Homework 5

Section 3.2 #13. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t>0$. Then show that $y = c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.

<u>Answer:</u> $y_1 = t^2 \quad t^2y'' - 2y = t^2(2) - 2t^2 = 0$

$y_2 = t^{-1}, \; y_2' = -t^{-2}, \; y_2'' = 2t^{-3}, \quad t^2y'' - 2y' = t^2 2t^{-3} - 2t^{-1} = 0$

$y = c_1y_1 + c_2y_2 = c_1t^2 + c_2t^{-1}$

$y' = 2c_1t + c_2(-1)t^{-2}, \quad y'' = 2c_1 + c_2 2t^{-3}$

$t^2y'' - 2y = t^2(2c_1 + 2c_2t^{-3}) - 2(c_1t^2 + c_2t^{-1})$

$= 2\cancel{c_1t^2} + 2\cancel{c_2t^{-1}} - 2\cancel{c_1t^2} - 2\cancel{c_2t^{-1}} = 0. \quad t>0.$

Section 3.2 Problem 14. Verify that $y_1(t) = 1$, and $y_2(t) = t^{1/2}$ are two solutions of the differential equation $yy'' + (y')^2 = 0$ for $t>0$. Then show that $y = c_1y_1(t) + c_2y_2(t)$ is not, in general, a solution of this equation. Explain why this result doesn't contradict Theorem 3.2.2.

<u>Answer:</u> $y_1 = 1 \quad yy'' + (y')^2 = 1\cdot 0 + 0^2 = 0$

$y_2 = t^{1/2}, \; y_2' = \frac{1}{2}t^{-1/2}, \; y_2'' = -\frac{1}{4}t^{-3/2}$

$yy'' + (y')^2 = t^{1/2}\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{1}{2}t^{-1/2}\right)^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$

$\Rightarrow$ Both y_1, y_2 are solutions of the above diff. eqn.

$y = c_1 + c_2t^{1/2}, \quad y' = \frac{1}{2}c_2t^{-1/2}, \quad y'' = -\frac{1}{4}c_2t^{-3/2}$

$$yy'' + (y')^2 = (c_1 + c_2 t^{1/2})\left(-\frac{1}{4} c_2 t^{-3/2}\right) + \left(\frac{1}{2} c_2 t^{-1/2}\right)^2$$

$$= -\frac{1}{4} c_1 c_2 t^{-3/2} - \cancel{\frac{1}{4} c_2^2 t^{-1}} + \cancel{\frac{1}{4} c_2^2 t^{-1}}$$

$$= -\frac{1}{4} c_1 c_2 t^{-3/2}$$ which vanishes only if $c_1 = 0$ or $c_2 = 0$

$\Rightarrow y = c_1 y_1 + c_2 y_2$ is not, in general, a solution of the above differential eqn.

It doesn't contradict the theorem because the differential equation here is not linear.

Section 3.2 # 16. Can $y = \sin(t^2)$ be a solution on an interval containing $t=0$ of an equation $y'' + p(t) y' + q(t) y = 0$ ~~at~~ with continuous coefficients? Explain your answer.

Answer: No.

$y = \sin(t^2)$ satisfies the initial condition $y(0) = 0, y'(0) = 0$.

Let's assume $y = \sin(t^2)$ is a solution of $y'' + p(t) y' + q(t) y = 0$ for some $p(t), q(t)$ continuous in an open interval containing $t = 0$.

$\Rightarrow y = \sin(t^2)$ solves the initial value problem.

$$\begin{cases} y'' + p(t) y' + q(t) y = 0 \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

Notice that $y_2 = 0$ is also a trivial solution of the same problem. Then $\sin(t^2) = 0$ by the uniqueness of solutions (theorem 3.2.1)

This is a contradiction.

Section 3.2 #17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t)=e^{2t}$, find $g(t)$

Answer: $W(f,g)(t) = f(t)g'(t) - f'(t)g(t)$

$= e^{2t}g'(t) - 2e^{2t}g(t) = e^{2t}(g'(t)-2g(t))$

$= 3e^{4t}$

$\Rightarrow g'(t) - 2g(t) = 3e^{2t}$

Integrating factors: $\mu(t) = e^{-2t}$

$\frac{d}{dt}\left[e^{-2t}g(t)\right] = 3 \Rightarrow e^{-2t}g(t) = 3t + C$

$g(t) = 3te^{2t} + Ce^{2t}$

Section 3.2 #36. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does it imply about the coefficients $p(t)$ and $q(t)$?

Answer:

We know $W(y_1, y_2) = Ce^{-\int p(t)\,dt}$

It is constant if $\int p(t)dt$ is a constant, which happens if and only if $p(t) \equiv 0$ (due to the constant of integration)

Then $p(t) \equiv 0$ and $q(t)$ can be anything.

Section 3.3 13-16. In each of the problems 13-16 find the general ~~eq~~ solution of the given differential equation.

(13.) $y'' + 2y' + 1.25y = 0$

Answer: Char. poly: $r^2 + 2r + 1.25 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4\times1\times1.25}}{2} = \frac{-2 \pm \sqrt{4-5}}{2} = \frac{-2 \pm i}{2} = -1 \pm \frac{1}{2}i$$

$\Rightarrow y = c_1 e^{-t}\cos(\tfrac{1}{2}t) + c_2 e^{-t}\sin(\tfrac{1}{4}t)$.

14. $9y'' + 9y' - 4y = 0$ Char. poly: $9r^2 + 9r - 4 = 0$

Answer: $$r = \frac{-9 \pm \sqrt{81 - 4\times9\times(-4)}}{2\times9} = \frac{-9 \pm \sqrt{225}}{18} = \frac{-9 \pm 15}{18}$$

$r_1 = \frac{-24}{18} = -\frac{4}{3}$, $r_2 = \frac{6}{18} = \frac{1}{3}$

$\Rightarrow y = c_1 e^{-\frac{4}{3}t} + c_2 e^{\frac{1}{3}t}$

15. $y'' + y' + 1.25y = 0$ Char poly: $r^2 + r + 1.25 = 0$

Answer: $$r = \frac{-1 \pm \sqrt{1 - 4\times1\cdot1.15}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

$\Rightarrow y = c_1 e^{-\frac{1}{2}t}\cos(t) + c_2 e^{-\frac{1}{2}t}\sin t$.

16. $y'' + 4y' + 6.25y = 0$ Char poly: $r^2 + 4r + 6.25 = 0$

Answer: $$r = \frac{-4 \pm \sqrt{16 - 4\times1\times6.25}}{2} = \frac{-4 \pm \sqrt{16-25}}{2} = \frac{-4 \pm \sqrt{-9}}{2} = \frac{-4 \pm 3i}{2} = -2 \pm \frac{3}{2}i$$

$y = c_1 e^{-2t}\cos(\tfrac{3}{2}t) + c_2 e^{-2t}\sin(\tfrac{3}{2}t)$.