2 Hwll. 1
2014

Homework 11

Section 5.5 #5, 6, 7.

(a) Show that the given differential equation has a regular singular point at $x=0$.

(b) Determine the indicial equation, the recorrence relation, and the roots of the indicial equation.

(c) Find the series solution $(x>0)$ corresponding to the largest root.

(d) If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

#5. $\quad 3x^2y'' + 2xy' + x^2 y = 0$

Answer:

(a) $p(x) = \frac{2}{3}\frac{1}{x}$, $q(x) = \frac{1}{3}$

Singular point $x=0$

$x\,p(x) = \frac{3}{2}$ analytical, $x^2 q(x) = \frac{1}{3}x^2$ analytical

$\Rightarrow x=0$ is a regular singular point.

(b) $y = x^r \sum_{n=0}^{\infty} a_n x^n$ $\quad p_0 = \frac{2}{3}$, $q_0 = 0$.

$\Rightarrow$ Indicial equation is $r^2 - r + \frac{2}{3}r = 0$

$\Rightarrow r^2 - \frac{1}{3}r \Rightarrow r(3r-1) = 0 \quad r = 0, \frac{1}{3}$.

$y' = \sum a_n (r+n) x^{n+r-1}$ $\quad y'' = \sum a_n (r+n)(r+n-1) x^{r+n-2}$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (r+n)(r+n-1)\, x^{r+n-2} + \frac{2}{3}\sum_{n=0}^{\infty} a_n (r+n)\, x^{r+n-2} + \frac{1}{3}\sum a_n x^{r+n} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} \left[a_{n+2}(r+n+2)(r+n+1)\cancel{x} + \frac{2}{3} a_{n+2}(r+n+2) + \frac{1}{3} a_n \right] x^{r+n} = 0$$

$$\boxed{\begin{array}{l} \text{Collecting } x^{r-1}: \\ a_1 (r+1)(r) + \frac{2}{3} a_1 (r+1) = 0 \\ \Rightarrow a_1 = 0 \end{array}}$$

$$\Rightarrow a_{n+2}\left[(r+n+2)\left(r+n+\frac{5}{3}\right) \right] = -\frac{1}{3} a_n$$

$$\Rightarrow a_{n+2} = -\frac{a_n}{(r+n+2)(3r+3n+5)}$$

(c) Largest root: $r = \frac{1}{3}$ $\Rightarrow a_{n+2} = -\dfrac{a_n}{(n+2+\frac{1}{3})(3n+6)} = -\dfrac{a_n}{(3n+7)(n+2)}$

$n=0$ $\quad a_2 = -\dfrac{a_0}{(7)(2)} = -\dfrac{1}{14} a_0$

$n=1$ $\quad a_3 = 0$

$n=2$ $\quad a_4 = -\dfrac{a_2}{13 \cdot 4} = \dfrac{1}{13 \cdot 14 \cdot 4} a_0 = \dfrac{1}{13 \cdot 14} \dfrac{1}{2^2} a_0$

$n=3$ $\quad a_5 = 0$

$n=4$ $\quad a_6 = -\dfrac{a_4}{19 \cdot 6} = -\dfrac{1}{13 \cdot 14 \cdot 19} \dfrac{1}{24} a_0$

$$y_1 = x^{1/3}\left[1 - \frac{1}{14}x^2 + \frac{1}{7 \cdot 13}\frac{1}{2!}\frac{x^4}{2^2} - \frac{1}{7 \cdot 13 \cdot 19}\frac{1}{48}x^6 + \dots \right].$$

(d) $r = 0$ $\Rightarrow$ $a_{n+2} = -\dfrac{a_n}{(n+2)(3n+5)}$

$n=0$ $\quad a_2 = -\dfrac{a_0}{2 \cdot 5} = -\dfrac{1}{10} a_0$

$n=1$ $\quad a_3 = 0$

$n=2 \quad a_4 = -\dfrac{a_2}{4\cdot 11} = +\dfrac{1}{4\cdot 10\cdot 11} a_0$

$n=3 \quad a_5 = 0$

$n=4 \quad a_6 = -\dfrac{a_4}{6\cdot 17} = -\dfrac{1}{4\cdot 6\cdot 10\cdot 11\cdot 17} a_0$

$\Rightarrow y_2 = 1 - \dfrac{1}{10}x^2 + \dfrac{1}{4\cdot 10\cdot 11}x^4 - \dfrac{1}{4\cdot 6\cdot 10\cdot 11\cdot 17}x^6 + \dots$

#6 $\quad x^2y'' + xy' + (x-2)y = 0$

Answer:

(a) $p(x) = \dfrac{1}{x}, \quad q(x) = \dfrac{1}{x} - \dfrac{2}{x^2}$. $\quad$ Sing. pt. $x=0$

$x^2q(x) = x - 2 \xrightarrow[x\to 0]{} -2 \Rightarrow q_0 = -2$

(b) $xp(x) = 1 \Rightarrow p_0 = 1$

$\Rightarrow$ Indicial equation $\quad r^2 - r + r - 2 = 0 \Rightarrow r = \pm\sqrt{2}$.

$y = x^r \sum a_n x^n$

$\Rightarrow x^2 \sum a_n (r+n)(r+n-1) x^{r+n-2} + x \sum a_n (r+n) x^{r+n-1} + (x-2)\sum a_n x^{r+n}$

$= \sum_{n=0}^{\infty} \left(a_n (r+n)^2 - 2a_n\right) x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n}$

~~n=1 $a_1(r+1)^2 - 2a_1$~~ $\Rightarrow$ For $n \geq 1$:

$a_n = \dfrac{a_{n-1}}{2-(r+n)^2} = -\dfrac{a_{n-1}}{\cancel{r^2}+2rn+n^2\cancel{-2}} = -\dfrac{a_{n-1}}{n(n+2r)}$.

(c) ~~[illegible]~~ $r = \sqrt{2}$

$n=1 \qquad a_1 = \dfrac{-a_0}{1+2\sqrt{2}}$

$n=2 \qquad a_2 = -\dfrac{a_1}{2(2+2\sqrt{2})} = \dfrac{a_0}{2(2+2\sqrt{2})(1+2\sqrt{2})}$

$n=3 \qquad a_3 = -\dfrac{a_2}{3(3+2\sqrt{2})} = -\dfrac{a_0}{3\cancel{2}(3+2\sqrt{2})(2+2\sqrt{2})(1+2\sqrt{2})}.$

$$\Rightarrow \quad y_1 = x^{\sqrt{2}}\left[1 - \frac{x}{1+2\sqrt{2}} + \frac{x^2}{2!(1+2\sqrt{2})(2+2\sqrt{2})} + \cdots \right.$$

$$\left. + \frac{(-1)^n \, x^n}{n!\,(1+2\sqrt{2})(2+2\sqrt{2})\cdots(n+2\sqrt{2})} + \cdots\right].$$

(d) $r=-\sqrt{2} \qquad a_n = -\dfrac{a_{n-1}}{n(n-2\sqrt{2})}$

$n=1 \qquad a_1 = -\dfrac{a_0}{1-2\sqrt{2}}$

$n=2 \qquad a_2 = -\dfrac{a_1}{2(2-2\sqrt{2})} = \dfrac{a_0}{2(1-2\sqrt{2})(2-2\sqrt{2})}$

$n=3 \qquad a_3 = -\dfrac{a_2}{3(3-2\sqrt{2})} = -\dfrac{a_0}{3!(1-2\sqrt{2})(2-2\sqrt{2})(3-2\sqrt{2})}$

$$\Rightarrow \quad y_2 = x^{-\sqrt{2}}\left[1 - \frac{x}{1-2\sqrt{2}} + \frac{x^2}{2!(1-2\sqrt{2})(2-2\sqrt{2})} + \cdots \right.$$

$$\left. + \frac{(-1)^n \, x^n}{n!\,(1-2\sqrt{2})(2-2\sqrt{2})\cdots(n-2\sqrt{2})} + \cdots\right]$$

#7 $xy'' + (1-x)y' - y = 0.$

Answer: $p = \frac{1-x}{x}$, $q(x) = -\frac{1}{x}$

$x=0$ is a singular point

(a) $x\,p(x) = 1-x \xrightarrow[x\to 0]{} 1 \Rightarrow p_0 = 1$

$x^2 q(x) = -x \xrightarrow[x\to 0]{} 0 \Rightarrow q_0 = 0$

$x\,p(x)$ and $x^2 q(x)$ are clearly analytical at $x=0$.

$\Rightarrow x=0$ is a regular singular point.

(b) Indicial equation: $r^2 \cancel{-r} \cancel{+r} = 0 \Rightarrow$ ~~[illegible]~~ $r^2 = 0$

~~[illegible]~~. $r=0$ is a double root.

$y = x^r \sum a_n x^n$

$\Rightarrow \sum a_n (r+n)(r+n-1) x^{r+n-2} + \left(\frac{1}{x}-1\right) \sum a_n (r+n) x^{r+n-1} - \frac{1}{x} \sum a_n x^{r+n}$

$= \sum a_n(r+n)(r+n-1)x^{r+n-2} + \sum a_n (r+n) x^{r+n-2} - \sum a_n (r+n) x^{r+n-1}$

$= \sum_{n=-2} a_{n+2} (r+n+2)(r+n+1) x^{r+n} - \sum a_n x^{r+n-1}$

$= \sum_{n=-2}^{\infty} a_{n+2}(r+n+2)(r+n+1)x^{r+n} + \sum_{n=-2}^{\infty} a_{n+2} (r+n+2) x^{r+n} - \sum_{n=-1}^{\infty} a_{n+1} (r+n+1) x^{r+n} - \sum_{n=-1}^{\infty} a_{n+1} x^{r+n}$

$n=-2$ gives the indicial equation above.

~~for~~ for $n \geq -1$:

$a_{n+2}(r+n+2)(r+n+2) - a_{n+1}(r+n+2) = 0$

$\Rightarrow a_{n+2} = \frac{a_{n+1}}{r+n+2} \quad n \geq -1$ or $a_n = \frac{a_{n-1}}{r+n}, \ n \geq 1.$

(c) $r=0 \Rightarrow a_n = \dfrac{a_{n-1}}{n}$

$a_1 = a_0$, $a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2}$, $a_3 = \dfrac{a_2}{3} = \dfrac{a_0}{3!}$

$\Rightarrow a_n = \dfrac{1}{n!}$

$y_1 = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = e^x$.

Section 5.5 #14 The Bessel equation of order zero is: $x^2y'' + xy' + x^2y = 0$

(a) Show that $x=0$ is a regular singular point:

Answer: $p(x) = \frac{1}{x}$, $q(x) = 1$

$xp(x) = 1$ and $x^2q(x) = x^2$ are analytical functions at $x=0$

and $p(x) = \frac{1}{x}$ blows up at $x=0$

$\Rightarrow x_0 = 0$ is a regular singular point.

(b) Show that the roots of the indicial equation are $r_1 = r_2 = 0$.

$xp(x) = 1 \xrightarrow[\text{as } x\to 0]{} 1 \Rightarrow p_0 = 1$

$x^2q(x) = x^2 \xrightarrow[\text{as } x\to 0]{} 0 \Rightarrow q_0 = 0$

Indicial equation: $r^2 - r + p_0 r + q_0 = r^2 - r + r + 0 = r^2$

$\Rightarrow$ The roots are $r_1 = r_2 = 0$.

(c) Show that one solution for $x>0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Answer: Let's get the recurrence relation:

$$x^2 \sum a_n (r+n)(r+n-1) x^{r+n-2} + x \sum a_n (r+n) x^{r+n-1} + x^2 \sum a_n x^{r+n}$$

$$= \sum a_n (r+n)(r+n-1) x^{r+n} + \sum a_n (r+n) x^{r+n} + \sum a_n x^{r+n+2}$$

$$= \sum_{n=0}^{\infty} \left(a_n (r+n)^2\right) x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n}$$

$$= a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty} \left(a_n (r+n)^2 + a_{n-2}\right) x^{r+n}$$

$\Rightarrow$ $r=0$, $a_1 = 0$

and $a_n = -\dfrac{a_{n-2}}{(r+n)^2}$ for $n \geq 2$

$r_1 = r_2 = 0 \Rightarrow a_n = -\dfrac{a_{n-2}}{n^2}$

$n=2$ $\quad a_2 = -\dfrac{a_0}{4}$ $\qquad$ $n=3$ $\quad a_3 = 0$

$n=4$ $\quad a_4 = -\dfrac{a_2}{4^2} = \dfrac{a_0}{2^2 4^2} = \dfrac{a_0}{2^4 (2!)^2}$

$n=5$ $\quad a_5 = 0$

$n=6$ $\quad a_6 = -\dfrac{a_4}{6^2} = -\dfrac{a_0}{6^2 2^4 (2!)^2} = -\dfrac{a_0}{3^2 2^{4+2} (2!)^2} = -\dfrac{a_0}{2^{2\cdot 3} (3!)^2}$

In general, we can guess $a_{2n} = \dfrac{(-1)^n}{2^{2n} (n!)^2}$

If we want to prove it by induction,

$n=1$ $a_2 = (-1)\frac{1}{4}$ ✓

Assume it for n, and show it for $n+1$:

$$a_{2(n+1)} = -\frac{a_{2n}}{(2(n+1))^2} = -\frac{1}{(2(n+1))^2} \cdot \frac{(-1)^n}{2^{2n}(n!)^2} = \frac{(-1)^{n+1}}{2^{2(n+1)}((n+1)!)^2}$$ ✓

$$\therefore y_1 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = J_0(x)$$

(d) Show that the series of $J_0(x)$ converges for all x. The function J_0 is known as the Bessel function of the first kind of order zero.

$$a_n = \frac{(-1)^n}{2^{2n}(n!)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{2^{2(n+1)}((n+1)!)^2}}{\frac{1}{2^{2n}(n!)^2}}\right| = \frac{\cancel{2^{2n}}\cancel{(n!)^2}}{\cancel{2^{2n}} \cdot 4(n+1)^2\cancel{(n!)^2}}$$

$$= \frac{1}{4(n+1)^2} \xrightarrow[\text{as } n\to\infty]{} 0 \Rightarrow L = 0 \Rightarrow \rho = \infty$$

$\Rightarrow$ The radius of convergence is $+\infty$.

Section 5.5 Problem 15 Referring to Problem 14, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.

Hint: If $y_2(x) = J_0(x)v(x)$, then

$$y_2(x) = J_0(x)\int \frac{dx}{x\,J_0(x)^2}\,dx.$$

Find the first term in the series expansion of $\frac{1}{x\,J_0(x)^2}$.

Answer:

By the method of reduction of order, v satisfies

$$J_0 v'' + (2J_0' + pJ_0)v' = J_0 v'' + (2J_0' + \tfrac{1}{x}J_0)v' = 0$$

Let $\omega = v'$ $\Rightarrow$ $J_0\omega' + (2J_0' + \tfrac{1}{x}J_0)\omega = 0$

$$\Rightarrow \frac{\omega'}{\omega} = -2\frac{J_0'}{J_0} - \frac{1}{x}$$

Integrating on both sides (and ignoring the constants of integration) we get

$$\ln\omega = -2\int(\ln J_0)'\,dx - \ln|x| = -2\ln J_0 - \ln|x|$$

$$\Rightarrow \omega = e^{-2\ln J_0}\,e^{-\ln|x|} = \frac{1}{x\,J_0(x)^2}$$

$$\Rightarrow v' = \frac{1}{x\,J_0(x)^2} \Rightarrow v = \int\frac{dx}{x\,J_0(x)^2}$$

$$\Rightarrow y_2 = J_0\int\frac{dx}{x\,J_0(x)^2}$$

$x J_0(x)^2$ is analytical at $x=0$ and its series expansion converges everywhere.

The first term is

$$x J_0(x)^2 = x\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right)^2 = x\left(1+2\frac{(-1)}{4}x^2+\dots\right)$$

$$= x-\frac{1}{2}x^3+\dots$$

The function $\frac{1}{J_0(x)^2}$ also has a Taylor expansion and we compute its coefficients

$$\frac{1}{J_0(x)^2} = (b_0+b_1x+\dots),$$

as follows:

$$(b_0+b_1x+\dots)(\cancel{x-\frac{1}{2}x} \; 1-\frac{1}{2}x^2+\dots) = 1$$

Collect ~~order of~~ coefficients with same exponent:

x^0: $b_0 = 1$

x^1: $b_1 = 0$ etc.

x^2: $b_0\cdot\left(-\frac{1}{2}\right)+b_2 = 0 \Rightarrow b_2 = \frac{1}{2}b_0 = \frac{1}{2}$

$$\Rightarrow \frac{1}{J_0(x)^2} = 1+\frac{1}{2}x^2+\dots$$

$$\Rightarrow \frac{1}{xJ_0(x)^2} = \frac{1}{x}+\frac{1}{2}x+\dots$$

$$\Rightarrow y_2 = J_0(x)\cdot\int\frac{dx}{xJ_0(x)} = \left(1-\frac{1}{4}x+\dots\right)\left(\ln x+\frac{x^2}{4}+\dots\right)$$

$\Rightarrow y_2$ contains a logarithmic term.