Section 5.6 Problems #1, 3

(a) Find all the regular singular points of the given differential equation

(b) Determine the indicial equation and the exponent of the singularity for each regular singular point.

**#1** $x y'' + 2xy' + 6e^x y = 0$

Answer: $y'' + 2y' + \frac{6e^x}{x} y = 0$

$p = 2, \quad q = 6\frac{e^x}{x}$

Singular point: $x = 0$

$x\,p(x) = 2x$ and $x^2 q(x) = 6xe^x$ are clearly analytical

$\Rightarrow x = 0$ is a regular singular point.

$p_0 = 0, \quad q_0 = 0 \quad \Rightarrow r^2 - r = 0 \quad \Rightarrow r = 0, 1.$

**#3** $x(x-1)y'' + 6x^2 y' + 3y = 0$

Answer: $y'' + \frac{6x}{x-1} y' + \frac{3}{x(x-1)} y = 0 \qquad p = \frac{6x}{x-1}, \quad q = \frac{3}{x(x-1)}$

Singular points $x = 0, 1.$

$x = 0$: $\quad x\,p(x) = \frac{6x^2}{x-1} = 6x^2 \sum x^n$ analytical

$x^2 q(x) = \frac{3x}{x-1} = 3x \sum x^n$ analytical

$p_0 = 0, \quad q_0 = 0 \quad \Rightarrow r(r-1) = 0 \quad r = 0, 1.$

$x=1$ $\quad (x-1)p(x) = 6x$ analyticcal

$(x-1)^2 q(x) = \dfrac{3(x-1)}{x}$ analytical at $x=0$.

$p_0 = 6 \cdot 1 = 6, \quad q_0 = 0$

$\Rightarrow r(r-1) + 6r = 0 \quad \Rightarrow r(r+5) = 0 \quad r = 0, -5.$

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In each of the problems 14 and 15,

(a) Show that $x=0$ is a regular singular point of the given differential equation.

(b) Find the exponent of the singular point $x=0$.

(c) Find the first three non zero terms in each of the two solutions (not multiples of each other) about $x=0$.

#14 $\quad xy'' + 2xy' + 6e^x y = 0$

(a) and (b) are already done in #1.

Answer:

(c). $\quad y = x^r \sum a_n x^n$

$r=1$ (take the largest first).

$y = \sum_{n=0}^{\infty} a_n x^{n+1} \quad y' = \sum a_n (n+1) x^n, \quad y'' = \sum a_n (n+1) n x^{n-1}$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+1) n \, x^n + 2 \sum_{n=0}^{\infty} a_n (n+1) x^{n+1} + 6\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

~~Highest expo~~ Lowest exponent: $x^0$

Collect order $x^0$: $\quad a_0 \cdot 1 \cdot 0 = 0 \Rightarrow$ No restrictions on $a_0$

Hw12.2

Collect terms $x^1$:

$a_1 \cdot 2 \cdot 1 + 2a_0 + 6a_0 = 0$

$\Rightarrow a_1 = -4a_0$

Collect terms $x^2$: $a_2 \cdot 3 \cdot 2 + 2a_1 \cdot 2 + 6a_0 + 6a_1 = 0$

$a_2 = -\frac{1}{6}(10a_1 + 6a_0) = -a_0 - \frac{10}{6}(-4)a_0 = \frac{20-3}{3}a_0 = \frac{17}{3}a_0$

Collect order $x^3$:

$a_3 \cdot 4 \cdot 3 + 2a_2 \cdot 3 + 6a_2 + 6a_1 + 3a_0 = 0$

$\Rightarrow a_3 = -\frac{1}{12}(12a_2 + 6a_1 + 3a_0)$

$= -a_2 - \frac{1}{2}(-4a_0) - \frac{1}{4}a_0 = \frac{-68+24-3}{12}a_0 = -\frac{47}{12}a_0$

$\Rightarrow y_1(x) = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \dots$

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According to theorem 5.6.1

$y_2 = a y_1(x) \ln x + x^0\left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$

$= a y_1(x) \ln x + 1 + \sum_{n=1}^{\infty} c_n x^n.$

$y_2' = a y_1'(x) \ln x + a y_1(x) \frac{1}{x} + \sum_{n=1}^{\infty} c_n n x^{n-1}$

$y_2'' = a y_1''(x) \ln x + a y_1'(x) \frac{1}{x} + a y_1'(x) \frac{1}{x} + a y_1 \left(-\frac{1}{x^2}\right) + \sum_{n=1}^{\infty} c_n n (n-1) x^{n-2}$

Since $y_1$ is a solution, the terms with $\ln x$ cancel out:

$x y_2'' + 2x y_2' + 6e^x y_2 = a\,(\underbrace{x y_1'' + 2x y_1' + 6e^x y_1}_{=0})\ln x + \dots$

Then the equation becomes:

$$2ay_1' - \frac{a}{x}y_1 + \sum_{n=1}^{\infty} c_n n(n-1)x^{n-1} + 6\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots\right)\left(1+\sum_{n=1}^{\infty} c_n x^n\right)$$
$$+ 2ay_1 + 2\sum_{n=1}^{\infty} c_n n x^n = 0$$

$$\Rightarrow 2a\left(1-8x+17x^2-\frac{47}{3}x^3+\dots\right) - a\left(1-4x+\frac{17}{3}x^2-\frac{47}{12}x^3+\dots\right)$$
$$+2a\left(x-4x^2+\frac{17}{3}x^3-\frac{47}{12}x^4+\dots\right) + 6\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots\right)\left(1+\sum_{n=1}^{\infty} c_n x^n\right) = 0.$$
$$+\sum_{n=1}^{\infty} c_n n(n-1)x^{n-1} + 2\sum_{n=1}^{\infty} c_n n x^n$$

Collect terms $x^0$:

$2a - a + 0 + 6 = 0 \Rightarrow a = -6$

Collect terms $x^1$:

$+12\cdot 8 - 24 - 12 + 2\cdot 1\cdot c_2 + 2c_1 + 6 + 6c_1 = 0$

$c_1$ would be a free parameter here. However, we can substract $y_1$ and assume $c_1 = 0$ $(y_1 = x - 4x^2 + \dots)$

$c_1 = 0$ $\quad 96 - 36 + 6 + 2c_2 = 0 \Rightarrow c_2 = -33$

Collect terms $x^2$:

Sorry, they don't vanish $\quad$ this one does.

$-12\times 17 + 2\times 17 + \underline{12\times 4} + 3\times 2\times c_3 + 2\times 2\, c_2 + 6c_2 + 6\cancel{c_1}^{0} + 3 = 0$

$\Rightarrow -13\cdot 12 + 2\cdot 17 + 3 + 6c_3 + 10\times(-33) = 0$

$\Rightarrow \quad c_3 = \frac{1}{6}(159 - 34 + 3 + 330) = \frac{449}{6}$

$\Rightarrow y_2 = -6\, y_1(x) \ln x + 1 - 33x^2 + \frac{449}{6}x^3 + \dots$

Hw12.3

#15. $x(x-1)y'' + 6x^2 y' + 3y = 0$.

Answer: Parts (a) and (b) were done previously.

Largest exponent: $r=1$

$y = x \cdot \sum a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$

$y' = \sum a_n (n+1) x^n \qquad y'' = \sum a_n (n+1) n \, x^{n-1}$

$(x^2-x) \sum a_n (n+1) n \, x^{n-1} + 6x^2 \sum a_n (n+1) x^n + 3 \sum a_n x^{n+1} = 0$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+1) n \, x^{n+1} - \sum_{n=0}^{\infty} a_n (n+1) n \, x^n + 6 \sum_{n=0}^{\infty} a_n (n+1) x^{n+2} + 3 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

Lowest order: $x^0$: $0 \cdot a_0 = 0 \Rightarrow a_0$ is a free parameter.

Collect terms $x^1$:

$0 \cdot a_0 - a_1 \cdot 2 \cdot 1 + 3a_0 = 0 \Rightarrow a_1 = \frac{3}{2} a_0$

Collect terms $x^2$:

$2 \cdot 1 \cdot a_1 - 2 \cdot 3 \cdot a_2 + 6 a_0 \cdot 1 + 3a_1 = 0$

$a_2 = \frac{1}{6}\left(3a_0 + 6a_0 + \frac{9}{2}a_0\right) = \frac{1}{6} \frac{18+9}{2} a_0 = \frac{27}{12} a_0 = \frac{9}{4} a_0$

Collect terms $x^3$:

$a_2 \cdot 3 \cdot 2 - 4 \cdot 3 a_3 + 6 \cdot 2 \cdot a_1 + 3a_2 = 0$

$a_3 = \frac{1}{12}\left(9 \cdot \frac{9}{4} + 6 \cdot 3\right) a_0 = \frac{153}{12 \cdot 4} a_0 = \frac{51}{16} a_0$

$\Rightarrow y_1 = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4$.

$r_2 = 0,\ r_1 = 1 \qquad r_1 - r_2 = 1$ and so differ by a natural number.

According to Theorem 5.6.1, $y_2$ has the form

$$y_2 = a\,y_1(x)\ln x + x^0 \cdot \left(1 + \sum_{n=1}^{\infty} c_n x^n\right)$$

$$= a\,y_1(x)\ln x + 1 + \sum_{n=1}^{\infty} c_n x^n$$

Need to substitute it into the ODE:

$$y_2' = a\,y_1'(x)\ln x + a\,y_1(x)\frac{1}{x} + \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y_2'' = a\,y_1''(x)\ln x + a\,y_1'(x)\frac{1}{x} + a\,y_1'\frac{1}{x} + a\,y_1\left(-\frac{1}{x^2}\right) + \sum_{n=1}^{\infty} c_n n(n-1)x^{n-2}$$

$$= a\,y_1''(x)\ln x + 2\,y_1'(x)\frac{1}{x} - \frac{a}{x^2}y_1(x) + \sum_{n=1}^{\infty} c_n n(n-1)x^{n-2}.$$

Again, the logarithmic part cancells out because $y_1$ is a solution. Since $y_1 = x + \frac{3}{2}x^2 + \dots$

we can assume $c_1 = 0$ here.

$$\Rightarrow \; 2(x-1)a\,y_1'(x) - \frac{a(x-1)}{x}y_1(x) + \sum c_n n(n-1)x^n - \sum c_n n(n-1)x^{n-1}$$

$$+ 6ax\,y_1 + \sum c_n n\,x^{n+1} + 3\left(1 + \sum c_n x^n\right) = 0$$

$$\Rightarrow \; 2a(x-1)\left(1 + 3x + \frac{27}{4}x^2 + \frac{51}{4}x^3 + \dots\right) - a(x-1)\left(1 + \frac{3}{2}x^1 + \frac{9}{4}x^2 + \frac{51}{16}x^3 + \dots\right)$$

$$+ 6a\left(x^2 + \frac{3}{2}x^3 + \frac{9}{4}x^4 + \frac{51}{16}x^5 + \dots\right) + \sum_{n=1}^{\infty} c_n n(n-1)x^n - \sum_{n=1}^{\infty} c_n n(n-1)x^{n-1}$$

$$+ \sum_{n=1}^{\infty} c_n n\,x^{n+1} + 3\left(1 + \sum_{n=1}^{\infty} c_n x^n\right) = 0.$$

Collect $x^0$:

$-2a + a - 0 + 3 = 0 \quad \Rightarrow a = 3$

Collect $x^1$:

$6 - 18 - 3 + \frac{9}{2} + 0 - c_2 \cdot 2 \cdot 1 + 0 + 3\cancel{c_1}^{\to 0} = 0$

$\Rightarrow c_2 = \frac{1}{2}\left(-15 + \frac{9}{2}\right) = \frac{1}{2}\,\frac{9-30}{2} = -\frac{21}{4}.$

Collect $x^2$:

$\underline{18} - 6 \cdot \frac{27}{4} - 3 \cdot \frac{3}{2} + 3 \cdot \frac{9}{4} + \underline{18} + c_2 \cdot 2 \cdot 1 - c_3 \cdot 3 \cdot 2 + \cancel{c_1}^{\to 0} + 3c_2 = 0$

$c_3 = \frac{1}{6}\left(36 - \frac{81}{2} - \frac{9}{2} + \frac{27}{4} + 5 \cdot \left(-\frac{21}{4}\right)\right)$

$= \frac{1}{6}\left(-9 - \frac{78}{4}\right) = -\frac{114}{6 \cdot 4} = -\frac{19}{4}$

$\Rightarrow$ ~~$y_2 = 3y_1(x)$~~ $\quad y_2 = 3y_1(x)\ln(x) + 1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 + \dots$

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Section 6.1 Problems 18 and 19. Use ~~the~~ integration by parts to find the Laplace transform of the following functions. Here $n$ is a positive integer and $a$ is a real constant.

\# 18 $\quad f(t) = t^n e^{at}$

Answer: $\mathcal{L}[\cancel{t}\, e^{at}] = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty$

$= \frac{1}{s-a}$ for $n = 0$.

$$\mathcal{L}[t^n e^{at}] = \int_0^\infty e^{-st} t^n e^{at}\, dt$$

$$= \int_0^\infty t^n e^{(a-s)t}\, dt = t^n \frac{e^{(a-s)t}}{a-s}\Big|_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{(a-s)t}}{a-s}\, dt$$

$$= 0 + \frac{1}{s-a}\, n\, \mathcal{L}[t^{n-1} e^{at}]$$

$$\Rightarrow \mathcal{L}[t^n e^{at}] = \frac{n}{s-a} \mathcal{L}[t^{n-1} e^{at}]$$

$$\Rightarrow \mathcal{L}[t\, e^{at}] = \frac{1}{(s-a)^2}$$

$$\mathcal{L}[t^2 e^{at}] = \frac{2}{(s-a)^3}$$

$$\mathcal{L}[t^3 e^{at}] = \frac{3!}{(s-a)^4}$$

$$\vdots$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}.$$

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#19 $f(t) = t^2 \sin(at)$. the Laplace transform of

Answer: Let's first compute

$t\sin(at)$ and $t\cos(at)$.

$$\mathcal{L}[t\sin(at)] = \int t\sin(at)\, e^{-st}\, dt = t\sin(at)\frac{e^{-st}}{-s}\Big|_0^\infty - \int (\sin(at) + at\cos(at))\frac{e^{-st}}{-s}\, dt$$

$$= 0 + \frac{1}{s}\mathcal{L}[\sin(at)] + \frac{a}{s}\int t\cos(at)e^{-st}\, dt$$

$$= \frac{1}{s}\mathcal{L}[\sin(at)] + \frac{a}{s}\left[t\cos(at)\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty (\cos(at) - at\sin(at))\frac{e^{-st}}{-s}\, dt\right.$$

$$= \frac{1}{s}\mathcal{L}[\sin(at)] + \frac{a}{s^2}\mathcal{L}[\cos(at)] - \frac{a^2}{s^2}\mathcal{L}[t\sin(at)]dt$$

Hw12.5

$$\Rightarrow \left(1+\frac{a^2}{s^2}\right)\mathcal{L}[t\sin(at)] = \frac{1}{s}\mathcal{L}[\sin(at)] + \frac{a}{s^2}\mathcal{L}[\cos(at)]$$

$$\Rightarrow \mathcal{L}[t\sin(at)] = \frac{s}{s^2+a^2}\mathcal{L}[\sin(at)] + \frac{a}{s^2+a^2}\mathcal{L}[\cos(at)].$$

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Let's now compute $\mathcal{L}[t\cos(at)]$

$$\mathcal{L}[t\cos(at)] = \int_0^\infty e^{-st}\, t\cos(at)\,dt$$

$$= t\cos(at)\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\left(\cos(at) - at\sin(at)\right)dt$$

$$= \frac{1}{s}\mathcal{L}[\cos(at)] - \frac{a}{s}\left(\cancel{\frac{e^{-st}}{-s}t\sin(at)\Big|_0^\infty}^{\to 0} - \int_0^\infty \frac{e^{-st}}{-s}\left(\sin(at) + at\cos(at)\right)dt\right.$$

$$= \frac{1}{s}\mathcal{L}[\cos(at)] + \frac{a}{s^2}\mathcal{L}[\sin(at)] - \frac{a^2}{s^2}\mathcal{L}[t\cos(at)]$$

$$\Rightarrow \mathcal{L}[t\cos(at)] = \frac{s}{s^2+a^2}\mathcal{L}[\cos(at)] - \frac{a}{s^2+a^2}\mathcal{L}[\sin(at)].$$

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}, \quad \mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$$

$$\Rightarrow \mathcal{L}[t\sin(at)] = \frac{s}{s^2+a^2}\frac{a}{s^2+a^2} + \frac{a}{s^2+a^2}\frac{s}{s^2+a^2} = \frac{2as}{(s^2+a^2)^2}.$$

$$\mathcal{L}[t\cos(at)] = \frac{s}{s^2+a^2}\frac{s}{s^2+a^2} - \frac{a}{s^2+a^2}\frac{a}{s^2+a^2} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Now, $\mathcal{L}[t^2\sin at] = \int_0^\infty e^{-st}t^2\sin at\,dt = \cancel{\frac{e^{-st}}{-s}t^2\sin at\Big|_0^\infty}^{\to 0}$

$$-\int_0^\infty \frac{e^{-st}}{-s}\left(2t\sin at + t^2 a\cos at\right)dt$$

$$= \frac{2}{s}\mathcal{L}[t\sin at] + \frac{a}{s}\int_0^\infty e^{-st}\, t^2 \cos at\, dt$$

$$= \frac{2}{s}\mathcal{L}[t\sin at] + \frac{a}{s}\left[\cancel{\frac{e^{-st}}{-s} t^2 \cos at\Big|_0^\infty} - \int_0^\infty \frac{e^{-st}}{-s}\left(2t\cos at - at^2 \sin at\right)dt\right.$$

$$= \frac{2}{s}\mathcal{L}[t\sin at] + \frac{2a}{s^2}\mathcal{L}[t\cos at] - \frac{a^2}{s^2}\mathcal{L}[t^2\sin at]$$

$$\Rightarrow \left(1+\frac{a^2}{s^2}\right)\mathcal{L}[t^2\sin at] = \frac{2}{s}\mathcal{L}[t\sin at] + \frac{2a}{s^2}\mathcal{L}[t\cos at]$$

$$\Rightarrow \mathcal{L}[t^2\sin at] = \frac{s^2}{s^2+a^2}\left(\frac{2}{s}\frac{2as}{(s^2+a^2)^2} + \frac{2a}{s^2}\frac{s^2-a^2}{(s^2+a^2)^2}\right)$$

$$= \frac{as^2}{(s^2+a^2)^3}\left(4 + \frac{2}{s^2}(s^2-a^2)\right) = \frac{4as^2+2as^2-2a^3}{(s^2+a^2)^3}$$

$$= \frac{6as^2-2a^3}{(s^2+a^2)^3} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}.$$

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Section 6.2 #21 Use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0$$

Answer: Apply Laplace transform to the equation:

$$\mathcal{L}[y''] = s\mathcal{L}[y'] - y'(0) = s\left(s\mathcal{L}[y] - y(0)\right) - 0 = s^2\mathcal{L}[y] - s$$

$$\Rightarrow s^2 Y - s - 2\left(sY(s) - 1\right) + 2Y = \mathcal{L}[\cos t] = \frac{s}{s^2+1}$$

Hw12.6

$$\Rightarrow (s^2-2s+2)Y - s + 2 = \frac{s}{s^2+1}$$

$$\Rightarrow Y = \frac{s-2}{s^2-2s+2} + \frac{s}{(s^2-2s+2)(s^2+1)}$$

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{as+b}{s^2-2s+2} + \frac{cs+d}{s^2+1}$$

$$= \frac{(as+b)(s^2+1)+(cs+d)(s^2-2s+2)}{(s^2-2s+2)(s^2+1)}$$

$$= \frac{s^3(a+c)+s^2(b+d-2c)+s(a+2c-2d)+b+2d}{(s^2-2s+2)(s^2+1)}$$

$a=-c$

$c=-a$ $\quad d=-2c$

$\Rightarrow$ $a+c=0$

$b+d-2c=0 \Rightarrow -d-2c=0$

$a+2c-2d=1 \Rightarrow c-2d=1 \quad c=2d+1 \Rightarrow 5c=1 \quad c=\frac{1}{5}$

$b+2d=0 \quad b=-2d$

$c=\frac{1}{5}$, $d=-\frac{2}{5}$, $b=\frac{4}{5}$, $a=-\frac{1}{5}$

$$\Rightarrow Y = \frac{s-2}{(s-1)^2+1} + \frac{1}{5}\frac{-s+4}{(s-1)^2+1} + \frac{1}{5}\frac{s-2}{s^2+1}$$

$$= \frac{\frac{4}{5}s-\frac{6}{5}}{(s-1)^2+1} + \frac{1}{5}\frac{s-2}{s^2+1} = \frac{4}{5}\frac{s-1}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}$$

$$\Rightarrow y(t) = \frac{4}{5}e^t\cos t - \frac{2}{5}e^t\sin t + \frac{1}{5}\cos t - \frac{2}{5}\sin t$$

Section 6.2 #24. Find the Laplace transform $Y(s) = \mathcal{L}[y]$ of the solution of the given initial value problem. A method of determining the inverse Laplace transform is developed in § 6.3.

$$y'' + 4y = \underbrace{\begin{cases} 1, & 0 \le t \le \pi \\ 0, & \pi \le t < \infty \end{cases}}_{g(t)} \qquad y(0) = 1,\ y'(0) = 0.$$

Answer:
Apply Laplace transform to the right hand side:

$$G(s) = \int_0^\infty g(t) e^{-st}\, dt = \int_0^\pi e^{-st}\, dt + \int_\pi^\infty 0\, dt$$

$$= \frac{e^{-st}}{-s}\Big|_0^\pi = \frac{e^{-s\pi}}{-s} + \frac{1}{s} = \frac{1 - e^{-s\pi}}{s}$$

$$\Rightarrow\ s(sY(s) - y(0)) - y'(0) + 4Y(s) = \frac{1 - e^{-s\pi}}{s}$$

$$\Rightarrow\ s^2 Y - s + 4Y = \frac{1 - e^{-s\pi}}{s}$$

$$\Rightarrow\ Y = \frac{s}{s^2 + 4} + \frac{1 - e^{-s\pi}}{s(s^2 + 4)}$$

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Section 6.4 # 1, 2. In each of the following problem
(a) Find the solution of the initial value problem
(b) Draw the graph of the solution and the forcing function; explain how they are related.

#1 $y'' + y = f(t) = \begin{cases} 1, & 0 \le t \le 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$

$y(0) = 0,\ y'(0) = 1.$

$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \ge c. \end{cases}$ $\quad c > 0.$

$f(t) = 1 - u_{3\pi} = u_0 - u_{3\pi}$

Apply Laplace transform:

$s(sY - y(0)) - y'(0) + Y = \int_0^{3\pi} e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_0^{3\pi} = \frac{1 - e^{-3\pi s}}{s}$

$\Rightarrow (s^2+1)Y - 1 = \frac{1 - e^{-3\pi s}}{s}$

$\Rightarrow Y = \frac{1}{s^2+1} + \frac{1 - e^{-3\pi s}}{s(s^2+1)}$

$H(s) = \frac{1}{s(s^2+1)} = \frac{a}{s} + \frac{b + cs}{s^2+1} = \frac{a(s^2+1) + s(b+cs)}{s(s^2+1)}$

$= \frac{s^2(a+c) + s(b) + a}{s(s^2+1)}$ $\quad \Rightarrow a = 1,\ b = 0,\ c = -1$

$\Rightarrow H(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-s}{s^2+1}$

$= \mathcal{L}[1] - \mathcal{L}[\cos t] = \mathcal{L}[1 - \cos t].$

Using the relation $\mathcal{L}[u_c(t)\, f(t-c)] = e^{-cs}\mathcal{L}[f]$, we get:

$$Y = \mathcal{L}[\sin t] + \mathcal{L}[1-\cos t] - \mathcal{L}[u_{3\pi}(t)(1+\cos t)]$$

Graph at the end

$\Rightarrow$ The solution is:

$$y = \sin t + 1 - \cos t - u_{3\pi}(t)(1+\cos t)$$

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#2 $\quad y'' + 2y' + 2y = h(t), \quad y(0) = 0, \quad y'(0) = 1$

Answer

$$h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi, \text{ and } t \ge 2\pi \end{cases}$$

$$\mathcal{L}[h(t)] = \int_{\pi}^{2\pi} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_{\pi}^{2\pi} = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

$$h(t) = u_{\pi} - u_{2\pi}$$

Apply Laplace transform:

$$s(sY - y(0)) - y'(0) + 2(sY - y(0)) + 2Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

$$s^2 Y - 1 + 2sY + 2Y = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

$$\Rightarrow Y = \frac{1}{s^2+2s+2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}$$

$$\cancel{H(s) = \frac{1}{s(s^2+2s+2)} = \frac{1}{s^2(s+2)} = \frac{a}{s+2} + \frac{bs+c}{s^2} = \frac{as^2 + (s+2)(bs+c)}{(s+2)s^2}}$$

HW12.8

$$H(s) = \frac{1}{s(s^2+2s+2)} = \frac{a}{s} + \frac{bs+c}{s^2+2s+2}$$

$$= \frac{a(s^2+2s+2) + s(bs+c)}{s(s^2+2s+2)} = \frac{s^2(a+b)+s(2a+c)+2a}{s(s^2+2s+2)}$$

$\Rightarrow a+b=0, \quad 2a+c=0, \quad 2a=1$

$a=\frac{1}{2}, \quad b=-\frac{1}{2}, \quad c=-1$

$$H(s) = \frac{1}{s(s^2+2s+2)} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}s-1}{s^2+2s+2}$$

$$= \frac{1}{2}\mathcal{L}[1] + \frac{-\frac{1}{2}s-1}{s^2+2s+1+1} = \frac{1}{2}\mathcal{L}[1] + \frac{-\frac{1}{2}s-1}{(s+1)^2+1}$$

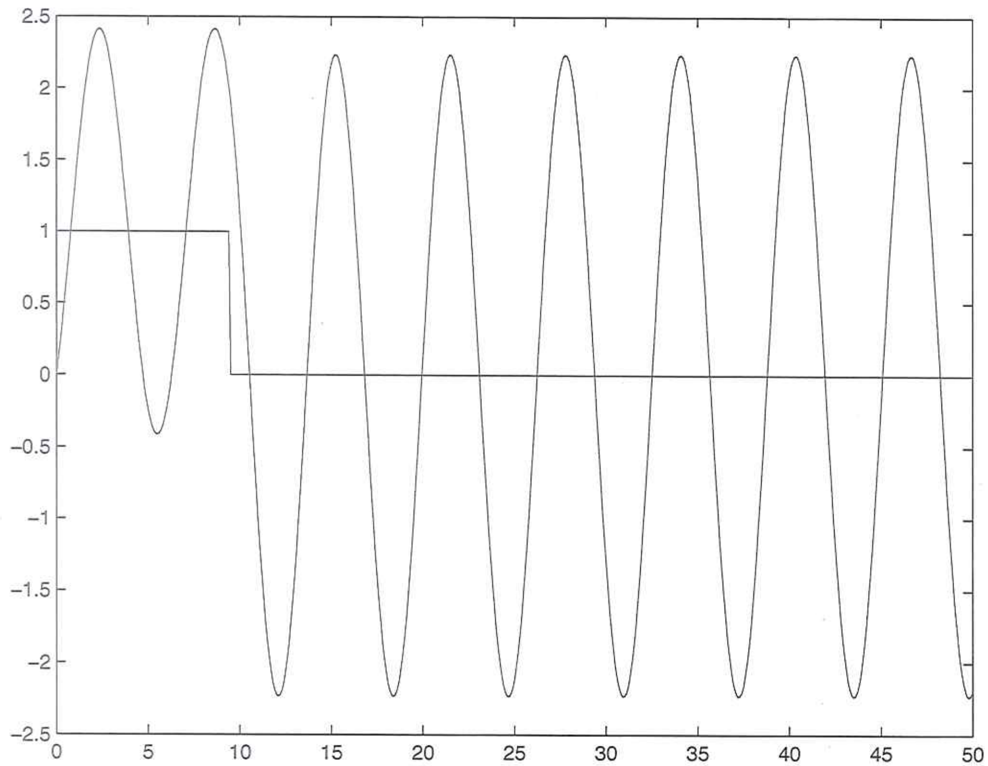
$$= \frac{1}{2}\mathcal{L}[1] - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

$$= \mathcal{L}\left[\frac{1}{2} - \frac{1}{2}\cdot e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t\right]$$

$$\frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\sin t]$$

$$\Rightarrow y(t) = e^{-t}\sin(t) + \frac{1}{2}u_\pi(t)\left(1-e^{-(t-\pi)}\cos(t-\pi) - e^{-(t-\pi)}\sin(t-\pi)\right)$$
$$- \frac{1}{2}u_{2\pi}(t)\left(1-e^{-(t-2\pi)}\cos(t-2\pi) - e^{-(t-2\pi)}\sin(t-2\pi)\right)$$

$$= e^{-t}\sin(t) + \frac{1}{2}u_\pi(t)\left(1+e^{-(t-\pi)}\cos t + e^{-(t-\pi)}\sin t\right)$$
$$- \frac{1}{2}u_{2\pi}(t)\left(1-e^{-(t-2\pi)}\cos t - e^{-(t-2\pi)}\sin t\right)$$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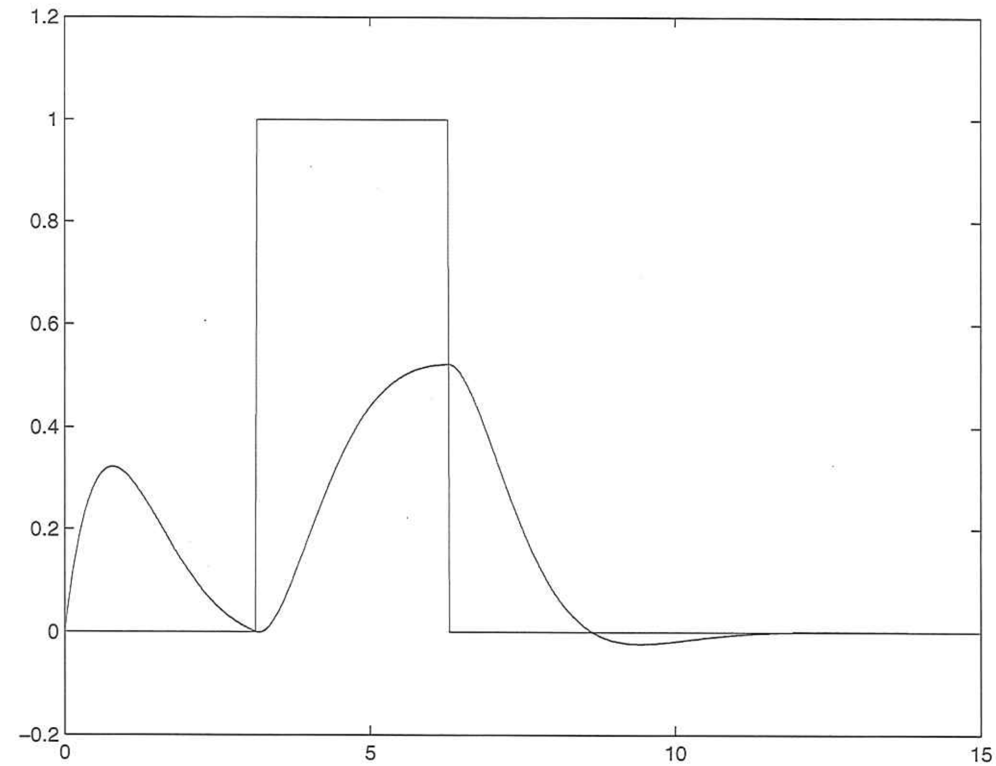
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