Section 4.1:

11, 16 Verify that the given functions are solutions of the ODE, and determine their Wronskian.

#11 $y''' + y' = 0$, $1, \cos t, \sin t$

Answer: $y_1 = 1 \Rightarrow y_1''' + y_1' = 0 + 0 = 0$

$y_2 = \cos t$ $\quad y_2''' + y_2' = \sin t - \sin t = 0$

$y_3 = \sin t$ $\quad y_3''' + y_3' = -\cos t + \cos t = 0$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1 \cdot (\sin^2 t + \cos^2 t) = 1.$$

#16 $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$, $x, x^2, \frac{1}{x}$

Answer: $y_1 = x$ $\quad x^3 y_1''' + x^2 y_1'' - 2x y_1' + 2y = 0 - 0 - 2x \cdot 1 + 2x = 0$

$y_2 = x^2$ $\quad x^3 y_2''' + x^2 y_2'' - 2x y_2' + 2y_2 = x^3 \cdot 0 + x^2 \cdot 2 - 2x \cdot 2x + 2x^2 = 0$

$y_3 = \frac{1}{x}$ $\quad y_3' = -x^{-2}$, $y_3'' = 2x^{-3}$, $y_3''' = -6x^{-4}$

$\Rightarrow x^3 y_3''' + x^2 y_3'' - 2x y_3' + 2y_3 = -6x^{-1} + 2x^{-1} + 2x^{-1} + 2x^{-1} = 0$

Problem: Consider the Hermite differential equation

$$y'' - 2xy' + y = 0, \quad y(0) = y_0, \quad y'(0) = y_0'.$$

Find the recurrence relation for the power series solution. Compute the first 7 terms.

Answer: $y = \sum_{n=0}^{\infty} a_n x^n$ $\quad y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$, $y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2) x^n - 2\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(a_{n+2}(n+1)(n+2) + (1-2n) a_n \right) x^n = 0$$

$$\Rightarrow a_{n+2}(n+1)(n+2) = (2n-1)a_n$$

$$a_{n+2} = \frac{2n-1}{(n+1)(n+2)} a_n .$$

$a_0 = y_0 , \quad a_1 = y_0' .$

$n=0 \quad a_2 = \frac{-1}{2} a_0 \qquad n=1 \quad a_3 = \frac{1}{2\cdot 3} a_1 = \frac{1}{6} a_1$

$n=2 \quad a_4 = \frac{3}{3\cdot 4} a_2 = \frac{1}{4}\left(-\frac{1}{2}a_0\right) = -\frac{1}{8}a_0$

$n=3 \quad a_5 = \frac{5}{4\cdot 5} a_3 = \frac{1}{4}\left(\frac{1}{6}a_1\right) = \frac{1}{24}a_1$

$n=4 \quad a_6 = \frac{7}{5\cdot 6} a_4 = \frac{7}{5\cdot 6}\left(-\frac{1}{8}a_0\right) = -\frac{7}{240} a_0$

$$\Rightarrow y = y_0 + y_0' x - \frac{1}{2} y_0 x^2 + \frac{1}{6} y_0' x^3 - \frac{1}{8} y_0 x^4 + \frac{1}{24} y_0' x^5 - \frac{7}{240} y_0 x^6 + \cdots$$

~~Section 5.1 Problems 21, 26 Rewrite the given expression as a sum whose generic term involves x^n.~~

Section 5.1 Problem 1, 4, 8. Determine the radius of convergence of the given power series;

#1: $\sum_{n=0}^{\infty} (x-3)^n \qquad a_n = 1 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = 1 \xrightarrow{\text{as } n\to\infty} 1$

$\Rightarrow \rho = 1$

#4 $\sum_{n=0}^{\infty} 2^n x^n \qquad a_n = 2^n \qquad \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2 \xrightarrow{\text{as } n\to\infty} 2$

$\Rightarrow L = 2 \Rightarrow \rho = \frac{1}{L} = \frac{1}{2}$

#8: $\sum_{n=0}^{\infty} \frac{n! x^n}{n^n} \qquad a_n = \frac{n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!\, n^n}{(n+1)^{n+1} n!}$

$$= \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+\frac{1}{n})^n} \xrightarrow[n\to\infty]{} \frac{1}{e}$$

$$\Rightarrow L = \frac{1}{e} \Rightarrow \rho = e$$

Section 5.1 #9, 12: Determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

#9 $\sin x$, $x_0 = 0$.

Answer: $\sin'(x) = \cos(x)$ $\quad \sin''(x) = -\sin(x)$ $\quad \sin^{(3)}(x) = -\cos x$ $\quad \sin^{(4)}(x) = \sin x$

$\Rightarrow \sin'(0) = 1$ $\quad \sin''(0) = 0$ $\quad \sin^{(3)}(0) = -1$ $\quad \sin^{(4)}(0) = 0$

$$\Rightarrow \sin^{(2k)}(0) = 0, \quad \sin^{(2k+1)}(0) = (-1)^k$$

$$\Rightarrow \quad a_n = \frac{\sin^{(n)}(0)}{n!} = \begin{cases} 0 & \text{if } n = 2k \text{ even} \\ \frac{(-1)^k}{(2k+1)!} & \text{if } n = 2k+1 \text{ odd} \end{cases}$$

$$\Rightarrow \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

#12: x^2, $x_0 = -1$

$$x^2 = (x+1-1)^2 = (x+1)^2 - 2(x+1) + 1 = (x-x_0)^2 - 2(x-x_0) + 1$$

$\Rightarrow a_0 = 1$, $a_1 = -2$, $a_3 = 1$, $a_n = 0$ for $n \geq 4$.

Section 5.2 #2, 4, 5, 8. In each of the following problems,

(a) Seek power series solutions of the given ODE about the given point x_0; find the recurrence relation

(b) Find the the first 4 terms of y_1, y_2.

(c) By evaluating the Wronskian, show that y_1, y_2 are fund. solns.

(d) if possible, find the general term in each sol.

#2: $y'' - xy' - y = 0, \quad x_0 = 0$

Answer: $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$-\sum a_n (x)^n$

$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - x\sum a_n (n)(x)^{n-1} - \sum a_n (x-1)^n$

$\sum a_{n+2}(n+2)(n+1)x^n - \sum a_n (n)(x)^n$

$= \sum_n a_{n+2}(n+2)(n+1) \quad -a_n)(x-1)^n$

$= \sum_n (a_{n+2}(n+2)(n+1) - a_n(n) \quad -a_n)(x-1)^n$

Sorry, I took $x_0 = 1$ initially.

$\Rightarrow \cancel{a_{n+2} = \frac{n}{(n+2)(n+1)}(a_n + a_{n+1})}$

$\Rightarrow a_{n+2} = \frac{a_n}{n+2}$.

$a_2 = \frac{a_0}{2} \qquad a_4 = \frac{a_1}{4} = \frac{a_0}{2 \cdot 4} = \frac{a_0}{2^2 \cdot (1 \cdot 2)},$

$a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6} = \frac{a_0}{2^3(1 \cdot 2 \cdot 3)} \Rightarrow \ldots a_{2k} = \frac{1}{2^k k!}$

$a_3 = \frac{a_1}{3} \qquad a_5 = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5} \qquad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7} = \frac{2 \cdot 4 \cdot 6}{7!} a_1 = \frac{2^3 \cdot 3!}{7!} a_1$

$\Rightarrow \ldots \quad a_{2k+1} = \frac{2^k k!}{(2k+1)!}$.

$\Rightarrow y_1 = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}, \quad y_2 = \sum_{k=0}^{\infty} \frac{2^k k! \, x^{2k+1}}{(2k+1)!}$

$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1$

$\Rightarrow W[y_1, y_2](0) = 1 \Rightarrow$ They form a fundamental set of solutions.

#4. $y'' + k^2 x^2 y = 0$, $x_0 = 0$, $k = \text{const}$.

Answer: $y = \sum a_n x^n \Rightarrow \sum a_{n+2}(n+2)(n+1)x^n + k^2 \sum a_n x^{n+2} = 0$

$$\Rightarrow \cancel{4}\, 2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left(a_{n+2}(n+2)(n+1) + k^2 a_{n-2}\right)x^n = 0$$

$\Rightarrow a_2 = 0 \quad a_3 = 0$ and

$$a_{n+2} = -\frac{k^2}{(n+2)(n+1)} a_{n-2} \qquad n \geq 2$$

$n=2$ $a_4 = -\frac{k^2}{4.3} a_0$ $\quad n=3$ $a_5 = \frac{-k^2}{5\cdot 4} a_1$

$n=4$ $a_6 = -\frac{k^2}{6.\cancel{3}} a_2 = 0$ $\quad n=5$ $a_7 = -\frac{k^2}{7\cdot 6} a_3 = 0$

$n=6$ $a_8 = -\frac{k^2}{8.7} a_4 = +\frac{k^4}{3.4.7.8} a_0$

$n=7$ $a_9 = -\frac{k^2}{9.8} a_5 = \frac{k^4}{4.5\cdot 8\cdot 9} a_1$ $\quad n=8$ $a_{10} = -\frac{k^2}{10\cdot 9} a_6 = 0$

$n=9$ $a_{11} = -\frac{k^2}{11\cdot 10} a_7 = \cancel{\frac{k^4}{6\cdot 7\cdot 10\cdot 11}} 0$

$n=10$ $a_{12} = -\frac{k^2}{12\cdot 11} a_8 = -\frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12} a_0$

$n=11$ $a_{13} = -\frac{k^2}{13\cdot 12} a_9 = -\frac{k^6}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13} a_1$

$$\Rightarrow y_1 = 1 - \frac{k^2}{3\cdot 4}x^4 + \frac{k^4}{3\cdot 4\cdot 7\cdot 8}x^4 - \frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}x^{12} + \cdots$$

$$y_2 = x - \frac{k^2}{4\cdot 5}x^5 + \frac{k^4}{4\cdot 5\cdot 8\cdot 9}x^9 - \frac{k^6}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13}x^{13} + \cdots$$

Similarly, $W[y_1, y_2](0) = 1 \neq 0 \Rightarrow$ form fund. sol.

\# 5 $\quad (1-x)y'' + y = 0, \quad x_0 = 0. \qquad y = \sum a_n x^n$

$$(1-x)\sum a_n n(n-1)x^{n-2} + \sum a_n x^n$$

$$= \sum a_n n(n-1)x^{n-2} - \sum a_n n(n-1)x^{n-1} + \sum a_n x^n$$

$$= \sum a_{n+2}(n+2)(n+1)x^n - \sum a_{n+1}(n+1)n\,x^n + \sum a_n x^n$$

$$\cancel{\Rightarrow \sum \left(a_{n+2}(n+1)n + a_n(n^2+3n+3)\right)x^n} = 0$$

$$\Rightarrow \quad a_{n+2}(n+2)(n+1) - a_{n+1}n(n+1) + a_n = 0$$

$$\Rightarrow \quad a_{n+2} = \frac{a_{n+1}n(n+1) - a_n}{(n+2)(n+1)}$$

$n=0 \qquad a_2 = \dfrac{-a_0}{2} = -\dfrac{1}{2}a_0$

$n=1 \qquad a_3 = \dfrac{a_2\cdot 2 - a_1}{3\cdot 2} = \dfrac{1}{3}a_2 - \dfrac{1}{6}a_1 = -\dfrac{1}{6}a_0 - \dfrac{1}{6}a_1$

$n=2 \qquad a_4 = \dfrac{a_3\cdot 3\cdot 2 - a_2}{4\cdot 3} = \dfrac{1}{2}a_3 - \dfrac{1}{12}a_2 = \dfrac{1}{2}\left(-\dfrac{1}{6}a_0 - \dfrac{1}{6}a_1\right) - \dfrac{1}{12}\left(-\dfrac{1}{2}a_0\right)$

$$= -\frac{1}{24}a_0 - \frac{1}{12}a_1$$

$n=3 \qquad a_5 = \dfrac{a_4\cdot 4\cdot 3 - a_3}{5\cdot 4} = \dfrac{3}{5}\left(-\dfrac{1}{24}a_0 - \dfrac{1}{12}a_1\right) - \dfrac{1}{20}\left(-\dfrac{1}{6}a_0 - \dfrac{1}{6}a_1\right)$

$$= \frac{4}{5\cdot 24}a_0 - \frac{1}{24}a_1$$

$\Rightarrow y_0 = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \cdots$

$W[y_0, y_1](0) = 1 \neq 0. \Rightarrow$ They form a fund. set of solns.

HW8. 4

#8 $xy'' + y' + xy = 0$, $x_0 = 1$

Answer:

$y = \sum a_n (x-1)^n$

$x \sum a_n (n)(n-1)(x-1)^{n-2} + \sum a_n n (x-1)^{n-1} + x \sum a_n (x-1)^n$

$x \to x-1+1$ $\qquad x \to x-1+1$

$= \sum a_n n(n-1)(x-1)^{n-1} + \sum a_n n(n-1)(x-1)^{n-2}$

$+ \sum n a_n (x-1)^{n-1} + \sum a_n (x-1)^{n+1} + \sum a_n (x-1)^n$

$= \sum (n+1) n a_{n+1} (x-1)^n + \sum (n+2)(n+1) a_{n+2} (x-1)^n$

$+ \sum (n+1) a_{n+1} (x-1)^n + \sum a_{n-1} (x-1)^n + \sum a_n (x-1)^n$

$\Rightarrow a_{n+2}(n+2)(n+1) + (n+1)^2 a_{n+1} + a_n + a_{n-1} = 0$

$n \geq 1 \qquad a_{-1} = 0$

$\Rightarrow a_{n+2} = \dfrac{-(n+1)^2 a_{n+1} - a_n - a_{n-1}}{(n+2)(n+1)}$

for $n=0$ $\quad a_2 = \dfrac{-a_1 - a_0}{2}$

$n=1$ $\quad a_3 = \dfrac{-4a_2 - a_1 - a_0}{3 \cdot 2} = \dfrac{2a_1 + 2a_0 - a_1 - a_0}{6} = \dfrac{a_0}{6} + \dfrac{a_1}{6}$

$n=2$ $\quad a_4 = \dfrac{-9a_3 - a_2 - a_1}{4 \cdot 3} = -\dfrac{3}{4}\left(\dfrac{a_0}{6} + \dfrac{a_1}{6}\right) - \dfrac{1}{12}\left(-\dfrac{1}{2}a_0 - \dfrac{1}{2}a_1\right) - \dfrac{1}{12}a_1$

$= -\dfrac{1}{12}a_0 - \dfrac{1}{6}a_1$

$\Rightarrow y_1 = 1 - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{6}(x-1)^3 - \dfrac{1}{12}(x-1)^4 + \dots$

$y_2 = (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{6}(x-1)^3 - \dfrac{1}{6}(x-1)^4 + \dots$

$W[y_1, y_2](1) = 1 \Rightarrow$ They form a fund. set of solns.

Section 5.1 Problems 21, 26. Rewrite the given expression as a sum whose generic term involves x^n.

Answer:

#21: $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$

#26: $\displaystyle\sum_{n=1}^{\infty} n a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n$

$\displaystyle= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^{n+1}$

$\displaystyle= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n$

$\displaystyle= a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right]x^n.$