Homework 13:

Problem: Determine the equilibrium temperature distribution for a one-dimensional rod with constant properties with the following sources and boundary conditions:

(a) $Q = 0, \; u(0) = 0, \; u(L) = T.$

Answer: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q \qquad Q = 0.$

Since we look for equilibrium solutions, $u = u(x)$ doesn't depend on time.

$\Rightarrow \frac{\partial u}{\partial t} = 0 \quad \Rightarrow \quad \frac{du}{dx^2} = 0 \quad \Rightarrow u(x) = C_1 x + C_2$

$u(0) = C_2 = 0, \quad u(x) = C_1 x$

$u(L) = C_1 L = T \quad \Rightarrow C_1 = \frac{T}{L}$

$\Rightarrow \quad u(x) = \frac{T}{L} x$ is the equilibrium distribution with those boundary conditions.

(b) $Q = 0, \; u(0) = T, \; \frac{\partial u}{\partial x}(L) = \alpha.$

Answer: By the same argument, $u(x) = C_1 x + C_2$

$\frac{\partial u}{\partial x} = C_1 = \alpha, \quad u(0) = C_2 = T$

$\Rightarrow \quad u(x) = \alpha x + T.$

Section 10.5 #8 Find the solution of the heat conduction problem:

$u_t = \frac{1}{4} u_{xx}$, $0 < x < 2$, $t > 0$

$u(0,t) = 0$, $u(2,t) = 0$, $t > 0$.

$u(x,0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin \pi x + 4\sin 2\pi x$, $0 < x < 2$

Answer: $L = 2$, $k = \frac{1}{4}$

$$u(x,0) = 2\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{2\pi x}{L}\right) + 4\sin\left(\frac{4\pi x}{L}\right)$$

Using the principle of superposition, we get:

$$u(x,t) = 2\sin\left(\frac{\pi x}{2}\right)e^{-\frac{1}{4}\left(\frac{\pi}{2}\right)^2 t} - \sin(\pi x)e^{-\frac{1}{4}(\pi)^2 t} + 4\sin(2\pi x)e^{-\frac{1}{4}(2\pi)^2 t}$$

$$= 2\sin\left(\frac{\pi x}{2}\right)e^{-\frac{\pi^2}{16}t} - \sin(\pi x)e^{-\frac{\pi^2}{4}t} + 4\sin(2\pi x)e^{-\pi^2 t}.$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at $0°C$ for all $t > 0$. In each of Problems 9 and 10 find the expression of the temperature $u(x,t)$ if the initial temperature distribution in the rod is the given function. $k = 1$

#9 $u(x,0) = 50°C$. $0 < x < 40$.

Answer: We need to find the sine Fourier coefficients of the constant function $u(x,0) = 50°C = f(x)$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{20}\int_0^{40} 50\sin\frac{n\pi x}{L}\,dx = \frac{5}{2}\,\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{40}$$

$$= \frac{5}{2}\,\frac{-40}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$\Rightarrow\ 50 = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi)}{n}\sin\frac{n\pi x}{40}$$

$$\Rightarrow\ u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi)}{n}\,e^{-\left(\frac{n\pi}{40}\right)^2 t}\sin\frac{n\pi x}{40}.$$

— o —

10 $$u(x,0) = \begin{cases} x, & 0\le x<20 \\ 40-x, & 20\le x\le 40. \end{cases} = f(x)$$

Answer: Need to compute the sine-Fourier ~~sea~~ coefficients.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad L = 40,\ n=1$$

$$\int_0^{20} x\sin\frac{n\pi x}{L}\,dx = x\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\Big|_0^{20} - \int_0^{20}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= -20\cdot\frac{40}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L}{n\pi}\,\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^{20}$$

$$= -\frac{800}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{20}^{40} (40-x)\sin\frac{n\pi x}{L}\,dx = 40\,\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{20}^{40}$$

$$-x\cdot\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{20}^{40} + \int_{20}^{40}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{40^2}{n\pi}\left(\cos\frac{n\pi}{2} - \cancel{\cos(n\pi)}\right) + \frac{L}{n\pi}\left(\cancel{40\cos(n\pi)} - 20\cos\left(\frac{n\pi}{2}\right)\right)$$

$$-\frac{L}{n\pi}\,\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{20}^{40}$$

$$= \frac{40\cdot 20}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cancel{\sin(n\pi)}^{0} - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$\Rightarrow B_n = \frac{2}{L}\left(\cancel{-\frac{800}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right.$$

$$\left. + \cancel{\frac{800}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{1}{20^2}\,\frac{40^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{40}\right)^2 t}$$

$$= \frac{160}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}\sin\left(\frac{n\pi x}{40}\right)e^{-\left(\frac{n\pi}{40}\right)^2 t}$$

Problem: Numerically compute solutions of the heat equation

$$u_t = k u_{xx}, \quad 0 \le x \le 1, \quad k = 1$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = f(x) = \begin{cases} 0, & 0 \le x < 0.5 \\ x - 0.5, & 0.5 \le x < 0.75 \\ 1 - x, & 0.75 \le x < 1 \end{cases}$$

using the numerical scheme

$$u_j^{(m+1)} = u_j^{(m)} + s\left(u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}\right), \quad j = 1, \dots, N-1$$

$$s = \frac{k\,\Delta t}{\Delta x^2}$$

with $\Delta x = 0.1$ ($N = 10$)

Do various s (discuss stability):

$s = 0.49$, $s = 0.5$, $s = 0.51$, $s = 0.52$.

Answer:

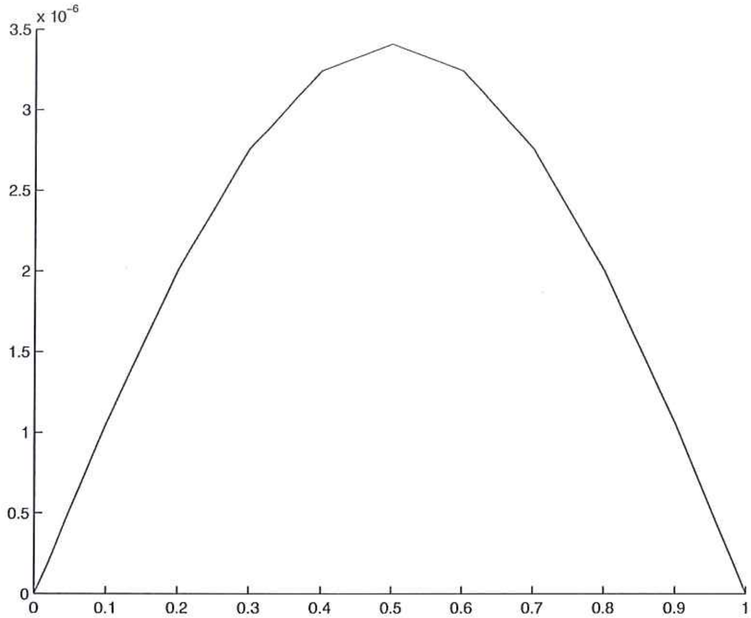
See the attached plots and code.

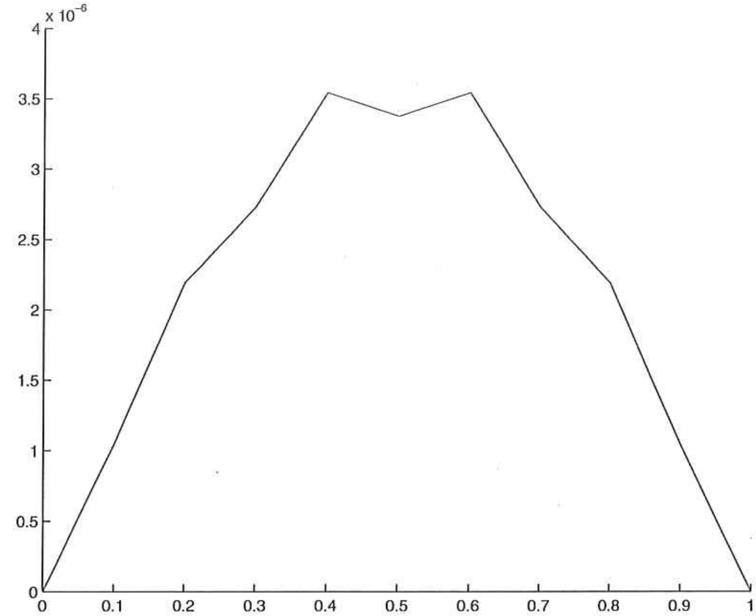
We note:

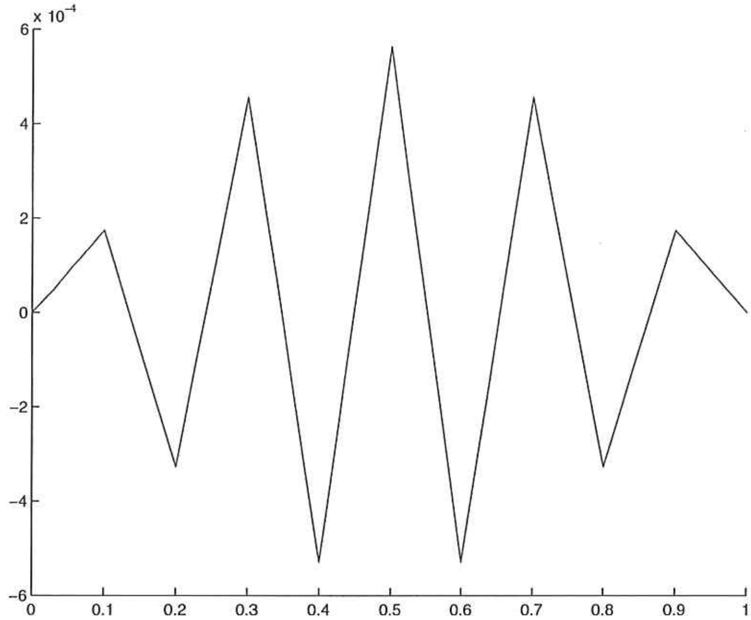
Instabilities if $s > \frac{1}{2}$

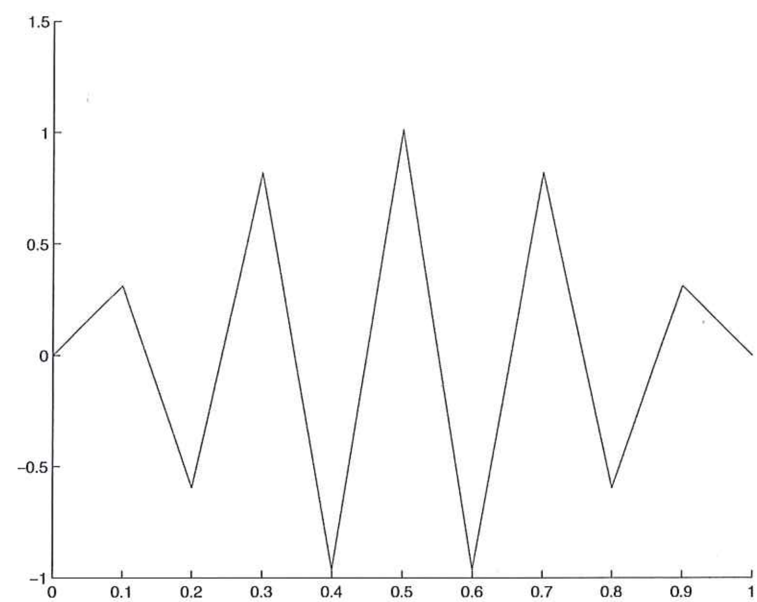
Stabilities if $s < \frac{1}{2}$

Pure oscillations if $s = \frac{1}{2}$


x 10^-6
3.5
3
2.5
2
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


x 10^-6
4
3.5
3
2.5
2
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


x 10^-4
6
4
2
0
−2
−4
−6
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


1.5
1
0.5
0
−0.5
−1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1