Hw 3.1

Homework 3

Section 2.2 Problems 1,2,5,6. In each of the following problems solve the given differential equation.

1. $y' = \frac{x^2}{y}$

Answer: We need to assume $y \neq 0$ because it is a denominator on the right hand side

$\Rightarrow \; y\,dy = x^2\,dx \;\Rightarrow\; \frac{y^2}{2} = \frac{x^3}{3} + c_1$

$3y^2 - 2x^3 = 6c_1 = c$

$3y^2 - 2x^3 = c. \quad y \neq 0$

— o —

2. $y' = \frac{x^2}{y(1+x^3)}$

Answer: Here we need to assume $y \neq 0$ and $1+x^3 \neq -1 \Leftrightarrow x \neq -1$.

$y\,dy = \frac{x^2}{1+x^3}\,dx = \frac{\frac{1}{3}d(x^3)}{1+x^3}$

$\Rightarrow \frac{y^2}{2} = \frac{1}{3}\ln|1+x^3| + c_1$

$\Rightarrow 3y^2 - 2\ln|1+x^3| = c$, where $c = 6c_1$, $y \neq 0$, $x \neq -1$

— o —

5. $y' = (\cos^2 x)(\cos^2 2y)$

Answer: We would like to divide by $\cos^2 2y$

So we need to (temporarily) assume that $\cos 2y \neq 0$

$\Rightarrow \frac{1}{\cos^2 2y}\,dy = \cos^2 x\,dx$

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$\Rightarrow \cos^2 x = \frac{1+\cos(2x)}{2}$

$\Rightarrow \int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + c$

and $\int \frac{1}{\cos^2 2y}\, dy = \frac{1}{2}\tan 2y$

$\Rightarrow \frac{1}{2}\tan(2y) = \frac{x}{2} + \frac{1}{4}\sin(2x) + C, \quad \cos^2 2y \neq 0$

Let's now consider the special case we omitted:

$\cos^2 2y = 0 \Rightarrow 2y = \frac{\pi}{2} + n\pi \Rightarrow y = \frac{\pi}{4} + \frac{n}{2}\pi$, n integer.

$\frac{dy}{dx} = \cos^2 x \cos^2 2y = 0 \Rightarrow y = \text{constant} = \frac{\pi}{4} + \frac{n}{2}\pi$

Then the solutions are:

$\tan 2y = x + \frac{1}{2}\sin 2x + 2C, \quad \cos^2 2y \neq 0$

or $\quad y = \frac{\pi}{2} + \frac{n}{2}\pi$, n = an integer

---

6. $xy' = (1-y^2)^{1/2}$

Answer:

If $1-y^2 > 0 \Rightarrow (1-y^2)^{-1/2} dy = \frac{1}{x} dx$

$\Rightarrow \int (1-y^2)^{-1/2} dy = \ln|x| + C, \quad x \neq 0$ (ln is not defined at $x=0$).

$\Rightarrow \sin^{-1} y = \ln|x| + C \Rightarrow y = \sin(\ln|x| + C)$. $x \neq 0$, $|y| < 1$

so that $1-y^2 > 0$.

We ignored the case $1-y^2 = 0$

$y = \pm 1$ are both constant solutions, clearly.

The solution are:

$y = \sin(\ln|x| + C)$, $|x| < 1$, $|y| < 1$

or $y = \pm 1$.

Hw 3.2

Section 2.2 Problem 21: Solve the initial value problem:

$y' = (1+3x^2)(3y^2-6y)$, $y(0)=1$.

and determine the ~~value~~ interval in which the solution is valid.

Hint: To find the interval of definition, look for points where the integral curve has vertical line.

Answer:

Let's separate variables:

$(3y^2-6y)dy = (1+3x^2)dx$

$\Rightarrow \int(3y^2-6y)dy = \int(1+3x^2)dx + C$

$\Rightarrow y^3 - 3y^2 = x + x^3 + C$

The initial condition is $y(0)=1$

$\Rightarrow 1^3 - 3 = C \Rightarrow C = -2$

$\Rightarrow y^3 - 3y^2 - x - x^3 + 2 = 0$.

Integral curves have vertical line where $y' = \infty$

This occurs when the denominator $3y^2 - 6y = 0$ vanishes.

$\Rightarrow 3y(y-2) = 0 \Rightarrow y = 0, 2$.

Substitute in the implicit eqn:

$y=0$ $\quad 0 - 0 - x - x^3 + 2 = 0 \Rightarrow x + x^3 = 2 \Rightarrow x \neq 1$ in particular

$y=2$ $\quad 8 - 12 - x - x^3 + 2 = 0 \Rightarrow x + x^3 = -2 \Rightarrow x \neq -1$ in particular.

$\Rightarrow$ The solution is valid for $|x| < 1$.

Section 2.2 Problem 30: Solve $\frac{dy}{dx} = \frac{y-4x}{x-y}$

by transforming this homogeneous equation into a separable equation as seen in class.

(a) $\frac{dy}{dx} = \frac{(y-4x)\cdot(\frac{1}{x})}{(x-y)\cdot(\frac{1}{x})} = \frac{\frac{y}{x}-4}{1-y/x}$

$\Rightarrow$ This is an homogeneous equation.

(b) Define $v = \frac{y}{x} \Rightarrow y = xv$

$\Rightarrow y' = \frac{d}{dx}(xv) = v(x) + x\frac{dv}{dx}$

(c) $v + x\frac{dv}{dx} = \frac{\frac{y}{x}-4}{1-y/x} = \frac{v-4}{1-v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v-4-v(1-v)}{1-v} = \frac{v-4-v+v^2}{1-v}$

$= \frac{v^2-4}{1-v}$.

$\Rightarrow \frac{1-v}{v^2-4}\,dv = \frac{1}{x}\,dx \Rightarrow$ The equation is separable.

(d) Integrating we can solve for v:

$\frac{-1}{v-2} + \frac{1}{v+2} = \frac{-v-2+v-2}{v^2-4} = \frac{-4}{v^2-4}$

$\Rightarrow \frac{1-v}{v^2-4} = (1-v)\left(\frac{1}{4}\frac{1}{v-2} - \frac{1}{4}\frac{1}{v+2}\right) = \frac{1}{4}\frac{1-v}{v-2} - \frac{1}{4}\frac{1-v}{v+2}$

$= -\frac{1}{4}\frac{v-1}{v-2} + \frac{1}{4}\frac{v-1}{v+2} = -\frac{1}{4}\left(\frac{v-2+1}{v-2}\right) + \frac{1}{4}\frac{v+2-3}{v+2}$

$= -\frac{1}{4}\left(1+\frac{1}{v-2}\right) + \frac{1}{4}\left(1-\frac{3}{v+2}\right) = -\frac{1}{4}\frac{1}{v-2} - \frac{3}{4}\frac{1}{v+2}$

Integrating we get:

$$-\frac{1}{4}\ln|v-2|-\frac{3}{4}\ln|v+2| = \ln|x| + C_1$$

$$\ln|v-2|+3\ln|v+2| = -4\ln|x| + C_2$$

$$\ln|(v-2)(v+2)^3| = -4\ln|x|+C_2$$

$$\Rightarrow (v-2)(v+2)^3 = \frac{C_3}{x^4}$$

(e) $v=\frac{y}{x} \Rightarrow \left(\frac{y}{x}-2\right)\left(\frac{y}{x}+2\right)^3 = \frac{C_3}{x^4}$ Multiply by x^4

$$\Rightarrow (y-2x)(y+2x)^3 = C$$

(f) For slope field, see separate file.

Section 2.3 Problem 3: A tank originally contains 100 gal of fresh water. Then water containing 1/2 lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 minutes the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of the 10 min.

time

Answer:

First 10 minutes: $r = 2$ gal/min.

Let $Q(t)$ be the amount of salt at time t, $t < 10$ min.

Concentration of salt in water coming in: $\frac{1}{2}$ lb.

Initial amount of salt: $Q_0 = 0$ lb/~~gal~~

Similarly as seen in class, Q satisfies:

$$\frac{dQ}{dt} = \frac{r}{2} - \frac{rQ}{100} = 1 - \frac{Q}{50}$$

$$\mu = e^{t/50} \qquad \frac{d}{dt}\left[e^{t/50} Q\right] = e^{t/50}$$

$$e^{t/50} Q = 50 e^{t/50} + C \qquad Q(0) = 0 \Rightarrow C = -50$$

$$\Rightarrow Q = 50 - 50 e^{-t/50}$$

At $t = 10$ min, $Q(10) = 50\left(1 - e^{-1/5}\right)$

For the next 10 minutes:

$r = 2$ gal/min, concentration of water coming in: 0 lb/gal

$Q_0 = 50\left(1 - e^{-1/5}\right)$, $t_0 = 10$ min

$\Rightarrow$ During the last 10 minutes

$$\frac{dQ}{dt} = 0 - \frac{Q}{50} = -\frac{Q}{50}$$

$$Q(t) = c e^{-t/50} \qquad Q(10) = c e^{-\frac{1}{5}} = 50\left(1 - e^{-1/50}\right)$$

$$\Rightarrow c = 50 e^{1/5}\left(1 - e^{-1/5}\right)$$

$$\Rightarrow Q(20) = 50 e^{1/5}\left(1 - e^{-1/5}\right) e^{-20/50} = 50 e^{-1/5}\left(1 - e^{-1/5}\right)$$

$$\approx 7.42 \text{ lb.}$$

HW3.4

Section 2.3 Problem 29: Suppose that a rocket is launched straight up from the surface of earth with initial velocity $v_0 = \sqrt{2gR}$, where R is the radius of earth. Neglect air resistance.

(a) Find an expression for the velocity v in terms of the distance x from the surface of earth.

Answer:

According to the solution found in class:

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x+R}} = \pm\sqrt{\frac{2gR^2}{x+R}}$$

(b) Find the time required for the rocket to go 240,000 mi (the approximate distance from the earth to the moon). Assume $R = 4000$ mi.

Answer:

Notice: $v = \frac{dx}{dt}$, x = displacement away from earth.

$\Rightarrow \frac{dx}{dt} = R\sqrt{\frac{2g}{x+R}}$ (positive sign because $v > 0$ away from earth)

$\Rightarrow (x+R)^{1/2}\,dx = R\sqrt{2g}\,dt$

$\Rightarrow \frac{(x+R)^{3/2}}{3/2} = R\sqrt{2g}\,t + C$ $\quad x(0) = 0$

$\Rightarrow \frac{2}{3}R^{3/2} = C$ $\quad \Rightarrow \frac{2}{3}(x+R)^{3/2} = R\sqrt{2g}\,t + \frac{2}{3}R^{3/2}$

$(x+R)^{3/2} = \frac{3}{2}R\sqrt{2g}\,t + R^{3/2}$ $\quad \Rightarrow x+R = \left[R^{3/2} + \frac{3}{2}R\sqrt{2g}\,t\right]^{2/3}$

$$x = \left[R^{3/2} + \frac{3}{2} R\sqrt{2g}\, t\right]^{2/3} - R$$

or $$t = \frac{2}{3}\left((x+R)^{3/2} - R^{3/2}\right)\frac{1}{R\sqrt{2g}}$$

$$= \frac{2}{3}\left(\left(\frac{x}{R}+1\right)^{3/2} - 1\right)\frac{R^{3/2}}{R\sqrt{2g}} = \frac{2}{3}\left(\left(\frac{x}{R}+1\right)^{3/2} - 1\right)\sqrt{\frac{R}{2g}}$$

we need $x = 240{,}000$ mi, $R = 4000$ mi

$$\Rightarrow \frac{x}{R} = 60$$

$$\frac{R}{2g} = \frac{4000\,\text{mi}}{2\times 9.81\frac{\text{m}}{\text{s}^2}} = \frac{4000\times 1609.34\,\cancel{\text{m}}}{2\times 9.81\frac{\cancel{\text{m}}}{\text{s}^2}} = 3.281\times 10^5\ \text{s}^2$$

$$= 3.281\times 10^5\ \frac{\text{hr}^2}{60^4}$$

$$\Rightarrow \sqrt{\frac{R}{2g}} = 0.1591\ \text{hr}$$

$$\Rightarrow t = \frac{2}{3}\left(61^{3/2} - 1\right)\times 0.1591\,\text{hr} \approx 50.4268\ \text{hrs.}$$

Section 2.4 Problem 16: Solve the initial value problem

$$y' = \frac{t^2}{y(1+t^3)}, \quad y(0) = y_0,$$

and determine how the interval on which the solution exists depends on the initial value y_0.

Answer: $$y\,dy = \frac{t^2}{1+t^3}\,dt$$

$$\frac{y^2}{2} = \int \frac{t^2}{1+t^3}\,dt + c \qquad u = t^3 \quad du = 3t^2\,dt$$

$$\Rightarrow \int \frac{t^2}{1+t^3}\,dt = \frac{1}{3}\int \frac{du}{1+u} = \frac{1}{3}\ln|1+u| = \frac{1}{3}\ln|1+t^3|$$

$$\Rightarrow \frac{y^2}{2} = \frac{1}{3}\ln|1+t^3| + c$$

$$\frac{y_0^2}{2} = \frac{1}{3}\ln 1 + c = c \Rightarrow c = \frac{y_0^2}{2}$$

$$y^2 = \frac{2}{3}\ln|1+t^3| + y_0^2 \Rightarrow y = \pm\sqrt{\tfrac{2}{3}\ln|1+t^3| + y_0^2}$$

In fact $y = \text{sign}(y_0)\sqrt{\tfrac{2}{3}\ln|1+t^3| + y_0^2}$

For the solution to exists, we need the expression inside the square roots to be positive.

$$\frac{2}{3}\ln|1+t^3| + y_0^2 \geq 0 \Leftrightarrow \ln|1+t^3| \geq -\frac{3}{2}y_0^2$$

$$\Leftrightarrow |1+t^3| \geq e^{-\frac{3}{2}y_0^2}$$

Since the interval must include $t = 0$, then

$$1+t^3 \geq e^{-3/2\, y_0^2} \Leftrightarrow t^3 \geq e^{-3/2\, y_0^2} - 1$$

Notice $e^{-3/2\, y_0^2} - 1 < 0$

$$e^{-3/2\, y_0^2} - 1 \leq t^3 < \infty$$

Section 2.5 Problem 15 Suppose that a certain population obeys the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$

(a) If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.

Answer: $y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

Find τ such that $\dfrac{y_0 K}{y_0 + (K-y_0)e^{-r\tau}} = 2y_0$

$\Rightarrow \dfrac{K}{y_0 + (K-y_0)e^{-r\tau}} = 2 \Rightarrow y_0 + (K-y_0)e^{-r\tau} = \dfrac{K}{2}$

$y_0 = \dfrac{K}{3} \Rightarrow \dfrac{K}{3} + (K - \dfrac{K}{3})e^{-r\tau} = \dfrac{K}{2}$

$\Rightarrow +\dfrac{2}{3} K e^{-r\tau} = \dfrac{K}{2} - \dfrac{K}{3} = \dfrac{3-2}{6}K = \dfrac{K}{6}$

$\Rightarrow e^{-r\tau} = \dfrac{1}{4} \quad -r\tau = \ln\dfrac{1}{4} = -\ln 4 \Rightarrow \tau = \dfrac{\ln 4}{r}$

If $r = \dfrac{0.025}{\text{year}} \Rightarrow \tau = \dfrac{\ln 4}{0.025}$ year $\simeq 55.4518$

(b) If $y_0/K = \alpha$, find the time T at which $\dfrac{y(T)}{K} = \beta$.
where $0 < \alpha, \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or $\beta \to 1$.
Find T for $r = 0.025$/year, $\alpha = 0.1$ and $\beta = 0.9$

$\beta = \dfrac{y(T)}{K} = \dfrac{y_0 K}{(y_0 + (K-y_0)e^{-rT})K} = \dfrac{y_0}{y_0 + (K-y_0)e^{-rT}} = \dfrac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rT}}$

$= \dfrac{\alpha}{\alpha + (1-\alpha)e^{-rT}} \Rightarrow \alpha + (1-\alpha)e^{-rT} = \dfrac{\alpha}{\beta} \Rightarrow (1-\alpha)e^{-rT} = \dfrac{\alpha}{\beta} - \alpha = \dfrac{\alpha(1-\beta)}{\beta}$

$\Rightarrow e^{-rT} = \dfrac{\alpha}{1-\alpha}\,\dfrac{1-\beta}{\beta} \Rightarrow -rT = \ln \dfrac{\alpha}{1-\alpha}\,\dfrac{1-\beta}{\beta}$

$T = \dfrac{1}{r}\ln\dfrac{1-\alpha}{\alpha}\,\dfrac{\beta}{1-\beta}$ If $\alpha \to \infty \Rightarrow \dfrac{1-\alpha}{\alpha} \to \infty \Rightarrow T \to \infty$
If $\beta \to 1 \Rightarrow \dfrac{\beta}{1-\beta} \to \infty \Rightarrow T \to \infty$

For $r = 0.025$ /year, $\alpha = 0.1$, $\beta = 0.9$

$T = \dfrac{\text{year}}{0.025} \ln \dfrac{0.9}{0.1}\,\dfrac{0.9}{0.1} \approx 175.7780$ years.