(Hw 9.1

2014

Homework #9

Section 5.2 #19

(a) By making a change of variables $x-1=t$ and assuming y has a Taylor series in powers of t, find two series solutions of

$$y''+(x-1)^2y'+(x^2-1)y=0$$

in powers of $x-1$

Answer: $\frac{dy}{dt}=\frac{dy}{dx}$ $\quad x^2-1=(t+1)^2-1=t^2+2t$

$$\Rightarrow \frac{d^2y}{dt^2}+t^2\frac{dy}{dt}+(t^2+2t)y=0.$$

$$y(t)=\sum a_n t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)t^n+\sum a_n n\, t^{n+1}+\sum a_n t^{n+2}+2\sum a_n t^{n+1}=0$$

$$=\sum_{n=2}^{\infty}\Big(a_{n+2}(n+2)(n+1)+a_{n-1}(n-1)+a_{n-2}+2a_{n-1}\Big)t^n$$
$$+2a_2+6a_3t+2a_0t=0$$

$$\Rightarrow a_2=0,\quad a_3=-\frac{1}{3}a_0$$

and $\quad a_{n+2}=\dfrac{-(n+1)a_{n-1}-a_{n-2}}{(n+2)(n+1)}$

$n=2\qquad a_4=\dfrac{-3a_1-a_0}{4\cdot 3}=-\frac{1}{4}a_1-\frac{1}{12}a_0$

$n=3\qquad a_5=\dfrac{-4a_2-a_1}{5\cdot 4}=-\frac{1}{5}\cdot 0-\frac{1}{20}a_1=-\frac{1}{20}a_1$

$n=4 \quad a_6 = \dfrac{-5a_3 - a_2}{6\cdot 5} = -\dfrac{1}{6}\left(-\dfrac{1}{3}\right)a_0 - 0 = \dfrac{1}{18}a_0$

$n=5 \quad a_7 = \dfrac{-6\cdot a_4 - a_3}{7\cdot 6} = -\dfrac{1}{7}\left(-\dfrac{1}{12}a_0 - \dfrac{1}{4}a_1\right) - \dfrac{1}{7\cdot 6}\left(-\dfrac{1}{3}a_0\right)$

$= \left(\dfrac{1}{7\cdot 12} + \dfrac{1}{7\cdot 18}\right)a_0 + \dfrac{1}{28}a_1$

$\Rightarrow y_1 = 1 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{18}(x-1)^6 + \dots$

$y_2 = (x-1) - \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{28}(x-1)^7 + \dots$

(b) Show that you obtain the same result by assuming y has a Taylor series in powers of $x-1$ and also expressing the coefficients x^2-1 in powers of $x-1$

$y = \sum a_n (x-1)^n$

$x^2 - 1 = (x-1+1)^2 - 1 = (x-1)^2 + 2(x-1)$

$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum a_n n (x-1)^{n+1} + \left((x-1)^2 + 2(x-1)\right)\sum a_n (x-1)^n$

The recurrence relation would be the same.

Hw9.2

Section 5.2 #20 Show directly, using the ratio test, that the two series solutions of Airy's equation about $x=0$ converge for all x.

Answer:

First series' coefficienst: $a_{3n} = \dfrac{1}{2\cdot 3\cdots(3n-1)(3n)}$

$$\frac{a_{3(n+1)}}{a_{3n}} = \frac{\dfrac{1}{2\cdot 3\cdots(3n-1)(3n)(3n+1)(3n+2)(3n+3)}}{\dfrac{1}{2\cdot 3\cdots(3n-1)(3n)}}$$

$$= \frac{1}{(3n+1)(3n+2)(3n+3)} \xrightarrow[\text{as } n\to\infty]{} 0$$

$\Rightarrow L = 0 \quad \rho = \left(\frac{1}{L}\right)^{1/3} = \infty.$

The radius of convergence is ∞.

Second series: $a_{3n+1} = \dfrac{1}{3\cdot 4\cdots(3n)(3n+1)}$

$$\frac{a_{3(n+1)+1}}{a_{3n+1}} = \frac{\dfrac{1}{3\cdot 4\cdots(3n)(3n+1)(3n+2)(3n+3)(3n+4)}}{\dfrac{1}{3\cdot 4\cdots(3n)(3n+1)}}$$

$$= \frac{1}{(3n+2)(3n+3)(3n+4)} \xrightarrow[n\to\infty]{} 0 \Rightarrow L = 0$$

$\Rightarrow \rho = \left(\frac{1}{L}\right)^{1/3} = \infty \Rightarrow$ The radius of convergence is ∞.

5.2 Problem 21 The Hermite equation

The equation $y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty$

where λ is a constant, is known as the Hermite equation. It is an important equation in mathematical physics.

(a) Find the first four terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.

Answer: $y = \sum a_n x^n \quad y' = \sum n a_n x^{n-1}$

$$\Rightarrow \sum a_{n+2}(n+2)(n+1)x^n - 2\sum n a_n x^n + \lambda \sum a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + (\lambda - 2n) a_n \right] x^n = 0$$

$\Rightarrow$ The recurrence relation is:

$$a_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)} a_n$$

$n = 0$: $a_2 = \dfrac{-\lambda}{2} a_0$

$n = 1$: $a_3 = \dfrac{2 - \lambda}{3 \cdot 2} a_1 = -\dfrac{\lambda - 2}{6} a_1$

$n = 2$: $a_4 = \dfrac{4 - \lambda}{4 \cdot 3} a_2 = +\dfrac{\lambda(\lambda - 4)}{2 \cdot 3 \cdot 4} a_0$

$n = 3$: $a_5 = \dfrac{6 - \lambda}{5 \cdot 4} a_3 = \dfrac{(\lambda - 2)(\lambda - 6)}{6 \cdot 5 \cdot 4} a_1 = \dfrac{(\lambda - 2)(\lambda - 6)}{5!} a_1$

Hw4.3

$n=4: \quad a_6 = \dfrac{8-\lambda}{6\cdot 5} a_4 = \dfrac{-\lambda(\lambda-4)(\lambda-8)}{6!} a_0$

$n=5: \quad a_7 = \dfrac{10-\lambda}{7\cdot 6} a_5 = -\dfrac{(\lambda-10)(\lambda-6)(\lambda-2)}{7!} a_1$

$\Rightarrow \quad y_1 = 1 - \dfrac{\lambda}{2!} + \dfrac{\lambda(\lambda-4)}{4!} x^4 - \dfrac{(\lambda)(\lambda-4)(\lambda-8)}{6!} x^6 + \cdots$

$y_2 = x - \dfrac{\lambda-2}{3!} x^3 + \dfrac{(\lambda-2)(\lambda-6)}{5!} x^5 - \dfrac{(\lambda-2)(\lambda-6)(\lambda-10)}{7!} x^7 + \cdots$

(b) Observe that if λ is a non-negative even integer, then or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6, 8$ and 10. Note that each polynomial is determined up to a multiplicative constant.

Answer: Assume $a_0 \neq 0$, $a_1 \neq 0$

Suppose λ is a non-negative even integer.

$\Rightarrow \lambda = 2k$ for some $k \in \mathbb{N} \cup \{0\}$ (i.e. $k = 0, 1, 2, 3, \ldots$)

$\Rightarrow a_{k+2} = \dfrac{2k-\lambda}{(k+2)(k+1)} a_k = \dfrac{2k-2k}{(k+2)(k+1)} a_k = 0$

Therefore $a_{k+4} = 0$, $a_{k+6} = 0$, ...

If k is itself even $\Rightarrow$ y_1 is polynomial the polynomial.

Otherwise, y_2 is the polynomial.

Cases: $\lambda = 0 \Rightarrow a_2 = 0 \Rightarrow y_1 = a_0 =$ constant

Case $\lambda = 2 \Rightarrow a_2 = \frac{-2}{2} a_0 = -a_0$, $a_3 = \frac{2-\lambda}{3\cdot 2} a_1 = 0 \Rightarrow y_2 = a_1 x$ polynomial degree 1.

$\lambda = 4$ $\quad a_2 = -\frac{\lambda}{2} a_0 = -2a_0$

$$a_4 = \frac{\lambda(\lambda-4)}{4!} a_0 = 0$$

$\Rightarrow y_1 = a_0(1 - 2x^2)$ polynomial of degree 2.

$\lambda = 6 \Rightarrow a_3 = -\frac{\lambda-2}{6} a_1 = -\frac{4}{6} a_1 = -\frac{2}{3} a_1$

$$a_5 = \frac{(\lambda-2)(\lambda-6)}{5!} a_1 = 0$$

$\Rightarrow y_2 = a_1(x - \frac{2}{3}x^3)$ polynomial of degree 3.

$\lambda = 8 \Rightarrow a_2 = -\frac{\lambda}{2} a_0 = -4a_0$ $\quad a_4 = \frac{\lambda(\lambda-4)}{4!} a_0 = \frac{8\cdot 4}{4\cdot 3\cdot 2} a_0$

$$= \frac{8}{6} a_0 = \frac{4}{3} a_0$$

$$a_6 = -\frac{\lambda(\lambda-4)(\lambda-8)}{6!} a_0 = 0$$

$\Rightarrow y_1 = a_0(1 - 4x^2 + \frac{4}{3}x^4)$ poly of deg. 4.

$\lambda = 10$ $\quad a_3 = -\frac{\lambda-2}{6} a_1 = -\frac{8}{6} a_1 = -\frac{4}{3} a_1$

$$a_5 = \frac{(\lambda-2)(\lambda-6)}{5!} a_1 = \frac{8\cdot 4}{5\cdot 4\cdot 3\cdot 2} a_1 = \frac{8}{5\cdot 3\cdot 2} a_1 = \frac{4}{15} a_1$$

$$a_7 = -\frac{(\lambda-10)(\lambda-6)(\lambda-2)}{7!} a_1 = 0$$

$\Rightarrow y_2 = a_1(x - \frac{4}{3}x^3 + \frac{4}{15}x^5)$ poly of degree 5.

(c) The Hermite poly $H_n(x)$ is defined as the poly solution of the Hermite eqn. with $\lambda = 2n$ for which the coeff. of x^n is 2^n

Find $H_0(x), \ldots, H_5(x)$.

HW9.4

Answer:

$n=0$ $a_0=1$ $H_0(x)=1$

$n=1, \lambda=2, a_1=2$ $H_1(x)=2x$

$n=2, \lambda=4, a_0=-2$ $H_2(x)=-2+4x^2$

$n=3, \lambda=6, a_1=-\frac{3}{2}2^3$ $H_3(x)=-12x+8x^3$

$n=4, \lambda=8, a_0=\frac{3}{4}2^4$ $H_4(x)=12-48x^2+16x^4$

$n=5, \lambda=10, a_1=\frac{15}{4}2^5$ $H_5(x)=120x-160x^3+32x^5$