Hw6. 1
2014

Section 3.3 Problem 23 Consider the initial value problem:

$$3u'' - u' + 2u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

(a) Find the solution $u(t)$ of this problem

Answer: Char. poly: $3r^2 - r + 2 = 0$

$$r = \frac{1 \pm \sqrt{1 - 4\times3\times2}}{2\cdot3} = \frac{1 \pm \sqrt{-23}}{6} = \frac{1}{6} \pm i\frac{\sqrt{23}}{6}$$

General solution:

$$u = c_1 e^{\frac{t}{6}} \cos\left(\frac{\sqrt{23}}{6}t\right) + c_2 e^{\frac{t}{6}} \sin\left(\frac{\sqrt{23}}{6}t\right)$$

$$u(0) = c_1 = 2$$

$$u'(t) = c_1 \frac{1}{6} e^{\frac{t}{6}} \cos\left(\frac{\sqrt{23}}{6}t\right) + c_1\left(-\frac{\sqrt{23}}{6}\right) e^{\frac{t}{6}} \sin\left(\frac{\sqrt{23}}{6}t\right)$$

$$+ c_2 \frac{1}{6} e^{\frac{t}{6}} \sin\left(\frac{\sqrt{23}}{6}t\right) + c_2 \frac{\sqrt{23}}{6} e^{\frac{t}{6}} \cos\left(\frac{\sqrt{23}}{6}t\right)$$

$$u'(0) = \frac{1}{6}c_1 + \frac{\sqrt{23}}{6}c_2 = \frac{2}{6} + \frac{\sqrt{23}}{6}c_2 = 0$$

$$\Rightarrow c_1 = 2, \quad c_2 = -\frac{2}{\sqrt{23}}$$

$$u(t) = 2e^{\frac{t}{6}} \cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}} e^{\frac{t}{6}} \sin\left(\frac{\sqrt{23}}{6}t\right).$$

(b) For $t>0$ find the first time at which $|u(t)|=10$

Answer: For a function like this, we need to compute the graph with your preferred software.

See the graph at the end of the manuscript.

The answer is $t \approx 10.759$.

Section 3.3 #26 Consider the initial value problem

$$y'' + 2ay' + (a^2+1)y = 0, \quad y(0)=1, \quad y'(0)=0$$

(a) Find the solution $y(t)$ of this problem.

Answer: Char poly: $r^2 + 2ar + (a^2+1) = 0$

$$r = \frac{-2a \pm \sqrt{4a^2 - 4(a^2+1)}}{2} = \frac{-2a \pm 2i}{2} = -a \pm i$$

$$\Rightarrow \quad y(t) = c_1 e^{-at}\cos(t) + c_2 e^{-at}\sin(t).$$

~~(b) For $a=1$, find the smallest T such that $|y(t)| < 0.1$ for $t > T$.~~

Forgot the initial conditions:

$$y(0) = c_1 = 1, \qquad y'(t) = -c_1 a e^{-at}\cos(t) + c_1 e^{-at}(-\sin t) - c_2 a e^{-at}\sin t + c_2 e^{-at}\cos t$$

$$y'(0) = -ac_1 + c_2 = -a + c_2 = 0 \Rightarrow c_2 = a$$

$$\Rightarrow y(t) = e^{-at}\cos(t) + a e^{-at}\sin(t)$$

Hw 6.2

(b) For $a=1$ find the smallest T such that $|y(t)|<0.1$ for $t>T$

Answer: We can use some inequalities here:

$a=1$ $\quad |y(t)| = |e^{-at}\cos(t) + e^{-at}\sin(t)| \le e^{-at}(|\cos(t)| + |\sin(t)|)$

$\le 2e^{-at} < 0.1 \iff t > -\ln(0.1/2)$

However, $T = -\ln(0.1/2) = 2.9957$ is not the optimal

If we look at the graph (at the end), we see that the optimal T is $T = 1.876$

(c) Repeat part (b) for $a = 1/4, 1/2$ and 2.

Answer: Look at the graphs:

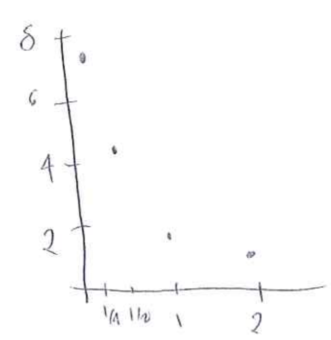
$a = 1/4, \quad T = 7.4284,$

$a = 1/2, \quad T = 4.3003,$

$\alpha = 2, \quad T = 1.5116$

(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

Answer:



T decreases a a increases

This is because e^{-at} decays exponentially.

Section 3.3 #27 Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$

Answer: By definition:

$$W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t)$$

$$= e^{\lambda t}\cos\mu t\left(\cancel{\lambda e^{\lambda t}\sin\mu t} + \underline{e^{\lambda t}\mu\cos\mu t}\right)$$

$$-\left(\cancel{\lambda e^{\lambda t}\cos\mu t} + e^{\lambda t}\underline{(-\mu\sin\mu t)}\right)e^{\lambda t}\sin\mu t$$

$$= e^{2\lambda t}\mu\left(\cos^2\mu t + \sin^2\mu t\right) = \mu e^{2\lambda t}$$

Section 3.3 #29: Using Euler's formula, show that

$$\cos t = \frac{e^{it}+e^{-it}}{2}, \quad \sin t = \frac{e^{it}-e^{-it}}{2}$$

Answer: $$\frac{e^{it}+e^{-it}}{2} = \frac{1}{2}\left(\cos t + i\sin t + \cos(-t) + i\sin(-t)\right)$$ $\sin(-t) = -\sin t$

$$= \frac{1}{2}\left(\cos t + \cancel{i\sin t} + \cos t - \cancel{i\sin t}\right) = \cos t.$$

Section 3.3 #32: Let the real-valued functions p and q be continuous on the open interval I, and let $\mu = u(t) + i\,v(t)$ be a complex-valued solution of $y'' + p(t)y + q(t)y = 0$, where u and v are real-valued functions. Show that both u and v are solutions.

Answer: $y = u + iv$ $y' = u' + iv'$, $y'' = u'' + iv''$

$$\Rightarrow 0 = u'' + iv'' + p(t)(u' + iv') + q(t)(u + iv)$$

$$= \underbrace{u'' + p(t)u' + q(t)u}_{\text{real valued}} + \underset{\substack{\uparrow \\ \text{purely imaginary}}}{i} \cdot [\underbrace{v'' + p(t)v' + q(t)v}_{\text{real valued}}]$$

The only way for that to be zero is if each the real part and the imaginary part are zero.

$$\Rightarrow u'' + p(t)u' + q(t)u = 0$$
$$v'' + p(t)v' + q(t)v = 0.$$

Section 3.4 Problems 7-10: Find the general solution of the given differential equation.

Answer: #7 $4y'' + 17y' + 4y = 0$

Char poly: $4r^2 + 17r + 4 = 0$

$$r = \frac{-17 \pm \sqrt{289 - 4 \times 4 \times 4}}{2 \cdot 4} = -\frac{17 \pm 15}{8} = -4, -1/4$$

$$\Rightarrow y = c_1 e^{-4t} + c_2 e^{-\frac{1}{4}t}.$$

\# 8 $16y'' + 24y' + 9y = 0$ Char poly: $16r^2 + 24r + 9 = 0$

$$r = \frac{-24 \pm \sqrt{576 - 576}}{2 \cdot 16} = -3/4$$

$$\Rightarrow y = c_1 e^{-3/4\,t} + c_2 t e^{-3/4\,t} \longleftarrow \text{repeated roots}$$

#9 $25y'' - 20y' + 4y = 0$

Char poly: $25r^2 - 20r + 4 = 0$

$$r = \frac{20 \pm \sqrt{400 - 400}}{2 \cdot 25} = 2/5 \leftarrow \text{Double root.}$$

$$y = c_1 e^{2/5 t} + c_2 t e^{2/5 t}.$$

#10 $2y'' + 2y' + y = 0$

Char poly: $2r^2 + 2r + 1 = 0$ $\quad r = \frac{-2 \pm \sqrt{4 - 8}}{2 \cdot 2} = \frac{-2 \pm 2i}{4} = -\frac{1}{2} \pm \frac{1}{2} i$

$$y = c_1 e^{-t/2} \cos(t/2) + c_2 e^{-t/2} \sin(t/2).$$

Section 3.4 #27 - 28. Use the method of reduction of order to find a second solution of the given differential equation.

27 $\quad x y'' - y' + 4x^3 y = 0, \quad x > 0, \quad y_1(x) = \sin x^2$

Answer: $\quad y'' - \frac{1}{x} y' + 4x^2 y = 0 \qquad p(x) = -\frac{1}{x}, \quad q(x) = 2x^2$

Assume $y_2 = v(x) y_1 = v(x) \sin x^2$

$\Rightarrow y_2' = v' \sin x^2 + v \, 2x \cos x^2$

$y_2'' = v'' \sin x^2 + v' 2x \cos x^2 + v' 2x \cos x^2 + v \left(2 \cos x^2 + 2x(-2x \sin x^2)\right)$

$= v'' \sin x^2 + v' 2x \cos x^2 + v (2 \cos x^2 - 4x^2 \sin x^2)$

$= v'' \sin x^2 + 4x \cos x^2 v' + v(2 \cos x^2 - 4x^2 \sin x^2)$

$0 = y_2'' - \frac{1}{x} y_2' + 4x^2 y_2 = v'' \sin x^2 + 4x \cos x^2 v' + v(2\cancel{\cos x^2} - \cancel{4x^2 \sin x^2})$

$- \frac{1}{x}\left(v' \sin x^2 + v \, 2x \cancel{\cos x^2}\right) + \cancel{4x^2 v \sin x^2}$

Hw6.4

$\Rightarrow v'' \sin x^2 + \left(4x\cos x^2 - \frac{1}{x}\sin x^2\right) v' = 0$

<u>Or</u> you can just use (correctly) the formulas derived in class:

$y_1 v'' + (2y_1' + p y_1) v' = 0$

$2y_1' + p y_1 = 2 \cdot 2x \cos x^2 + \left(-\frac{1}{x}\right) \sin x^2 = 4x\cos x^2 - \frac{1}{x}\sin x^2$

$\Rightarrow \quad v'' \sin x^2 + \left(4x \cos x^2 - \frac{1}{x}\sin x^2\right) v' = 0$

$w = v' \quad \Rightarrow \quad w' \sin x^2 + \left(4x\cos x^2 - \frac{1}{x}\sin x^2\right) w = 0$

$\Rightarrow \quad \frac{dw}{w} = \frac{\frac{1}{x}\sin x^2 - 4x\cos x^2}{\sin x^2} dx = \left(\frac{1}{x} - 4x \cot x^2\right) dx$

Integrating both sides we get:

$\ln|w| = \ln|x| - 2\int \cot(x^2)\, dx^2 = \ln|x| - 2\ln|\sin x^2| + c_1$

$\Rightarrow w = c_2 \frac{e^{\ln|x|}}{e^{2\ln|\sin(x^2)|}} = c_2 \frac{x}{(\sin(x^2))^2}$

$\Rightarrow v' = c_2 \frac{x}{\sin^2(x^2)} \quad \Rightarrow \quad dv = c_2 \frac{x\,dx}{\sin^2(x^2)}$

$v = -c_2 \cot(x^2) + c_3$

$\Rightarrow \quad y_2 = \cot(x^2) \cdot \sin(x^2) = \cos(x^2)$

All of that for this answer??

#28: $(x-1)y'' - xy' + y = 0, \quad x>1, \quad y_1(x) = e^x$

$y_2 = v(t)y_1$

$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0 \quad \Rightarrow p(x) = \frac{-x}{x-1}, \quad q(x) = \frac{1}{x-1}$

$2y_1' + py_1 = 2 \cdot e^x + \frac{-x}{x-1}e^x = e^x \frac{2x-2-x}{x-1} = e^x \frac{x-2}{x-1}.$

$\Rightarrow e^x v'' + e^x \frac{x-2}{x-1}v' = 0 \qquad w = v'$

$\Rightarrow w' + \frac{x-2}{x-1}w = 0 \quad \Rightarrow \frac{dw}{w} = \frac{2-x}{x-1}dx$

$\Rightarrow \ln|w| = \int \frac{2-(x-1)-1}{x-1}dx = \ln|x-1| - x + c_1$

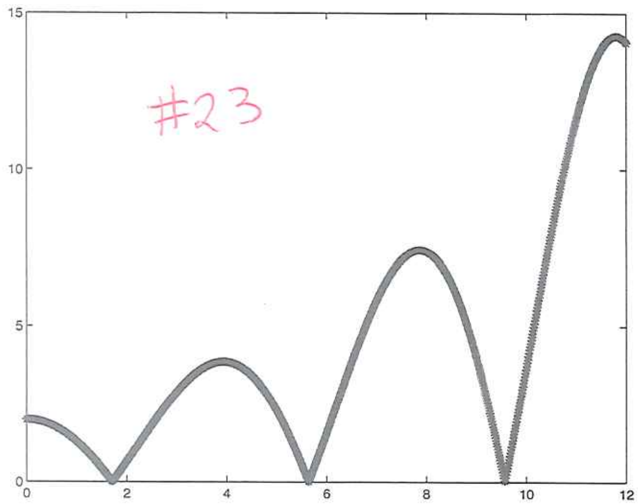
$w = c_2 \frac{e^{\ln|x-1|}}{e^x} = c_2 \frac{|x-1|}{e^x} \quad \Rightarrow v' = c_2 \frac{x-1}{e^x}$

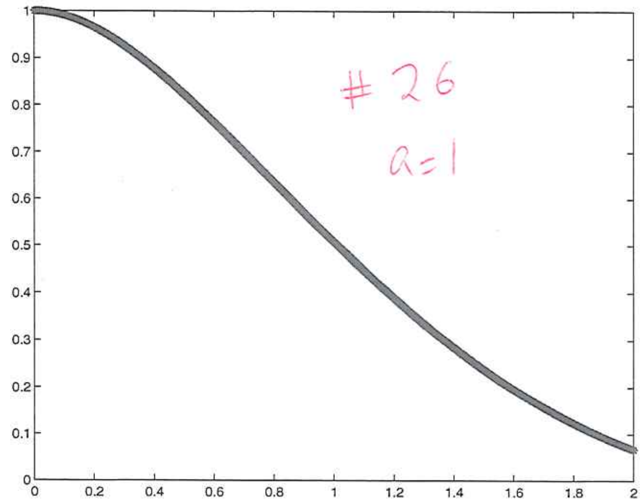
$v = c_2\left(\int xe^{-x}dx - \int e^{-x}dx\right) = c_2\left(x(-e^{-x}) - \int -e^{-x}dx - \int e^{-x}dx\right)$

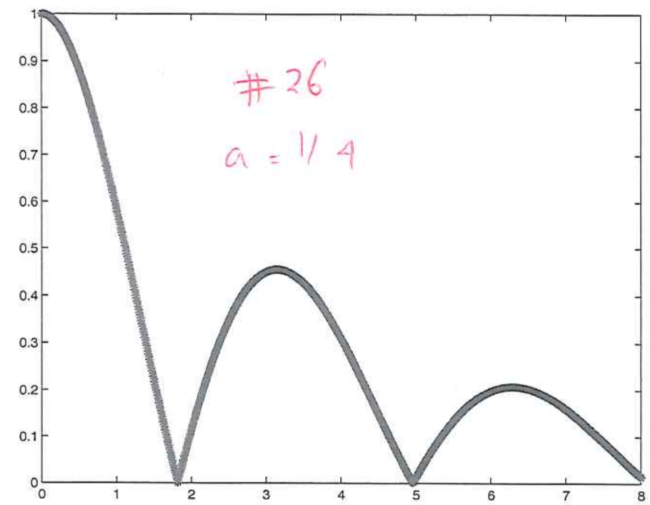
$= -c_2 xe^{-x} + c_3$

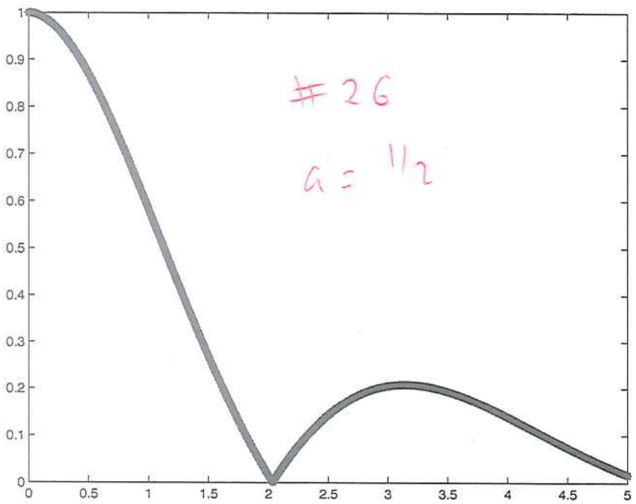
$\Rightarrow y_2 = xe^{-x}y_1 = xe^{-x}e^x = x$

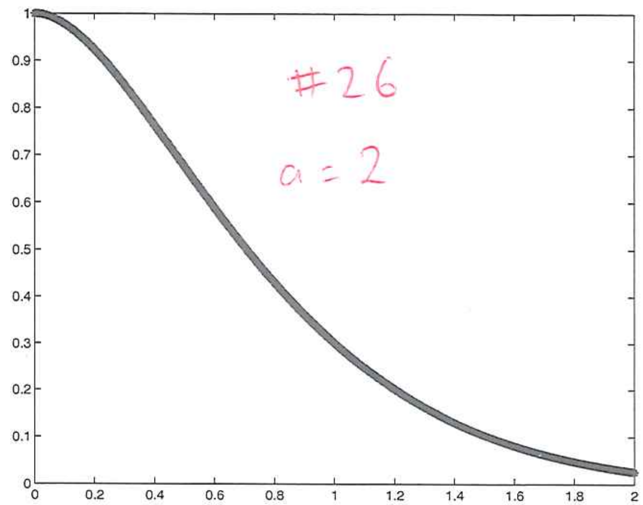
$\therefore y_2 = x.$


#23


#26
a=1


#26
a = 1/4


#26
a = 1/2


#26
a = 2