Hw7.1

2014

Homework 7

Section 3.5 #9-12: Find the general solution of the given differential equation.

#9: $u'' + \omega_0^2 u = \cos \omega t$. $\omega^2 \neq \omega_0^2$.

Answer: Guess: $u = A\cos\omega t + B\sin\omega t$

$u' = -A\omega \sin\omega t + B\omega\cos\omega t$

$u'' = -A\omega^2\cos\omega t - B\omega^2 \sin\omega t$

Substitute it into ODE:

$-A\omega^2\cos\omega t - B\omega^2\sin\omega t + \omega_0^2(A\cos\omega t + B\sin\omega t) = \cos\omega t$

$\Rightarrow A(\omega_0^2-\omega^2)\cos\omega t + B(\omega_0^2-\omega^2)\sin\omega t = \cos\omega t$

$\Rightarrow B=0, \quad A = \dfrac{1}{\omega_0^2-\omega^2}$.

$\Rightarrow u_p = \dfrac{1}{\omega_0^2-\omega^2}\cos\omega t$ is a particular solution, $\omega^2 \neq \omega_0^2$.

Homogeneos solution: Char. poly: $r^2 + \omega_0^2 = 0 \Rightarrow r = \pm i\omega_0$

$\Rightarrow$ Homogeneous solution: $c_1\cos\omega_0 t + c_2\sin\omega_0 t$

General solution: $u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \dfrac{1}{\omega_0^2-\omega^2}\cos\omega t$

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#10: $u'' + \omega_0^2 u = \cos\omega_0 t$

Answer: $r = \pm i\omega_0$ are the solutions of the characteristic polynomial.

The same guess from #9 doesn't work.

Need to try $u_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$

$\Rightarrow u' = A\cos\omega_0 t + B\sin\omega_0 t + t(-A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t)$

$u'' = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$

$\quad - A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t + t(-A\omega_0^2 \cos\omega_0 t - B\omega_0^2 \sin\omega_0 t)$

$\quad = -2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t + t(-A\omega_0^2 \cos\omega_0 t - B\omega_0^2 \sin\omega_0 t)$

Substitute it into ODE:

$-2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t + t(-A\omega_0^2 \cancel{\cos\omega_0 t} - B\omega_0^2 \cancel{\sin\omega_0 t})$

$+ \omega_0^2 (t(A\cancel{\cos\omega_0 t} + B\cancel{\sin\omega_0 t})) = \cos\omega_0 t$

$\Rightarrow \quad A = 0, \quad B = \dfrac{1}{2\omega_0}$, if $\omega_0 \neq 0$

In the special case where $\omega_0 = 0$, the ODE becomes:

$u'' = 1$ whose particular solution can be $\dfrac{t^2}{2}$.

The general solution is:

$$\begin{cases} c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \dfrac{1}{2\omega_0} t \sin\omega_0 t & \text{if } \omega_0 \neq 0 \\ c_1 + c_2 t + \dfrac{t^2}{2} & \text{if } \omega_0 = 0 \end{cases}$$

#11: $y'' + y' + 4y = 2\sinh t$, $\quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

Answer: Char poly: $r^2 + r + 4 = 0$

Roots: $r = \dfrac{-1 \pm \sqrt{1 - 4 \times 4}}{2} = \dfrac{-1 \pm \sqrt{15}\, i}{2}$.

Homogeneous sol: $c_1 e^{-t/2} \cos \sqrt{15}/2\, t + c_2 e^{-t/2} \sin \frac{\sqrt{15}}{2} t$.

Since neithe ± 1 are roots of char. polynomial, our gues can be $y_p = Ae^t + Be^{-t}$

$\Rightarrow y_p' = Ae^t - Be^{-t}$, $y_p'' = Ae^t + Be^{-t}$

$\Rightarrow Ae^t + \cancel{Be^{-t}} + Ae^t - \cancel{Be^{-t}} + 4(Ae^t + Be^{-t}) = e^t - e^{-t}$

$\Rightarrow 6Ae^t + 4Be^{-t} = e^t - e^{-t} \Rightarrow A = \frac{1}{6}, B = -\frac{1}{4}$

$\Rightarrow$ General solution is:

$$y = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$$

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#12: $y'' - y' - 2y = \cosh(2t)$, $\cosh t = \frac{e^t + e^{-t}}{2}$

$= \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$

Answer: Char poly: $r^2 - r - 2 = 0$

Roots: $r = \frac{1 \pm \sqrt{1 - 4\times(-2)}}{2} = \frac{1 \pm 3}{2} = -1, 2$.

Homogeneous sol: $c_1 e^{-t} + c_2 e^{2t}$

Since 2 is root of the char. polynomial, the guess should be.

$y_p = Ae^{-2t} + Bte^{2t}$ $\quad y_p' = -2Ae^{-2t} + Be^{2t} + Bt\,2e^{2t}$

$= -2Ae^{-2t} + Be^{2t} + 2Bte^{2t}$

$y_p'' = 4Ae^{-2t} + 2Be^{2t} + 2Be^{2t} + 4Bte^{2t} = 4Ae^{-2t} + 4Be^{2t} + 4Bte^{2t}$

Substitute it into ODE:

$4Ae^{-2t} + 4Be^{2t} + \cancel{4Bte^{2t}} \cancel{+2Ae^{-2t}} - Be^{2t} - \cancel{2Bte^{2t}}$

$\cancel{-2Ae^{-2t}} - \cancel{2Bte^{2t}} = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$

$\Rightarrow 4Ae^{-2t} + 3Be^{2t} = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t}$

$\Rightarrow A = \frac{1}{8}, \quad B = \frac{1}{6}.$

The general solution is:

$$y = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{8}e^{-2t} + \frac{1}{6}te^{2t}.$$

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Section 3.5 #18 Find the solution of the given initial value problem:

$$y'' + 2y' + 5y = 4e^{-t}\cos(2t), \quad y(0) = 1, \; y'(0) = 0.$$

Answer: Char poly: $r^2 + 2y + 5 = 0$

Roots: $r = \dfrac{-2 \pm \sqrt{4 - 4\times 5}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

Homogeneous sol: $c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

Since $e^{-t}\cos(2t)$ is part of the homogeneous solution, our guess should be: $y_p = \left(Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)\right)t$

$\cancel{y_p' = \left(-Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t)\right)t + Ae^{-t}\cos 2t + Be^{-t}\sin 2t}$

$\cancel{y_p'' = Ae^{-t}\cos(2t) + 2Ae^{-t}\sin(2t) + 2Ae^{-t}\sin(2t) - 4Ae^{-t}\cos(2t)}$

Shortcut: $y_p' = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) + t\frac{d}{dt}\left(Ae^{-t}\cos 2t + Be^{-t}\sin 2t\right)$

$$y_p'' = \frac{d}{dt}\left(Ae^{-t}\cos 2t + Be^{-t}\sin 2t\right) + \frac{d}{dt}\left(Ae^{-t}\cos 2t + Be^{-t}\sin 2t\right) + t\frac{d^2}{dt^2}\left(Ae^{-t}\cos 2t + Be^{-2t}\sin 2t\right)$$

$$= 2\frac{d}{dt}\left(Ae^{-t}\cos 2t + Be^{-t}\sin 2t\right) + t\frac{d^2}{dt^2}\left(Ae^{-t}\cos 2t + Be^{-2t}\sin 2t\right)$$

Subsitute it into ODE:

$$2\frac{d}{dt}\left(Ae^{-t}\cos 2t + Be^{-t}\sin 2t\right) + t\frac{d^2}{dt^2}\left(Ae^{-t}\cancel{\cos 2t + Be^{-t}}\sin 2t\right)$$

$$+\,2\left(Ae^{-t}\cos 2t + Be^{-t}\sin 2t\right) + t\,2\frac{d}{dt}\left(Ae^{-t}\cancel{\cos 2t + Be^{-t}}\sin 2t\right)$$

$$+\,t\,5\left(\cancel{Ae^{-t}\cos 2t} + Be^{-t}\sin 2t\right)$$

$$= 4e^{-t}\cos 2t$$

Because $Ae^{-t}\cos 2t + Be^{-t}\sin 2t$ is homogeneous solution.

$$\Rightarrow 2\left(-\cancel{Ae^{-t}\cos 2t} - 2Ae^{-t}\sin 2t - \cancel{Be^{-t}\sin 2t} + 2Be^{-t}\cos 2t\right)$$

$$+\,2\cancel{Ae^{-t}\cos 2t} + \cancel{2Be^{-t}\sin 2t} = 4e^{-t}\cos 2t$$

$$\Rightarrow -4Ae^{-t}\sin 2t + 4Be^{-t}\cos 2t = 4e^{-t}\cos 2t$$

$$\Rightarrow A = 0,\quad B = 1 \quad\Rightarrow\quad y_p = te^{-t}\sin 2t$$

$\Rightarrow$ General sol: $y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + te^{-t}\sin 2t$

Initial conditions:

$$y(0) = c_1 = 1$$

$$y' = -c_1 e^{-t}\cos 2t - 2c_1 e^{-t}\sin 2t - c_2 e^{-t}\sin 2t + 2c_2 e^{-t}\cos 2t$$

$$+\,e^{-t}\sin 2t - te^{-t}\sin 2t + 2te^{-t}\cos 2t$$

$$y'(0) = -c_1 + 2c_2 = 0 \quad\Rightarrow\quad c_2 = \frac{c_1}{2} = \frac{1}{2}$$

$$\Rightarrow y = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t$$

Section 3.6 # 3, 4 Use the method of variations of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients:

#3: $y'' + 2y' + y = 3e^{-t}$

Answer: Char poly: $r^2 + 2r + 1 = 0$

Roots: $r = \dfrac{-2 \pm \sqrt{4 - 4 \times 1}}{2} = -1$

-1 is a double root. $y_1 = e^{-t}$, $y_2 = te^{-t}$.

$\Rightarrow$ Homogeneous sol: $c_1 e^{-t} + c_2 te^{-t}$

Wronskian:
$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2$
$= e^{-t}(e^{-t} - \cancel{te^{-t}}) + e^{-t}\cancel{te^{-t}}$
$= e^{-2t}$

Method of variations of parameters:

$y_p = u_1(t)e^{-t} + u_2(t)\, te^{-t}$.

$p(t) = 2$, $q(t) = 1$, $g(t) = 3e^{-t}$

$\Rightarrow u_1(t) = -\displaystyle\int \frac{y_2(t)\, g(t)}{W(t)}\, dt = -\int \frac{te^{-t}\, 3\cancel{e^{-t}}}{\cancel{e^{-2t}}}\, dt = -3\frac{t^2}{2} = -\frac{3}{2}t^2$

$u_2(t) = \displaystyle\int \frac{y_1(t)\, g(t)}{W(t)}\, dt = \int \frac{e^{-t}\, 3e^{-t}}{e^{-2t}}\, dt = 3t$

$\Rightarrow y_p = -\frac{3}{2}t^2 e^{-t} + 3t\, te^{-t} = \frac{3}{2}t^2 e^{-t}$.

This also gives us the ~~one~~ guess for the method of undetermined coefficients:

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Method of undetermined coefficients:

$y_p = At^2 e^{-t}$ $\quad y_p' = A\,2te^{-t} + At^2(-e^{-t})$

$= 2Ate^{-t} - At^2e^{-t}$

$y_p'' = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t}$

$= 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$

Substitute it into ODE:

$2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}} + 2\left(\cancel{2Ate^{-t}} - \cancel{At^2e^{-t}}\right) + \cancel{At^2e^{-t}}$

$= 3e^{-t}$

$\Rightarrow 2Ae^{-t} = 3e^{-t} \Rightarrow A = \frac{3}{2}$

$\therefore y_p = \frac{3}{2}t^2e^{-t}$ is a particular solution.

#4: $4y'' - 4y' + y = 16e^{t/2}$

Answer: Remember to divide by the term in front of y'':

$y'' - y' + \frac{1}{4}y = 4e^{t/2}$

Char poly: $r^2 - r + \frac{1}{4} = 0$ Roots: $r = \frac{1 \pm \sqrt{1 - 4\cdot 1\cdot \frac{1}{4}}}{2} = \frac{1}{2}$

$\frac{1}{2}$ is a double root:

Homogeneous solution: $c_1e^{t/2} + c_2te^{t/2}$

$y_1 = e^{t/2}, \quad y_2 = te^{t/2}$

$p(t) = -1, \quad q(t) = \frac{1}{4}, \quad g(t) = 4e^{t/2}$

$W(t) = e^{t/2}\left(e^{t/2} + \cancel{\frac{t}{2}e^{t/2}}\right) - \cancel{\frac{1}{2}e^{t/2}\,te^{t/2}} = e^t$

Method of variations of parameters:

$y_p = u_1(t)\, e^{t/2} + u_2(t)\, t e^{t/2}$

$u_1 = -\int \frac{y_2(t) g(t)}{W(t)}\, dt = -\int \frac{t\cancel{e^{t/2}}\, 4\cancel{e^{t/2}}}{\cancel{e^{t}}}\, dt = -2t^2$

$u_2 = \int \frac{y_1(t) g(t)}{W(t)}\, dt = \int \frac{\cancel{e^{t/2}}\, 4\cancel{e^{t/2}}}{\cancel{e^{t/2}}}\, dt = 4t$

$\Rightarrow y_p = -2t^2 e^{t/2} + 4t\, t e^{t/2} = 2t^2 e^{t/2}$.

Method of undetermined coefficients:

Guess: $y_p = A t^2 e^{t/2}$

$y_p' = 2Ate^{t/2} + \frac{1}{2} A t^2 e^{t/2}$

$y_p'' = 2Ae^{t/2} + Ate^{t/2} + Ate^{t/2} + \frac{1}{4} A t^2 e^{t/2}$

$= 2Ae^{t/2} + 2Ate^{t/2} + \frac{1}{4} A t^2 e^{t/2}$

Substitute it into ODE:

$2Ae^{t/2} + \cancel{2Ate^{t/2}} + \cancel{\frac{1}{4}At^2 e^{t/2}} - \cancel{2Ate^{t/2}} - \cancel{\frac{1}{2}At^2 e^{t/2}}$

$+ \frac{1}{4} \cancel{At^2 e^{t/2}} = 4e^{t/2} \quad \Rightarrow 2Ae^{t/2} = 4e^{t/2}$

$\Rightarrow A = 2$

$\therefore y_p = 2t^2 e^{t/2}$ is a particular solution.

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Section 3.6 #7 and 8: Find the general solution of the given differential equation:

#7: $y'' + 4y' + 4y = t^{-2} e^{-2t}, \quad t > 0.$

Answer: Char poly: $r^2 + 4r + 4 = 0$

Roots: $r = \dfrac{-4 \pm \sqrt{16 - 4 \times 4}}{2} = -2$

$r = -2$ is a double root

Homogeneneous solution: $c_1 e^{-2t} + c_2 t e^{-2t}$, $y_1 = e^{-2t}$, $y_2 = te^{-2t}$

$p(t) = 4, \quad q(t) = 4, \quad g(t) = t^{-2} e^{-2t}$

$$W[y_1, y_2](t) = e^{-2t}\left(e^{-2t} - 2\cancel{t}e^{-2t}\right) + 2e^{-2t}\,\cancel{te^{-2t}}$$

$$= e^{-4t}$$

$$y_p = u_1(t) e^{-2t} + u_2(t) t e^{-2t}$$

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(t)}\, dt = -\int \frac{t\cancel{e^{-2t}}\, t^{-2} \cancel{e^{-2t}}}{\cancel{e^{-4t}}}\, dt = -\ln t$$

$$u_2(t) = \int \frac{y_1(t) g(t)}{W(t)}\, dt = \int \frac{e^{-2t}\, t^{-2} e^{-2t}}{e^{-4t}}\, dt = -\frac{1}{t}$$

$$\Rightarrow y_p = -\ln t\, e^{-2t} - \frac{1}{t}\, t e^{-2t} = -\ln t\, e^{-2t} \underbrace{- e^{-2t}}_{\text{part of homogeneous solution.}}$$

General solution:

$$y = c_1 e^{-2t} + c_2 t e^{-2t} - \ln t\, e^{-2t}$$

#8 $y'' + 4y = 3\csc(2t)$, $0 < t < \pi/2$

Answer: Char poly: $r^2 + 4 = 0$

Roots: $r = \pm 2i$ Homog. solution: $c_1 \cos 2t + c_2 \sin 2t$

$y_1 = \cos 2t$, $y_2 = \sin 2t$

$p(t) = 0$, $q(t) = 4$, $g(t) = 3\csc(2t)$.

$W[y_1, y_2](t) = (\cos 2t)\, 2\cos 2t + 2\sin 2t\, 2\sin 2t = 2(\cos^2 2t + \sin^2 2t) = 2$.

$y_p = u_1(t)\cos 2t + u_2(t)\sin 2t$

$$u_2 = +\int \frac{y_1(t)g(t)}{W(t)}\,dt = +\int \frac{\cos 2t\; 3\csc(2t)}{2}\,dt = +\frac{3}{2}\int \cot 2t\,dt$$

$$= +\frac{3}{4}\ln|\sin 2t| = +\frac{1}{4}\ln \sin 2t$$ since $\sin 2t > 0$ for $0 < t < \pi/2$

$$u_1 = -\int \frac{y_2(t)g(t)}{W(t)}\,dt = -\int \frac{\sin 2t\, 3\csc 2t}{2}\,dt = -3t/2$$

$$\Rightarrow y_p = -\frac{3t}{2}\cos(2t) + \frac{3}{4}\ln(\sin 2t)\sin(2t)$$

General solution: $c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2}t\cos(2t) + \frac{3}{4}\ln(\sin 2t)\sin(2t)$.

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Section 3.6 # 14 and 17: Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equations; then find ~~the~~ a particular solution of the given non-homogeneous equation.

#14: $t^2 y'' - t(t+2)y' + (t+2)y = 2t^3$; $y_1 = t$, $y_2 = te^t$

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Answer: Divite by t^2:

$y'' - \frac{1}{t}(t+2)y' + \frac{1}{t^2}(t+2)y = 2t$

$y_1 = t \quad y_1'' - \frac{1}{t}(t+2)y_1' + \frac{1}{t^2}(t+2)y_1 = 0 - \frac{1}{t}(t+2) + \frac{1}{t}(t+2) = 0$

$y_2 = te^t \quad y_2' = e^t + te^t, \quad y_2'' = e^t + e^t + te^t = 2e^t + te^t$

$\Rightarrow y_2'' - \frac{1}{t}(t+2)y_2' + \frac{1}{t^2}(t+2)y_2 = 2e^t + te^t - \frac{1}{t}(t+2)(e^t + te^t) + \frac{1}{t^2}(t+2)te^t$

$= \cancel{2e^t} + \cancel{te^t} - \cancel{e^t} - \cancel{te^t} - \cancel{\frac{2}{t}e^t} - \cancel{2e^t} + \cancel{e^t} + \cancel{\frac{2}{t}e^t} = 0$

$\Rightarrow y_1, y_2$ are homogeneous solutrons.

Method of Variatron of Parameters:

$y_p = u_1(t)\,t + u_2(t)\,te^t$

$p(t) = -\frac{1}{t}(t+2), \quad q(t) = \frac{1}{t^2}(t+2), \quad g(t) = 2t.$

$W(t) = y_1 y_2' - y_1' y_2 = t(e^t + te^t) - te^t$

$= te^t + t^2e^t - te^t$

$= t^2 e^t$

$u_1 = -\int \frac{y_2(t)\,g(t)}{W(t)}\,dt = -\int \frac{te^t\,2t}{t^2\cancel{e^t}}\,dt = -2t$

$u_2 = \int \frac{y_1(t)\,g(t)}{W(t)}\,dt = \int \frac{t\,2t}{t^2 e^t}\,dt$

$= \int 2e^{-t}\,dt = -2e^{-t}$

$\Rightarrow y_p = -2t\,t - 2e^{-t}\,te^t = -2t^2 - 2t$

part of homog. sol.

$\Rightarrow y_p = -2t^2$ is also a partrcular solutron.

#17: $x^2y'' - 3xy' + 4y = x^2\ln x$, $x>0$, $y_1 = x^2$, $y_2 = x^2\ln x$

Answer: Divide by x^2 $\quad y'' - 3x^{-1}y' + 4x^{-2}y = \ln x$

$$y_1 = x^2 \Rightarrow y_1'' - 3x^{-1}y_1' + 4x^{-2}y_1 = 2 - 3x^{-1}(2x) + 4x^{-2}x^2 = 2-6+4 = 0$$

$$y_2 = x^2\ln x \quad y_2' = 2x\ln x + x^2\frac{1}{x} = 2x\ln x + x$$

$$y_2'' = 2\ln x + 2x\frac{1}{x} + 1 = 2\ln x + 2 + 1 = 2\ln x + 3$$

$$y_2'' - 3x^{-1}y_2' + 4x^{-2}y = 2\ln x + 3 - 3x^{-1}(2x\ln x + x) + 4x^{-2}x^2\ln x = 0.$$

$$= 2\cancel{\ln x} + \cancel{3} - 6\cancel{\ln x} - \cancel{3} + 4\cancel{\ln x}$$

Variation of parameters:

$p(x) = -3x^{-1}$, $q(x) = 4x^{-2}$, $g(x) = \ln x$

$$y_p = u_1(x)\,x^2 + u_2(x)\,x^2\ln x$$

$$W(x) = y_1y_2' - y_1'y_2 = x^2(2x\ln x + x) - 2x\,x^2\ln x$$

$$= 2\cancel{x^3\ln x} + x^3 - 2\cancel{x^3\ln x} = x^3.$$

$$u_1 = -\int\frac{y_2(x)\,g(x)}{W(x)}\,dx = -\int\frac{x^2\ln x\,\ln x}{x^3}\,dx = -\int\frac{(\ln x)^2}{x}\,dx$$

$$= -\frac{1}{3}(\ln x)^3$$

$$u_2 = \int\frac{y_1(x)\,g(x)}{W(x)}\,dx = \int\frac{x^2\ln x}{x^3} = \int\frac{\ln x}{x}\,dx = \frac{1}{2}(\ln(x))^2$$

$$y_p = -\frac{1}{3}(\ln x)^3x^2 + \frac{1}{2}(\ln x)^2x^2\ln x =$$

$$= \frac{1}{6}x^2(\ln x)^3$$