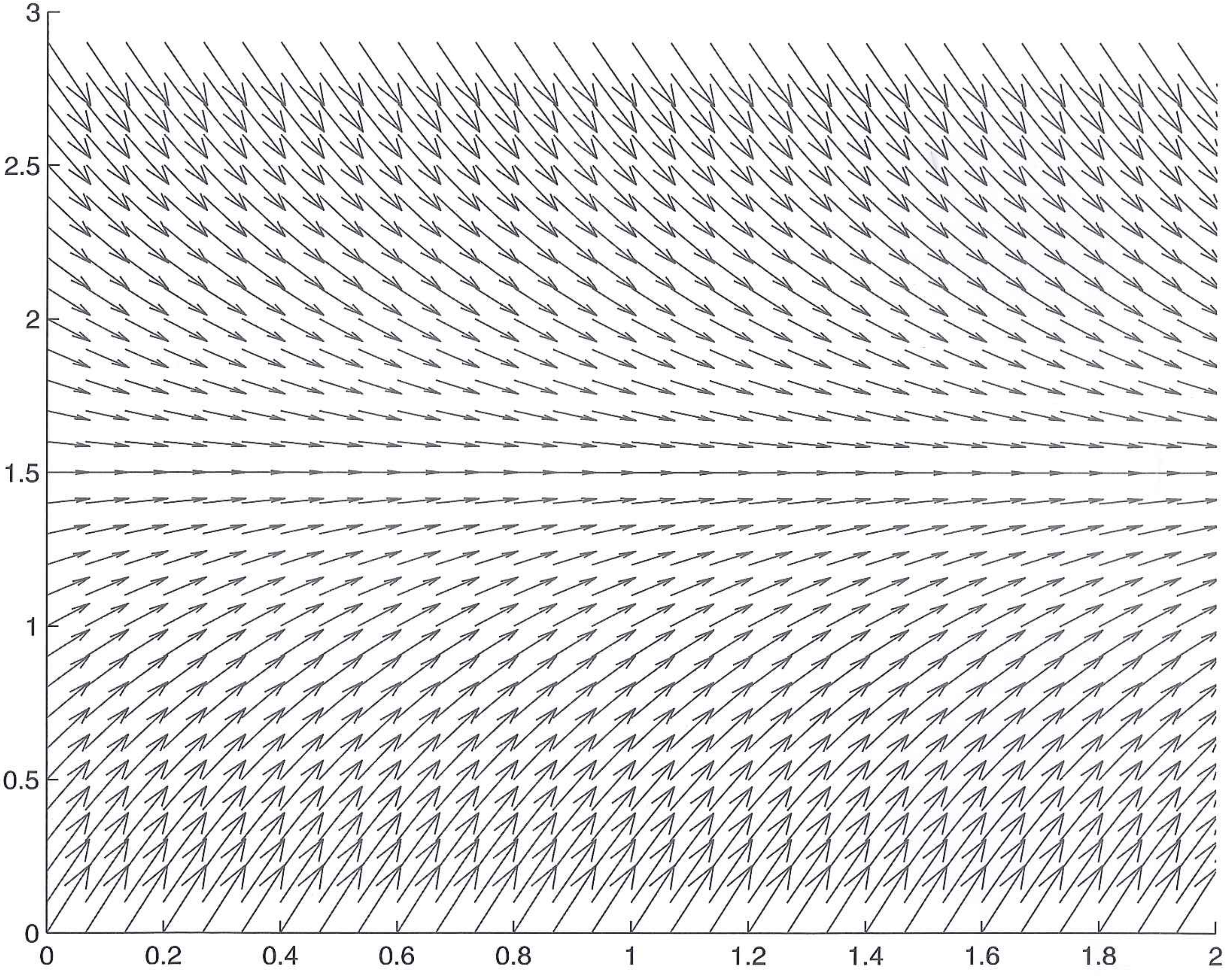
Let r be such rate in gal/min

$$\Rightarrow q(t) = 5\,kg\, e^{-\frac{r}{60000\,gal} t}$$

$$\Rightarrow \text{We need} \quad q(4h) = q(240\,min) = 5\,kg\, e^{-\frac{240\,min}{60000\,gal} r} = 1.2\,kg$$

$$\Rightarrow \quad -\frac{240\,min}{60000\,gal} r = +\ln(1.2/5) \Rightarrow r = -\frac{60000\,gal}{240\,min} \ln\left(\frac{1.2}{5}\right) = \frac{356.78\,gal}{min}$$

Problem 2



Problem 1, Section 1.2