Homework 2:

Problem: Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 , & x \le c \\ (x-c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x.

Answer:

For $x > c$, $y' = 2(x-c)$

$2\sqrt{y} = 2\sqrt{(x-c)^2} = 2(x-c)$ because $x - c > 0$

$\Rightarrow y' = 2\sqrt{y}$ for $x > c$

For $x < c$, $y(x) = 0$ in the interval $(-\infty, c)$

$\Rightarrow y'(x) = 0$, $2\sqrt{y(x)} = 0$ for $x > c$

$\Rightarrow y'(x) = 2\sqrt{y(x)}$ for $x > c$.

If $x = c$, then both left and right derivatives are zero and $y(c) = 0 \Rightarrow y'(c) = 2\sqrt{y(c)} = 0$.

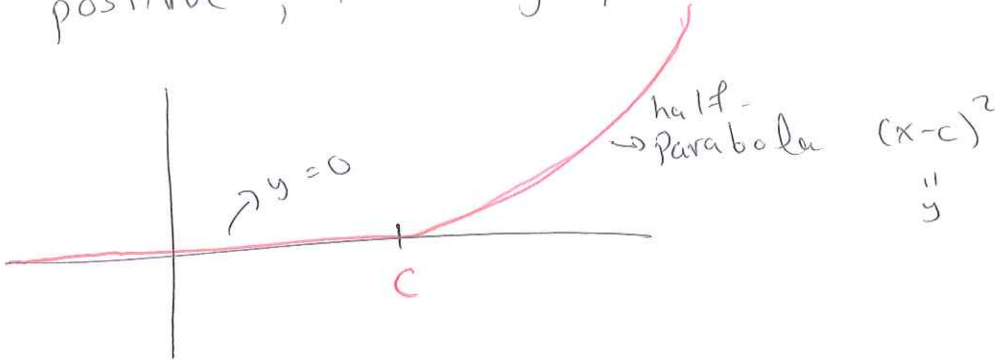
Therefore, $y' = 2\sqrt{y}$ for all x.

Problem Construct a figure illustrating the fact that the initial value problem $\begin{cases} y' = 2\sqrt{y} \\ y(0) = 0 \end{cases}$ has infinitely many solutions

Answer: Consider the previous problem.

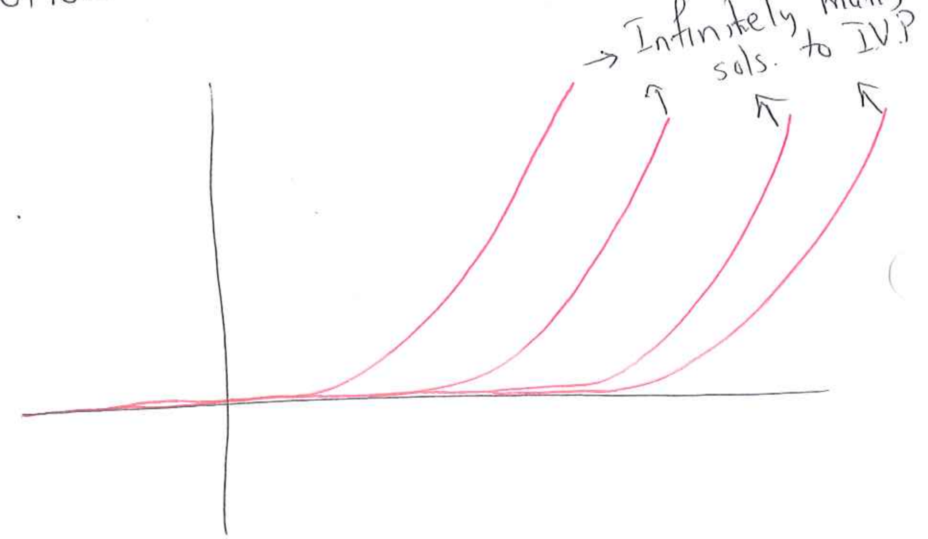
$$y(x) = \begin{cases} (x-c)^2 & \text{for } x \ge c \\ 0 & \text{for } x < c \end{cases}$$ are all solutions to the ODE.

For $c > 0$ positive, their graphs look like



Clearly, $y(0) = 0$ if $c > 0$.
Then, any of the above functions for any $c > 0$ provides/defines a solution to the initial value problem

$$\begin{cases} y' = 2\sqrt{y} \\ y(0) = 0 \end{cases}$$



Section 2.1 Problems 13-20. In each of the problems 13-20 find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}$, $y(0) = 1$.

Integrating factors: $\mu y' - \mu y = 2te^{2t}\mu$

$\frac{d}{dt}[\mu y] = \mu y' + \frac{d\mu}{dt} y \Rightarrow \frac{d\mu}{dt} = -\mu \Rightarrow$ choose $\mu(t) = e^{-t}$

$$\Rightarrow \frac{d}{dt}\left[e^{-t} y\right] = 2te^{2t}e^{-t} = 2te^{t}$$

$$e^{-t} y = \int 2te^{t}\,dt + c$$

integ. by parts

$$\int 2te^{t}\,dt = 2te^{t} - \int 2e^{t}\,dt = 2te^{t} - 2e^{t}$$

$$\Rightarrow e^{-t} y = 2(t-1)e^{t} + c \quad \Rightarrow \quad y = 2(t-1)e^{2t} + ce^{t}$$

$$1 = y(0) = -2 + c \quad \Rightarrow \quad c = 3$$

$$\Rightarrow y = 2(t-1)e^{2t} + 3e^{t}.$$

14.- $y' + 2y = te^{-2t}$, $y(1) = 0$.

$$\frac{d\mu}{dt} = 2\mu \quad \Rightarrow \text{choose } \mu(t) = e^{2t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{2t} y\right] = te^{-2t}e^{2t} = t$$

$$\Rightarrow e^{2t} y = \frac{t^2}{2} + c \quad \Rightarrow \quad y = \frac{t^2}{2}e^{-2t} + ce^{-2t}$$

$$0 = y(1) = \frac{1}{2}e^{-2} + ce^{-2} \quad \Rightarrow \quad c = -\frac{1}{2}$$

$$\Rightarrow y = \frac{t^2}{2}e^{-2t} - \frac{1}{2}e^{-2t}$$

15.- $ty' + 2y = t^2 - t + 1$, $y(1) = \frac{1}{2}$, $t > 0$.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \qquad \mu'(t) = \frac{2}{t}\mu \Rightarrow \ln|\mu| = 2\ln t$$

$$\Rightarrow \text{choose } \mu = e^{2\ln t} = t^2.$$

$$\Rightarrow \frac{d}{dt}\left[t^2 y\right] = \left(t-1+\frac{1}{t}\right)t^2 = t^3 - t^2 + t$$

$$\Rightarrow t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c$$

$$y = \frac{t^2}{4} - \frac{t^3}{3} + \frac{1}{2} + \frac{c}{t^2}$$

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \quad \Rightarrow c = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\Rightarrow y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$$

16.- $y' + \frac{2}{t} y = \frac{\cos t}{t^2}$, $y(\pi) = 0$, $t > 0$

$$\mu' = \frac{2}{t}\mu \quad \Rightarrow \quad \ln \mu = 2\ln t \quad \Rightarrow \quad \mu = t^2$$

$$\Rightarrow \frac{d}{dt}\left[t^2 y\right] = \frac{\cos t}{t^2} t^2 = \cos t$$

$$\Rightarrow t^2 y = \sin t + c \quad \Rightarrow y = \frac{\sin t}{t^2} + \frac{c}{t^2}$$

$$0 = y(\pi) = \frac{\sin \pi}{\pi^2} + \frac{c}{\pi^2} = \frac{c}{\pi^2} \quad \Rightarrow c = 0$$

$$\therefore \ y = \frac{\sin t}{t^2}.$$

17.- $y' - 2y = e^{2t}$, $y(0) = 2$

$$\frac{d\mu}{dt} = -2\mu \quad \Rightarrow \quad \mu(t) = e^{-2t} \quad \Rightarrow \frac{d}{dt}\left[e^{-2t} y\right] = e^{2t} e^{-2t} = 1$$

$$\Rightarrow e^{-2t} y = t + c \quad \Rightarrow y = te^{2t} + ce^{2t}$$

$$2 = y(0) = 0 + c \quad \Rightarrow c = 2 \quad \Rightarrow y = (t+2)e^{2t}$$

Hw2.3

18.- $t y' + 2y = \sin t, \quad y(\pi/2) = 1, \; t > 0$

$$y' + \frac{2}{t} y = \frac{\sin t}{t}$$

$\mu = t^2$ (from previous problems)

$$\Rightarrow \frac{d}{dt}[t^2 y] = \frac{\sin t}{t} t^2 = t \sin t$$

$$\Rightarrow t^2 y = \int t \sin t \, dt + c$$

int. by parts

$$\int t \sin t \, dt = -t \cos t - \int 1 \cdot (-\cos t) \, dt$$

$$= -t \cos t + \int \cos t \, dt = -t \cos t + \sin t$$

$$\Rightarrow t^2 y = -t \cos t + \sin t + c$$

$\Rightarrow$

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}$$

$$1 = y(\pi/2) = 0 + \frac{1 + c}{\left(\frac{\pi}{2}\right)^2} \Rightarrow c = \frac{\pi^2}{4} - 1$$

$$\Rightarrow y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{\frac{\pi^2}{4} - 1}{t^2}$$

19.- $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0, \; t < 0$

$$y' + \frac{4}{t} y = \frac{e^{-t}}{t^3} \qquad \frac{d\mu}{dt} = \frac{4}{t}\mu \Rightarrow \ln \mu = 4 \ln t$$

$$\Rightarrow \mu = e^{4 \ln t} = t^4 \qquad \Rightarrow \frac{d}{dt}[t^4 y] = \frac{e^{-t}}{t^3} t^4 = t e^{-t}$$

$$\Rightarrow t^4 y = \int t e^{-t} dt + c$$

$$\int t e^{-t}\,dt = -te^{-t} - \int 1.(-e^{-t})\,dt = -te^{-t} - e^{-t}$$

$$\Rightarrow y = \frac{-(t+1)e^{-t}}{t^4} + \frac{c}{t^4}$$

$$0 = y(-1) = \frac{c}{1} \Rightarrow c = 0$$

$$\Rightarrow y = \frac{-(t+1)e^{-t}}{t^4}, \quad t < 0.$$

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20.- $t y' + (t+1) y = t, \quad y(\ln(2)) = 1, \quad t > 0.$

$$\Rightarrow y' + \left(1 + \frac{1}{t}\right) y = 1 \qquad \frac{d\mu}{dt} = \left(1 + \frac{1}{t}\right)\mu$$

$$\Rightarrow \ln|\mu| = t + \ln t \Rightarrow \mu = e^{t + \ln t} = t e^t.$$

$$\Rightarrow \frac{d}{dt}\left[t e^t y\right] = t e^t$$

$$t e^t y = \int t e^t\,dt + c, \qquad \int t e^t\,dt = te^t - e^t = (t-1)e^t$$

$$\Rightarrow t e^t y = (t-1)e^t + c \Rightarrow y = \frac{t-1}{t} + \frac{c}{te^t}$$

$$1 = y(\ln(2)) = \frac{\ln 2 - 1}{\ln 2} + \frac{c}{2\ln 2} \Rightarrow 2\ln 2 = 2(\ln 2 - 1) + c \Rightarrow c = 2$$

$$\Rightarrow y = \frac{t-1}{t} + \frac{2}{te^t}.$$

Section 2.1 Problem 29: Consider the initial value problem:

$$y' + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behaviour for large t.

Answer:

$$\frac{d\mu}{dt} = \frac{1}{4}\mu \Rightarrow \text{ choose } \mu(t) = e^{\frac{1}{4}t}.$$

$$\Rightarrow \frac{d}{dt}\left[e^{\frac{1}{4}t} y\right] = (3 + 2\cos(2t))\, e^{\frac{1}{4}t}$$

$$\Rightarrow e^{\frac{1}{4}t} y = \int (3 + 2\cos(2t))\, e^{\frac{1}{4}t}\, dt + c$$

Trick: Do integration by parts twice:

$$\int \cos(2t)\, e^{\frac{1}{4}t}\, dt = \cos(2t)\, 4e^{\frac{1}{4}t} - \int -2\sin(2t)\, 4\, e^{\frac{1}{4}t}\, dt$$

$$= 4\cos(2t)\, e^{t/4} + 8\int \sin(2t)\, e^{t/4}\, dt$$

Now:

$$\int \sin(2t)\, e^{t/4}\, dt = \sin(2t)\, 4e^{t/4} - \int 2\cos(2t)\, 4e^{t/4}\, dt$$

Substitute above and get:

$$\int \cos(2t)\, e^{t/4}\, dt = 4\cos(2t)\, e^{t/4} + 8\left(4\sin(2t)\, e^{t/4} - 8\int \cos(2t)\, e^{t/4}\, dt\right)$$

$$= 4\cos(2t)\, e^{t/4} + 32\sin(2t)\, e^{t/4} - 64\int \cos(2t)\, e^{t/4}\, dt$$

Solve for $\int \cos(2t)\, e^{t/4}\, dt$

$$\int \cos(2t)\, e^{t/4}\, dt = \frac{4\cos(2t) + 32\sin(2t)}{65}\, e^{t/4}$$

$$\Rightarrow e^{t/4} y = 12 e^{t/4} + \frac{8\cos(2t)+64\sin(2t)}{65} e^{t/4} + c$$

$$\Rightarrow \quad y = 12 + \frac{8\cos(2t)+64\sin(2t)}{65} + c e^{-t/4}$$

$$0 = y(0) = 12 + \frac{8}{65} + c \Rightarrow \quad c = -12 - \frac{8}{65} = -\frac{788}{65}$$

$$\Rightarrow y = 12 + \frac{8\cos(2t)+64\sin(2t)}{65} - \frac{788}{65} e^{-t/4}$$

As $t \to \infty$, $e^{-t/4}$ decays to zero, and $\cos(\ )$, $\sin(\ )$ oscillate

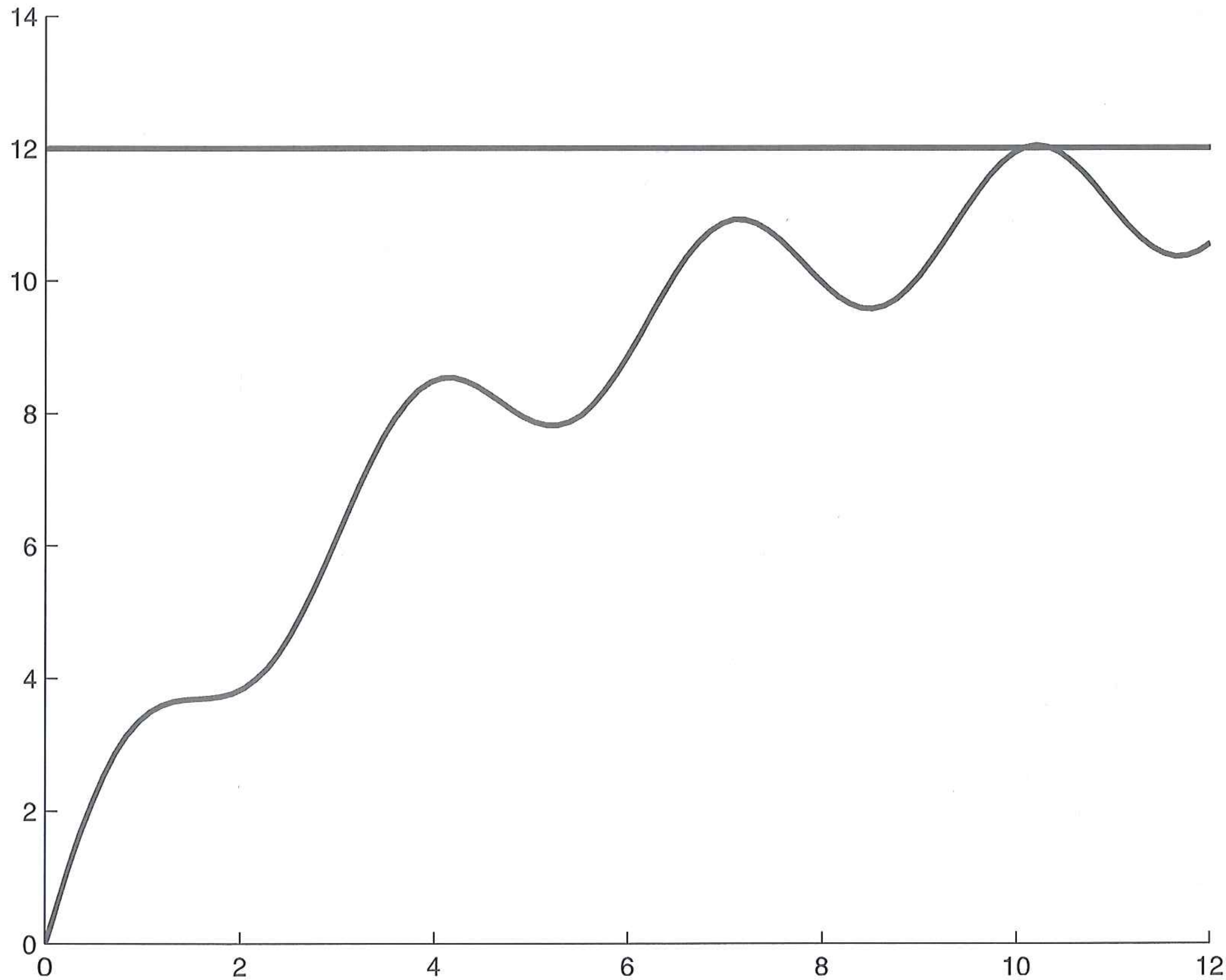
$\Rightarrow y$ oscillates around 12 as $t \to +\infty$.

(b) Determine the value of t for which the solution first intersects the line $y = 12$.

See attachment

$t \approx 10.065$

Hw2.5



Section 2.1 Problem 30: Find the value of y_0 for which the solution of the initial value problem:

$$y' - y = 1 + 3\sin(t), \quad y(0) = y_0$$

remains finite as $t \to \infty$.

$\frac{d\mu}{dt} = -\mu \Rightarrow \mu(t) = e^{-t} \Rightarrow \frac{d}{dt}[e^{-t}y] = (1 + 3\sin(t))e^{-t}$

$\Rightarrow e^{-t}y = \int (1 + 3\sin(t))e^{-t}\,dt$

We need integration by parts twice (again):

$\int \sin(t)e^{-t}\,dt = \sin(t)(-e^{-t}) - \int \cos t(-e^{-t})\,dt = -\sin t\, e^{-t} + \int \cos(t)e^{-t}\,dt$

$\int \cos(t)e^{-t}\,dt = \cos(t)(-e^{-t}) - \int -\sin(t)(-e^{-t})\,dt = -\cos(t)e^{-t} - \int \sin(t)e^{-t}\,dt$

Substitute and get:

$\int \sin(t)e^{-t}\,dt = -e^{-t}(\sin t + \cos t) - \int \sin(t)e^{-t}\,dt$

$\Rightarrow \int \sin(t)e^{-t}\,dt = -\frac{\sin t + \cos t}{2}e^{-t}$

$\Rightarrow e^{-t}y = -e^{-t} - \frac{3}{2}(\sin t + \cos t)e^{-t} + c$

$\Rightarrow y = -1 - \frac{3}{2}(\sin t + \cos t) + ce^{t}$

$y_0 = y(0) = -1 - \frac{3}{2} + c \Rightarrow c = y_0 + \frac{5}{2}$

$\Rightarrow y = -1 - \frac{3}{2}(\sin t + \cos t) + (y_0 + \frac{5}{2})e^{t}$.

$e^{t} \to \infty$ as $t \to +\infty \Rightarrow$ We need the coefficient in front to vanish to avoid growth as $t \to \infty$

$\Rightarrow y_0 = -5/2$.