Hw 4.1

Section 2.6 Determine whether each of the equations in Problems 1 through 4 is exact. If it is exact, find the solution:

1.- $(2x+3)+(2y-2)y'=0$

Answer: $M=2x+3$, $N=2y-2$ $\quad M_y=0$, $N_x=0 \Rightarrow$ exact

$M=\frac{\partial\psi}{\partial x}$, $N=\frac{\partial\psi}{\partial y}$ Find ψ:

$\frac{\partial\psi}{\partial x}=2x+3 \quad \psi=x^2+3x+h(y)$, $h'(y)=2y-2$

$\Rightarrow$ Choose $\psi=x^2+3x+y^2-2y$

$\Rightarrow$ The implicit equation is $x^2+3x+y^2-2y=C$, C = constant

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2.- $(2x+4y)+(2x-2y)y'=0$

Answer: $M=2x+4y$, $N=2x-2y$ $\quad M_y=4$, $N_x=2$

$\Rightarrow$ The equation is not in exact form.

3.- $(3x^2-2xy+2)dx+(6y^2-x^2+3)dy=0$

Answer:

$M=3x^2-2xy+2$, $N=6y^2-x^2+3$

$M_y=-2x$, $N_x=-2x$ $\Rightarrow$ it is in exact form

$M=\frac{\partial\psi}{\partial x}$, $N=\frac{\partial\psi}{\partial y}$ Find ψ

Integrate: $\psi=x^3-x^2y+2x+h(y)$

$N=\frac{\partial\psi}{\partial y}=-x^2+h'(y)=-x^2+6y^2+3 \Rightarrow h'(y)=6y^2+3$

$\Rightarrow h(y)=2y^3+3y+$ constant $\Rightarrow$ Choose $\psi=x^3-x^2y+2x+2y^3+3y$

$\Rightarrow$ The implicit equation is $x^3-x^2y+2x+2y^3+3y=C$

4.- $(2xy^2+2y)+(2x^2y+2x)y'=0$

Answer:

$M=2xy^2+2y, \quad N=2x^2y+2x$

$M_y=4xy+2, \quad N_x=4xy+2 \Rightarrow$ It is in exact form:

$M=\frac{\partial\psi}{\partial x}, N=\frac{\partial\psi}{\partial y}$ Find ψ:

$\frac{\partial\psi}{\partial x}=2xy^2+2y \Rightarrow \psi=x^2y^2+2xy+h(y)$

$\frac{\partial\psi}{\partial y}=2x^2y+2x+h'(y)=2x^2y+2x \Rightarrow h'(y)=0$

$\Rightarrow$ Choose $\psi=x^2y^2+2xy$

$\Rightarrow$ The Implicit equation. $x^2y^2+2xy=c$

Section 2.6 Problem 13: Solve the given initial value problem and determine at least approxmately where the solution is valid.

$$(2x-y)dx+(2y-x)dy=0, \quad y(1)=3$$

$M=2x-y, \; N=2y-x \quad M_y=-1, \; N_x=-1$

$\Rightarrow$ It is in exact form.

$M=\frac{\partial\psi}{\partial x}, \; N=\frac{\partial\psi}{\partial y},$ Find ψ.

$\frac{\partial\psi}{\partial x}=2x-y \Rightarrow \psi=x^2-xy+h(y)$

$\frac{\partial\psi}{\partial y}=-x+h'(y)=-x+2y \Rightarrow h'(y)=2y \Rightarrow h(y)=y^2+$ constant

$\Rightarrow$ Choose $\psi=x^2-xy+y^2$

$\Rightarrow$ Implicit solution is $x^2-xy+y^2=C$

Subsituting $x_0 = 1, y_0 = 3$ we get:

$1^2 - 1\times 3 + 3^2 = 1 - 3 + 9 = 7 = C$

$\Rightarrow x^2 - xy + y^2 = 7$

$\Rightarrow y^2 - xy + x^2 - 7 = 0$ ← quadratic equation:

$$y = \frac{x \pm \sqrt{x^2 - 4\times(x^2-7)}}{2} = \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

In fact, since $y(1) = 3$, we need to choose the "+" sign:

$$y = \frac{x + \sqrt{28 - 3x^2}}{2}$$

This is valid where $28 - 3x^2 \geq 0 \Leftrightarrow x^2 \leq \frac{28}{3}$

$\Leftrightarrow -\sqrt{28/3} \leq x \leq \sqrt{28/3}$

Section 2.6 Problem 20: Show that the given equation is not exact but becomes exact when multiplied by a given integrating factor. Then solve the equation

Answer: $\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0,$

$\mu = ye^{x}$

$M = \frac{\sin y}{y} - 2e^{-x}\sin x, \quad N = \frac{\cos y + 2e^{-x}\cos x}{y}$

$M_y = \frac{\cos y \, y - \sin y}{y^2}, \quad N_x = \frac{2}{y}\left(-xe^{x}\cos x - e^{-x}\sin x\right)$

$\Rightarrow M_y \neq N_x \Rightarrow$ Not in exact form

However, if we multiply by $\mu = ye^x$ we get:

$$(\sin y\, e^x - 2y \sin x)dx + (\cos y\, e^x + 2\cos x)dy = 0$$

$\tilde{M} = \sin y\, e^x - 2y\sin x$, $\tilde{N} = \cos y\, e^x + 2\cos x$

$\tilde{M}_y = \cos y\, e^x - 2\sin x$, $\tilde{N}_x = \cos y\, e^x - 2\sin x$

$\Rightarrow \tilde{M}_y = \tilde{N}_x$ $\Rightarrow$ It is in exact form:

$\tilde{M} = \frac{\partial \psi}{\partial x}$, $\tilde{N} = \frac{\partial \psi}{\partial y}$ Find ψ:

$\frac{\partial \psi}{\partial x} = \sin y\, e^x - 2y\sin x$ $\Rightarrow \psi = \sin y\, e^x + 2y\cos x + h(y)$

$\Rightarrow \frac{\partial \psi}{\partial y} = \cos y\, e^x + 2\cos x + h'(y) = \cos y\, e^x + 2\cos x \Rightarrow h'(y) = 0$

$\Rightarrow$ Choose $\psi = \sin y\, e^x + 2y\cos x$

$\Rightarrow$ The implicit equation is $\sin y\, e^x + 2y\cos x = c$.

Section 2.6 Problem 25: Find an integrating factor and solve the equation:

$$(3x^2y + 2xy + y^3)dx + (x^2+y^2)dy = 0.$$

Answer:

$M = 3x^2y + 2xy + y^3$, $N = x^2+y^2$

$M_y = 3x^2 + 2x + 3y^2$, $N_x = 2x$

$$\frac{M_y - N_x}{N} = \frac{3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}}{x^2+y^2} = \frac{3(x^2+y^2)}{x^2+y^2} = 3$$

$\Rightarrow$ The integrating factor needed satisfies $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = 3\mu$

$\Rightarrow$ Choose $\mu = e^{3x}$

HW4.3

$$\Rightarrow e^{3x}(3x^2y + 2xy + y^3)\,dx + e^{3x}(x^2+y^2)\,dy = 0$$

is exact

$$\tilde{M} = e^{3x}(3x^2y + 2xy + y^3), \quad \tilde{N} = e^{3x}(x^2+y^2)$$

$$\tilde{M}_y = e^{3x}(3x^2 + 2x + 3y^2), \quad \tilde{N}_x = 3e^{3x}(x^2+y^2) + e^{3x}2x = e^{3x}(3x^2+2x+3y^2)$$

$$\Rightarrow \tilde{M} = \frac{\partial \psi}{\partial x}, \quad \tilde{N} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \psi}{\partial x} = e^{3x}(3x^2y + 2xy + y^3)$$

$$\int e^{3x}x\,dx = \frac{1}{3}e^{3x}x - \int \frac{1}{3}e^{3x}\,dx = \frac{1}{3}e^{3x}(x-1)$$

$$\int e^{3x}x^2\,dx = \frac{1}{3}e^{3x}x^2 - \frac{2}{3}\int e^{3x}x = \frac{1}{3}e^{3x}x^2 - \frac{2}{9}e^{3x}(x-1)$$

$$\Rightarrow \psi = 3y\left(\frac{1}{3}e^{3x}x^2 - \cancel{\frac{2}{9}e^{3x}(x-1)}\right) + \cancel{2y\frac{1}{3}e^{3x}(x-1)} + \frac{1}{3}e^{3x}y^3 + h(y)$$

$$= ye^{3x}x^2 + \frac{1}{3}e^{3x}y^3 + h(y)$$

$$\frac{\partial \psi}{\partial y} = e^{3x}x^2 + e^{3x}y^2 + h'(y) = e^{3x}(x^2+y^2) + h'(y) = e^{3x}(x^2+y^2)$$

$$\Rightarrow h'(y) = 0 \qquad \Rightarrow \text{Choose} \quad \psi = ye^{3x}x^2 + \frac{1}{3}e^{3x}y^3$$

$\Rightarrow$ The implicit equation is $e^{3x}\left(yx^2 + \frac{1}{3}y^3\right) = C$

Section 2.7 Problem 3: Consider the following initial value problem $y' = 0.5 - t + 2y$, $y(0) = 1$.

(a) Find the approximate values of the solution at $t = 0.1, 0.2, 0.3$ and 0.4 using the Euler's method with $h = 0.1$

Answer: Euler's method is given by:

$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$ if the ODE is $y' = f(t, y)$

$\Rightarrow$ We get the following approximations for $h = 0.1$

t_n	y_n	$f(t_n, y_n)$
0	1	2.5
0.1	1.25	2.9
0.2	1.54	3.38
0.3	1.8780	3.956
0.4	2.2736	

⋮

See matlab code

(b) Repeat part (a) with $h = 0.05$. Compare the results with those in part (a)

t_n	y_n	$f(t_n, y_n)$
0	1	2.5
0.05	1.125	2.7
0.1	1.26	2.92
0.15	1.41	3.16
0.2	1.56	3.43
0.25	1.74	3.72
0.3	1.92	4.04
0.35	2.12	4.40
0.4	2.34	

(c) Repeat part (a) with $h = 0.025$. Compare the results with these in (a) and (b)

t_n	y_n	$f(t_n, y_n)$
0	1	2.5
0.025	1.0625	2.6
0.05	1.13	2.705
0.075	1.2	2.82
0.1	1.27	2.93
0.125	1.34	3.05
0.15	1.42	3.18
0.175	1.49	3.31

t_n	y_n	$f(t_n, y_n)$
0.2	1.57	3.45
0.225	1.66	3.6
0.25	1.75	3.76
0.275	1.85	3.92
0.3	1.95	4.09
0.325	2.05	4.27
0.35	2.16	4.46
0.375	2.27	4.66
0.4	2.38	

(d) Find the solution $y=\phi(t)$ of the given problem and evaluate $\phi(t)$ at $t=0.1, 0.2, 0.3$ and 0.4. Compare these values with the results of parts a, b, c.

Answer: $y'-2y+t-1/2=0$

$$\frac{d}{dt}\left[e^{-2t}y\right]=e^{-2t}(-t+1/2)$$

$$\Rightarrow e^{-2t}y=-\int e^{-2t}t\,dt+\frac{1}{4}e^{-2t}+C$$

$$=\frac{1}{2}e^{-2t}\,t+C$$

$$\Rightarrow y=\frac{1}{2}t+Ce^{2t} \quad \text{Since } y(0)=1$$

$$\Rightarrow C=1$$

$$\Rightarrow y=\frac{1}{2}t+e^{2t}$$

$y(0)=1$, $y(0.1)\simeq 1.27$, $y(0.2)\approx 1.59$

$y(0.3)\simeq 1.97$, $y(0.4)\approx 2.43$

Comparing with parts (a) – (c), we observe that the smaller the time step is, the better the approximation.

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Section 3.1 In each of the problems 9–11 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$

Answer: Look for solutions of the form: $y = e^{rx}$

$\Rightarrow$ Characteristic polynomial:

$r^2 + r - 2 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times (-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2}$

$\Rightarrow r_1 = 1, \; r_2 = -2$

$\Rightarrow e^x, \; e^{-2x}$ are two fundamental solution

$\Rightarrow y = c_1 e^x + c_2 e^{-2x}$ is the general solution

Apply initial conditions:

$y(0) = c_1 + c_2 = 1$

$y'(x) = c_1 e^x - 2c_2 e^{-2x} \qquad y'(0) = c_1 - 2c_2 = 1$

$c_2 = 1 - c_1 \qquad c_1 - 2(1 - c_1) = 1 \Rightarrow 3c_1 - 2 = 1$

$\Rightarrow c_1 = 1, \; c_2 = 0$

$\Rightarrow y = e^x$

$y \to \infty$ as $x \to +\infty$.

10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \; y'(0) = -1$

Answer: Characteristic polynomial: $r^2 + 4r + 3 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4 \times 3}}{2} = \frac{-4 \pm \sqrt{4}}{2} \qquad r_1 = -3, \; r_2 = -1$

$\Rightarrow y = c_1 e^{-3t} + c_2 e^{-t}$ is the general solution.

$y(0) = C_1 + C_2 = 2$

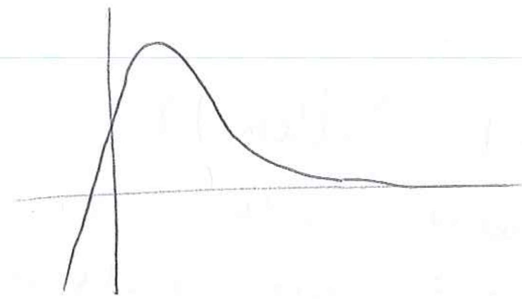
$y'(t) = -3C_1 e^{-3t} - C_2 e^{-t} \qquad y'(0) = -3C_1 - C_2 = -1$

$C_2 = 2 - C_1 \quad , \quad -3C_1 - (2 - C_1) = -2C_1 - 2 = -1 \Rightarrow C_1 = -\frac{1}{2}$

$C_2 = \frac{5}{2} \quad \Rightarrow y = -\frac{1}{2} e^{-3t} + \frac{5}{2} e^{-t}$

~~As $t \to +\infty$, $y \to 0$~~

The graph look like



$y \to 0$ as $t \to +\infty$

$y \to -\infty$ as $t \to -\infty$

11.. $6y'' - 5y' + y = 0, \; y(0) = 4, \; y'(0) = 0$

Answer: The characteristic polynomial is

$6r^2 - 5r + 1 = 0$

$$r = \frac{5 \pm \sqrt{25 - 4 \times 6 \times 1}}{2 \times 6} = \frac{5 \pm 1}{12}$$

$r_1 = \frac{1}{2}, \; r_2 = \frac{1}{3}$

$y = C_1 e^{1/2\, t} + C_2 e^{\frac{1}{3} t}$ is the general solution

$y(0) = C_1 + C_2 = +4$

$y'(t) = \frac{C_1}{2} e^{1/2\, t} + \frac{C_2}{3} e^{\frac{1}{3} t} \qquad y'(0) = \frac{C_1}{2} + \frac{C_2}{3} = 0$

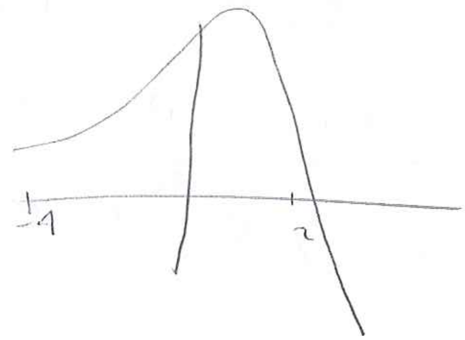
$\Rightarrow C_2 = -\frac{3}{2} C_1 \Rightarrow -\frac{1}{2} C_1 = +4 \Rightarrow C_1 = -8, \; C_2 = +12$

$\Rightarrow y = -8 e^{t/2} + 12 e^{t/3}$

Since $1/2 > 1/3$ $e^{t/2}$ grows faster

$\Rightarrow y \to -\infty$ as $t \to \infty$

The graph looks like



Section 3.1 Problem 17 Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$

Answer: 2 and -3 are roots of

$(r-2)(r+3) = r^2 + r - 6$, which is the characteristic polynomial of

$y'' + y' - 6y = 0$.

Section 3.1 Problem 21 Solve the initial value problem

$y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$

Answer: Characteristic polynomial:

$r^2 - r - 2 = 0$ $\quad r = \dfrac{1 \pm \sqrt{1 - 4 \times 1 \times (-2)}}{2} = \dfrac{1 \pm \sqrt{9}}{2}$

$r_1 = 2$, $r_2 = -1$

$y = c_1 e^{-t} + c_2 e^{2t}$ is the general solution

$y(0) = c_1 + c_2 = \alpha$

$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t}$ $\quad y'(0) = -c_1 + 2c_2 = 2$

$c_2 = \alpha - c_1$

$\Rightarrow \quad -c_1 + 2(\alpha - c_1) = -c_1 - 2c_1 + 2\alpha = -3c_1 + 2\alpha = 2$

$3c_1 = 2(\alpha - 1) \qquad c_1 = \frac{2}{3}(\alpha - 1)$

$c_2 = \alpha - \frac{2}{3}(\alpha - 1) = \frac{1}{3}\alpha + \frac{2}{3}$

$\Rightarrow \quad y = \frac{2}{3}(\alpha - 1) e^{-t} + \left(\frac{1}{3}\alpha + \frac{2}{3}\right) e^{2t}$

$e^{-t} \to 0$ as $t \to \infty$, and $e^{2t} \to +\infty$ as $t \to \infty$

$\Rightarrow$ Need $\frac{1}{3}\alpha + \frac{2}{3} = 0 \quad \Rightarrow \alpha = -2$.

Section 3.2 Compute the Wronskian of the following pair of functions and determine if the are linearly independent or linearly dependent on the real line:

Answer: $f(x) = \sin^2 x, \quad g(x) = 1 - \cos(2x)$

$W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$

$= \sin^2 x\,(2\sin(2x)) - 2\sin x\cos(x)(1 - \cos(2x))$

~~$4\sin^2 x$~~ Use $\sin(2x) = 2\sin x\cos x$ &

$\cos(2x) = \cos^2(x) - \sin^2 x$

$\Rightarrow W = 4\sin^3 x\cos x - 2\sin x\cos x(1 - \cos^2 x + \sin^2 x)$

$= 4\sin^3 x\cos x - 2\sin x\cos x(2\sin^2 x) = 0$.

$\Rightarrow f$ and g are linearly independent $\left(f = \frac{1}{2}g\right)$.