HW10.1

2014

Homework 10

Section 5.3 #7 and 8. Determine the lower bound for the radius of convergence of series solutions about each given point x_0 for the differential equation.

#7. $(1+x^3)y'' + 4xy' + y = 0$; $x_0 = 0$, $x_0 = 2$

$x_0 = 0$:

Answer:

$$y'' + \frac{4x}{1+x^3}y' + \frac{1}{1+x^3}y = 0.$$

$$\Rightarrow p(x) = \frac{4x}{1+x^3}, \quad q(x) = \frac{1}{1+x^3}$$

We will use Theorem 5.3.1 to answer ~~the question~~.

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n} \Rightarrow a_{3n} = (-1)^n$$

$$\Rightarrow \left|\frac{a_{3(n+1)}}{a_{3n}}\right| = |-1| \xrightarrow[\text{as } n\to\infty]{} +1 \quad \Rightarrow \rho = \frac{1}{\sqrt[3]{1}} = 1$$

and $\frac{4x}{1+x^3}$ has the same radius of convergence.

$\Rightarrow$ A lower bound for the radius of convergence is 1.

$x_0 = 2$.

Answer: We need to apply partial fractions.

In order to use partial fractions, we need to know the $\boxed{3}$ roots of the cubic polynomial $1+x^3$.

$1+x^3=0 \iff x=-1,\ e^{\pm i\pi/3} = \cos(\pi/3) \pm i\sin(\pi/3) = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

partial fractions

$\Rightarrow \frac{1}{1+x^3} = \frac{d_1}{x+1} + \frac{d_2}{x-e^{i\pi/3}} + \frac{d_3}{x-e^{-i\pi/3}}$ for some constants d_1, d_2, d_3

We then need to check the radius of convergence of the Taylor series of each fraction

$\frac{d_1}{x+1} = \frac{d_1}{3+(x-2)} = \frac{d_1/3}{1+\frac{(x-2)}{3}} = \frac{d_1}{3}\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n (x-2)^n$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$

$\Rightarrow$ radius of convergence is 3.

$\frac{d_2}{x-e^{+i\pi/3}} = \frac{d_2}{-e^{i\pi/3}+2+(x-2)} = \frac{d_2/(2-e^{i\pi/3})}{1+\frac{(x-2)}{(2-e^{i\pi/3})}}$

$= \frac{d_2}{2-e^{i\pi/3}} \sum \frac{(-1)^n}{(2-e^{i\pi/3})^n}(x-2)^n \Rightarrow a_n = \frac{(-1)^n}{(2-e^{i\pi/3})^n}\frac{d_2}{2-e^{i\pi/3}}$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{|2-e^{i\pi/3}|} \Rightarrow L = |2-e^{i\pi/3}| = \left|2-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3}$.

$\frac{d_3}{x-e^{-i\pi/3}} = \frac{d_3}{2-e^{-i\pi/3}+(x-2)} = \frac{d_3/(2-e^{-i\pi/3})}{1+\frac{(x-2)}{2-e^{-i\pi/3}}}$

$= \frac{d_3}{2-e^{-i\pi/3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2-e^{-i\pi/3})^n}(x-2)^n \Rightarrow a_n = \frac{d_3}{2-e^{-i\pi/3}}\left(\frac{(-1)^n}{(2-e^{-i\pi/3})}\right)^n$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{|2-e^{-i\pi/3}|} \Rightarrow L = \left|2-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3}$

$\Rightarrow$ A lower bound for the radius of convergence of a series solution is $\sqrt{3}$.

Section 5.3 #10 The Chebyshev differential equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

(a) Determine two solutions in powers of x for $|x|<1$ and show that they form a fundamental set of solutions.

Answer: $y = \sum a_n x^n$

$$(1-x^2)\sum a_n n(n-1)x^{n-2} - x\sum a_n n x^{n-1} + \alpha^2 \sum a_n x^n$$

$$= -\sum a_n n(n-1)x^n + \sum a_n n(n-1)x^{n-2} - \sum a_n n x^n + \alpha^2 \sum a_n x^n$$

$$= \sum \left[a_{n+2}(n+2)(n+1) + \left(-n(n-1) - n + \alpha^2\right) a_n \right] x^n = 0$$

$$\Rightarrow a_{n+2} = \frac{n^2 - \alpha^2}{(n+2)(n+1)} a_n \quad \leftarrow \text{Recurrence relation.}$$

$n=0$ $\quad a_2 = \frac{-\alpha^2}{2} a_0$ $\qquad n=1$ $\quad a_3 = \frac{1-\alpha^2}{3\cdot 2} a_1$

$n=2$ $\quad a_4 = \frac{4-\alpha^2}{4\cdot 3} a_2 = -\frac{(4-\alpha)\alpha^2}{4!} a_0$

$n=3$ $\quad a_5 = \frac{9-\alpha^2}{5\cdot 4} a_3 = \frac{(3^2-\alpha^2)(2^2-\alpha^2)}{5!} a_1$

$$\Rightarrow y_1 = 1 - \frac{\alpha^2}{2!}x^2 + \frac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \frac{(4^2-\alpha^2)(2^2-\alpha^2)\alpha^2}{6!}x^6 - \cdots$$

$$- \frac{\left[(2m-2)^2 - \alpha^2\right]\cdots(2^2-\alpha^2)}{(2m)!} x^{2m} - \cdots$$

$$y_2 = x + \frac{1-\alpha^2}{3!}x^3 + \frac{(3^2-\alpha^2)(1-\alpha^2)}{5!}x^5 + \cdots$$

$$+ \frac{[(2m-1)^2-\alpha^2]\cdots(1-\alpha^2)}{(2m+1)!}x^{2m+1} + \cdots$$

$W[y_1, y_2](0) = 1 \Rightarrow$ They form a fundamental set of solutions.

(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called Chebyshev polynomials. They are very useful in problems that require a polynomial approximation to a function defined on $-1 \le x \le 1$.

Answer:

If $\alpha = n$ n non-negative integer

$$\Rightarrow a_{n+2} = \frac{n^2-n^2}{(n+2)(n+1)}a_n = 0$$

If n is even $\Rightarrow y_1$ is a polynomial of degree n

If n is odd $\Rightarrow y_2$ " " " " " " .

(c) Find the polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

$n=0 \Rightarrow y_1 = 1$

$n=1 \Rightarrow y_2 = x$

$n=2 \Rightarrow y_1 = 1 - 2x^2$

$n=3 \quad y_2 = x - \frac{4}{3}x^3$.

Hw 10.3

Section 5.3 # 13 and 14 Find the first four non-zero terms in each of two power solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

#13 $(\cos x)\, y'' + x y' - 2y = 0.$

Answer:

$\cos(0) = 1 \Rightarrow x_0 = 0$ is a regular point.

Taylor expansion of $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$

$y = \sum a_n x^n$

$\Rightarrow \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right)\left(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \dots\right)$

$+ \left(a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \dots\right)$

$- 2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\right) = 0.$

Collect order x^0 terms:

$2a_2 - 2a_0 = 0 \Rightarrow a_2 = a_0.$

x^1: $6a_3 + a_1 - 2a_1 = 0 \Rightarrow a_3 = \frac{1}{6} a_1$

x^2: $-a_2 + 12a_4 + \cancel{2a_2} - \cancel{2a_2} = 0 \Rightarrow a_4 = \frac{1}{12} a_2 = \frac{1}{12} a_0.$

x^3: $20a_5 - 3a_3 + 3a_3 - 2a_3 = 0 \Rightarrow a_5 = \frac{1}{10} a_3 = \frac{1}{60} a_1$

x^4: $30a_6 - \frac{12}{2} a_4 + \frac{1}{12} a_2 + 4a_4 - 2a_4 = 0$

$\Rightarrow a_6 = \frac{1}{30}\left(4a_4 - \frac{1}{12} a_2\right) = \frac{2}{15}\frac{1}{12} a_0 - \frac{1}{30}\frac{1}{12} a_0 = \frac{1}{120} a_0$

$x^5: \quad 42a_7 - 10a_5 + \frac{1}{4}a_3 + 5a_5 - 2a_5 = 0$

$$\Rightarrow a_7 = \frac{1}{42}\left(7a_5 - \frac{1}{4}a_3\right) = \frac{1}{42}\left(\frac{7}{60}a_1 - \frac{1}{24}a_1\right)$$

$$= \frac{1}{560}a_1$$

$$\Rightarrow y_1 = 1 + x^2 + \frac{1}{12}x^4 + \frac{1}{120}x^6 + \cdots$$

$$y_2 = x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{560}x^7 + \cdots$$

#14 $\quad e^{-x}y'' + \ln(1+x)\,y' - xy = 0$

$$y'' + e^{x}\ln(1+x)\,y' - xe^{x}y = 0.$$

Answer:

At $x=0$, $\ln(1+x) = 0$

$x_0 = 0$ is a regular point.

We need the Taylor expansion of $p(x) = e^{x}\ln(1+x)$.

and $q(x) = xe^{x}$

$p'(x) = e^{x}\ln(1+x) + \dfrac{e^x}{1+x} \qquad p'(0) = 1$

$p''(x) = e^{x}\ln(1+x) + \dfrac{e^x}{1+x} + \dfrac{e^x(1+x) - e^x}{(1+x)^2} = e^{x}\ln(1+x) + \dfrac{e^x}{1+x} + \dfrac{xe^x}{(1+x)^2}$

$p''(0) = 1$

$p'''(x) = e^{x}\ln(1+x) + \dfrac{e^x}{1+x} + \dfrac{xe^x}{(1+x)^2} + \dfrac{(1+x)e^x(1+x)^2 - xe^x\,2(1+x)}{(1+x)^2}$

$= e^{x}\ln(1+x) + \dfrac{e^x}{1+x} + \dfrac{xe^x}{(1+x)^2} + \dfrac{(x^2+1)}{(1+x)^2}e^x$

$p'''(0) = 1 + 1 = 2$

Hw 10.4

$$p^{(4)}(x) = e^x \ln(1+x) + \frac{e^x}{1+x} + \frac{xe^x}{(1+x)^2} + \frac{(1+x^2)}{(1+x)^2}e^x$$
$$+ \frac{\left(2xe^x+(1+x^2)e^x\right)(1+x)^2 - (1+x^2)e^x}{(1+x)^4}$$

$$= e^x\ln(1+x) + \frac{e^x}{1+x} + \frac{xe^x}{(1+x)^2} + \frac{(1+x^2)}{(1+x)^2}e^x + \left(1 - \frac{1+x^2}{(1+x)^4}\right)e^x$$

$$p^{(4)}(0) = 1+1 = 2$$

$$\Rightarrow e^x \ln(1+x) = x + \frac{x^2}{2} + \frac{2}{3!}x^3 + \frac{2}{4!}x^4 + \dots$$
$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \dots$$

$$xe^x = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \dots$$

$\Rightarrow$ For $y = \sum a_n x^n$

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$
$$+ \left(x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{12}x^4 + \dots\right)\left(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots\right)$$
$$- \left(x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \dots\right)\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\right) = 0$$

Collect terms corresponding to:

x^0 : $2a_2 = 0 \Rightarrow a_2 = 0$

x^1 : $6a_3 + a_1 - a_0 \Rightarrow a_3 = \frac{1}{6}a_0 - \frac{1}{6}a_1$

x^2 : $12a_4 + 2a_2 + \frac{1}{2}a_1 - a_1 - a_0 = 0$

$\Rightarrow a_4 = \frac{1}{12}a_0 + \frac{1}{24}a_1$

x^3 : $20a_5 + 3a_3 + \cancel{a_2}^{0} + \frac{1}{3}a_1 - \cancel{a_2}^{0} - a_1 - \frac{a_0}{2} = 0$

$a_5 = \frac{1}{20}\left(-3a_3 + \frac{2}{3}a_1 + \frac{1}{2}a_0\right) = \frac{1}{20}\left(-\cancel{\tfrac{1}{2}a_0} + \frac{1}{2}a_1 + \frac{2}{3}a_1 + \cancel{\tfrac{1}{2}a_0}\right) = \frac{7}{120}a_1$

x^4: $30a_6 + 4a_4 + \frac{3}{2}a_3 + \frac{2}{3}\cancel{a_2}^{\to 0} + \frac{1}{12}a_1 - a_3 - \cancel{a_2}^{\to 0} - \frac{1}{2}a_1 - \frac{1}{6}a_0 = 0$

$a_6 = \frac{1}{30}\left(\frac{1}{6}a_0 + \frac{5}{12}a_1 - \frac{1}{2}a_3 - 4a_4\right)$

$= \frac{1}{30}\left(\frac{1}{6}a_0 + \frac{5}{12}a_1 - \frac{1}{12}a_0 + \frac{1}{12}a_1 - \frac{1}{3}a_0 - \frac{1}{6}a_1\right)$

$= \frac{1}{30}\left(-\frac{3}{12}a_0 + \frac{1}{3}a_1\right) = -\frac{1}{120}a_0 + \frac{1}{90}a_1$

$\Rightarrow y_1 = 1 + \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{120}x^6 + \dots$

$y_2 = x - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{7}{120}x^5 + \frac{1}{90}x^6 + \dots$

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Section 5.4 Problems 9, 10. Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

#9 $x^2y'' - 5xy' + 9y = 0$

Answer:

$y = x^r$ $\quad y' = rx^{r-1}$ $\quad y'' = r(r-1)x^{r-2}$ $\quad r = \text{const}.$

$\Rightarrow r(r-1)x^r - 5rx^r + 9x^r = 0$

$\Rightarrow r^2 - 6r + 9 = 0$ $\quad r = \frac{6 \pm \sqrt{36 - 4\cdot 9}}{2} = 3.$

$r = 3$ is a double root.

$\Rightarrow y_1 = x^3,\ y_2 = x^3 \ln|x|.$

$\Rightarrow$ The general solution is $y = C_1x^3 + C_2x^3\ln|x|.$

HW10.5

#10. $(x-2)^2 y'' + 5(x-2)y' + 8y = 0$

Answer: $y = (x-2)^r$ $\quad y' = r(x-2)^{r-1}$, $y'' = r(r-1)(x-2)^{r-2}$

$\Rightarrow r(r-1)(x-2)^r + 5r(x-2)^r + 8(x-2)^r = 0$

$\Rightarrow r^2 + 4r + 8 = 0 \quad r = \frac{-4 \pm \sqrt{16 - 4\cdot 8}}{2} = \frac{-4 \pm \sqrt{-16}}{2}$

$= -2 \pm 2i$

$\Rightarrow$ The two fundamental solutions are:

$y_1 = (x-2)^{-2} \cos(2\ln|x-2|)$, $y_2 = (x-2)^{-2} \sin(2\ln|x-2|)$

General solutions $y = c_1 (x-2)^{-2} \cos(2\ln|x-2|) + c_2 (x-2)^{-2} \sin(2\ln|x-2|)$.

Section 5.4 Problems 31, 32, 33 Find all singular points of the given equation and determine whether each one is regular or irregular.

#31 $x^2 y'' - 3\sin x\, y' + (1+x^2) y = 0$

Answer: $y'' - \frac{3\sin x}{x^2} y' + \frac{1+x^2}{x^2} y$

$p(x) = -\frac{3\sin x}{x^2}$, $q(x) = \frac{1+x^2}{x^2}$.

Singular point: $x = 0$ (where p and q blows up).

$x^2 q = 1 + x^2$ is analytical.

$x p(x) = -3 \frac{\sin x}{x} = -3 \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = -3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n}$

which has a radius of convergence $\rho = \infty$ at $x = 0$.

$\Rightarrow x p(x)$ is analytical

$x = 0$ is a regular singular pt.

#32 $x y'' + y' + \frac{\cot x}{x} y = 0$

Answer: $p(x) = \frac{1}{x}$, $q(x) = \frac{\cot x}{x} = \frac{1}{x \tan x}$

~~$x p(x)$~~ Singular points:

$x = 0$, $\tan x = 0 \iff \sin x = 0 \iff x = \pm n\pi$, $n = 1, 2, \ldots$

$\Rightarrow$ The singular points are: $x_0 = 0, \pm n\pi$.

$x p(x) = 1$ is always analytical

$x^2 q(x) = \frac{x}{\tan x} = \frac{x \cos x}{\sin x}$

For $x = 0$, $\frac{\sin x}{x}$ is ~~[illegible]~~ analytical, as seen in the previous problem, and $\frac{\sin x}{x} \to 1$ as $x \to 0$

$\Rightarrow$ $x^2 q(x)$ is analytical at $x = 0$.

However, at $x = \pm n\pi$, $n \geq 1$ integer, the denominator vanishes but the numerator doesn't.

$\Rightarrow$ $x = 0$ is regular, $x = \pm n\pi$, $n = 1, 2, \ldots$ are irregular.

#33 $\sin x \, y'' + x y' + 4y = 0$

Answer: $p(x) = \frac{x}{\sin x}$, $q(x) = \frac{4}{\sin x}$

$x p(x) = \frac{x^2}{\sin x}$, $x^2 q(x) = \frac{4x^2}{\sin x}$

Singular points: $x = 0, \pm n\pi$ $n = 1, 2, \ldots$

$\frac{\sin x}{x}$ is analytical at $x=0$ and $\frac{\sin x}{x} \longrightarrow 1$ as $x \to 0$

$\Rightarrow \frac{x}{\sin x}$ is analytical at $x=0$

$\Rightarrow x=0$ is a regular singular point.

However, for $x = \pm n\pi$, $n=1,2,\ldots$, only the denominator vanishes.

Therefore, $x=0$ is a ~~sing~~ regular singular point, and

$x = \pm n\pi$ $n=1,2,\ldots$ are irregular singular points.

Section 5.4 #36. Find all values of β for which all solutions of $x^2y''+\beta y=0$ approach zero as $x \to 0$.

$y = x^r \Rightarrow y' = rx^{r-1}$ $\quad y'' = r(r-1)x^{r-2}$

$\Rightarrow r(r-1)x^r + \beta x^r = 0$

$\Rightarrow r^2 - r + \beta = 0$ $\quad r = \frac{1 \pm \sqrt{1-4\beta}}{2}$

if $1-4\beta < 0 \Rightarrow y = c_1 x^{\frac{1}{2}} \cos\left(\frac{1}{2}\sqrt{4\beta-1}\ln|x|\right) + c_2 x^{\frac{1}{2}} \sin\left(\frac{1}{2}\sqrt{4\beta-1}\ln|x|\right)$

which approaches zero as $x \to 0$.

If $1-4\beta > 0 \Rightarrow$ need $\frac{1 \pm \sqrt{1-4\beta}}{2} > 0$

$\Leftrightarrow \sqrt{1-4\beta} < 1 \Leftrightarrow 1-4\beta < 1 \Leftrightarrow \beta > 0$.

$\Rightarrow$ Either $\beta > 0$ or $\beta > \frac{1}{4} \Leftrightarrow \beta > 0$.

Therefore, all ~~values~~ positive values $\beta > 0$ satisfy that all solutions of $x^2y''+\beta y=0$ approach zero as $x \to 0$.