Math 319 : Midterm 1

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* SHOW ALL OF YOUR WORK FOR FULL CREDIT

TOTAL NUMBER OF PAGES: 8

YOUR NAME:

Solution

Prob 1 /25	
Prob 2 /25	
Prob 3 /25	
Prob 4 /25	
TOTAL /100	

Good luck!

Problem 1.

(a) Find the general solution of the following ordinary differential equation

$$ty' - y = t^3 e^{-t}, t > 0$$

$y' - \frac{1}{t} y = t^2 e^{-t}$ Use integrating factors $\frac{d\mu}{dt} = -\frac{1}{t}\mu$

$\Rightarrow \ln|\mu| = -\ln t + c, t>0 \Rightarrow$ Choose $\mu = e^{-\ln t} = \frac{1}{t}$

$\Rightarrow \frac{d}{dt}\left[\frac{1}{t} y\right] = \frac{1}{t} t^2 e^{-t} = te^{-t}$

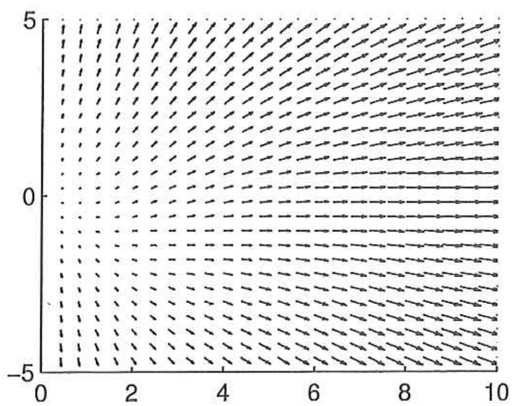
$\frac{1}{t} y = \int te^{-t} dt + c = -te^{-t} - \int -e^{-t} dt + c$ ← Integration by part

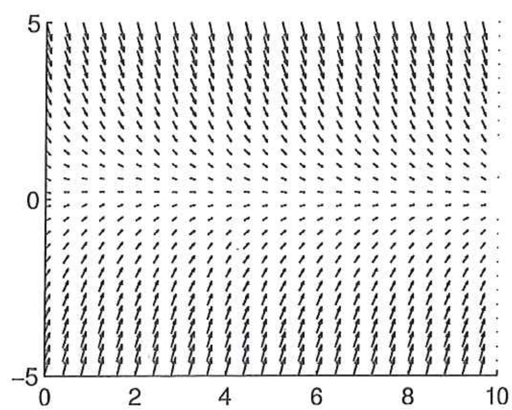
$= -(t+1) e^{-t} + c$

$\Rightarrow y = -t(t+1)e^{-t} + ct$

(b) Which of the following two slope fields corresponds to the ordinary differential equation above? **Why?**

Hint: Analyze the behavior of the solutions as $t \to \infty$ or near zero $t \approx 0$.





The one on the left

As $t \to \infty$ $t(t+1)e^{-t} \to 0$ and y looks like ct

Therefore, for large values of t, the graph of y must look like rays increasing or decreasing, but not converging to zero.

Problem 2.

(a) Write down the theorem of existence and uniqueness of solutions stated in class for first order ordinary differential equations

$$\frac{dy}{dt} = f(t, y).$$

Assume initial conditions $y(t_0) = y_0$. Suppose f and $\partial f/\partial y$ are both continuous functions on an open rectangle containing (t_0, y_0). Then there is an open interval where there exists one and only solution satisfying $y(t_0) = y_0$.

(b) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$\begin{cases} y' = \frac{-t+(t^2+4y)^{1/2}}{2} \\ y(2) = -1 \end{cases}$$

Where are these solutions valid?

$y_1 = 1-t \qquad y_1' = -1$ On the other hand:

$$\frac{-t+(t^2+4y)^{1/2}}{2} = \frac{-t+(t^2+4(1-t))^{1/2}}{2} = \frac{-t+(t^2-4t+4)^{1/2}}{2}$$

$$= \frac{-t+|t-2|}{2} = -1 \quad \text{for } t \geq 2.$$

$$\Rightarrow y_1' = \frac{-t+(t^2+4y_1)^{1/2}}{2} \quad \text{for } t \geq 2$$ ← where the solution is valid.

$y_2 = -t^2/4 \qquad y_2' = -\frac{1}{2}t$

$$\frac{-t+(t^2+4y)^{1/2}}{2} = \frac{-t+(t^2+4(-t^2/4))^{1/2}}{2} = -t/2 \quad \text{for all } t$$

$$\Rightarrow y_2' = \frac{-t+(t^2+4y_2)^{1/2}}{2} \quad \text{for all } t$$

y_1 is valid on $[2, \infty)$, y_2 is valid everywhere.

(c) Explain why the existence of these two solutions of the given problem does not contradict the uniqueness part of the theorem stated above.

In this case, $f(t,y) = \dfrac{-t+(t^2+4y)^{1/2}}{2}$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{4}(t^2+4y)^{-1/2}4 = (t^2+4y)^{-1/2}$

if $t_0=2, y_0=-1$ $\quad t^2+4y = 4-4=0$ and $\dfrac{\partial f}{\partial y}$ is discontinuous at $t_0=2, y_0=-1$, which violates the hypothesis of the theorem.

(d) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (b) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

$y = ct+c^2 \qquad y' = c$

$\dfrac{-t+(t^2+4y)^{1/2}}{2} = \dfrac{-t+(t^2+4(ct+c^2))^{1/2}}{2}$

$= \dfrac{-t+(t^2+4ct+(2c)^2)^{1/2}}{2} = \dfrac{-t+|t+2c|}{2}$

$= c = y'$ if $t \geq -2c$

If $c=-1 \Rightarrow y = 1-t$, $y(2) = -1$.

Suppose there is a constant c such that

$ct+c^2 = -t^2/4$

The function on the left is linear and the one ~~one~~ the right is quadratic

$\Rightarrow$ Not possible

If we derive twice, we get $0 = -\frac{1}{2}$, which is a contradiction.

Problem 3. Consider the initial value problem

$$\begin{cases} \frac{2x(-3+\sin(5y))}{1+x^2} + 5\cos(5y)\frac{dy}{dx} = 0 \\ y(0) = y_0. \end{cases}$$

(a) For what values of y_0 is a unique solution certain to exists in some interval containing $x_0 = 0$?

$$\frac{dy}{dx} = f(x,y), \quad \text{where } f(x,y) = \frac{2x(3-\sin(5y))}{5\cos(5y)}$$

$$\frac{\partial f}{\partial y} = \frac{2x}{5} \frac{-5\cos(5y)\,5\cos(5y) - (3-\sin(5y))(-25\sin(5y))}{25\cos^2(5y)}$$

Which are discontinuous at $5y = \frac{\pi}{2} + n\pi$ wher $\cos(5y) = 0$ vanishes.

$\Rightarrow$ If $y_0 \neq \frac{\pi}{10} + \frac{n}{5}\pi$

the solution is certain to exist locally

near $x_0 = 0$.

(b) Show that the equation above is not exact. Use integrating factors to make the differential equation exact and get the general (implicit) solution.

$$M = \frac{2x(-3+\sin(5y))}{1+x^2}, \quad N = 5\cos(5y)$$

$$M_y = \frac{2x}{1+x^2} 5\cos(5y), \quad N_x = 0 \quad \Rightarrow \text{Not exact}$$

$$\frac{M_y - N_x}{N} = \frac{\frac{2x}{1+x^2} 5\cos(5y)}{5\cos(5y)} = \frac{2x}{1+x^2}$$

$$\frac{d\mu}{dx} = \frac{2x}{1+x^2}\mu \Rightarrow \ln(\mu) = \ln(1+x^2) \Rightarrow \text{Choose } \mu = 1+x^2$$

$$\Rightarrow 2x(-3+\sin(5y)) + 5\cos(5y)(1+x^2)\frac{dy}{dx} = 0$$

is exact

$\frac{\partial \psi}{\partial x} = 2x(-3+\sin(5y)) \Rightarrow \psi = x^2(-3+5\sin(5y)) + h(y)$

$\frac{\partial \psi}{\partial y} = x^2 5\cos(5y) + h' = 5\cos(5y)(1+x^2)$

$\Rightarrow h'(y) = 5\cos(5y) \Rightarrow$ (choose) $h(y) = \sin(5y)$

$\Rightarrow \psi = x^2(-3+\sin(5y)) + \sin(5y) = -3x^2 + (1+x^2)\sin(5y) = C.$

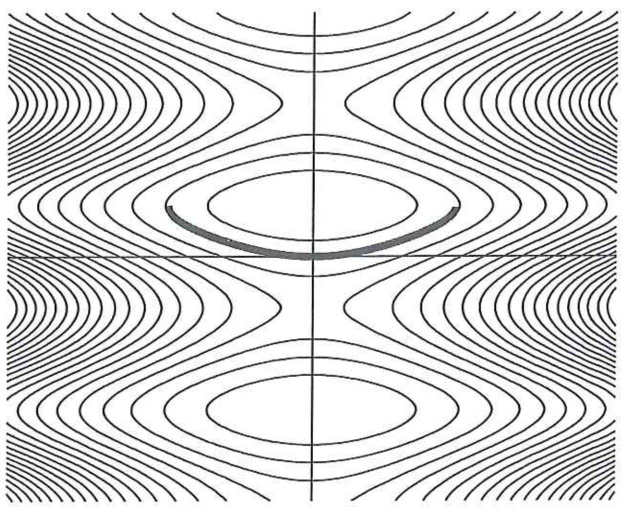
(d) Assume $y_0 = 0$. The figure below shows contours of solutions of the implicit equation. The thick line describes the graph of the solution with initial condition $y(0) = 0$. Determine the longest interval where the solution is valid.

Hint: Look for points of infinite derivatives $\frac{dy}{dx} = \infty$.

$y(0) = 0$

$\Rightarrow C = 0$

$\Rightarrow -3x^2 + (1+x^2)\sin(5y) = 0$



$\frac{dy}{dx} = \infty$ when $\cos(5y) = 0$ $\quad y = \frac{\pi}{10}$ for this solution.

$\Rightarrow -3x^2 + (1+x^2)\sin\left(\frac{\pi}{2}\right) = 0$

$\quad \underbrace{}_{-2x^2+1} \Rightarrow x = \pm\sqrt{1/2}$

Problem 4. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream , write a differential equation for the amount of the drug that is present in the bloodstream at any time.

Concentration 5mg/cm^3 rate : 100 cm^3/h

$$\frac{dQ}{dt} = 5\,\text{mg}/\text{cm}^3 \times 100\,\text{cm}^3/\text{h} - \frac{0.4}{\text{h}} Q, \quad t \text{ in } \text{h}.$$

$$= 500\,\text{mg}/\text{h} - \frac{0.4}{\text{h}} Q$$

(b) How much of the drug is present in the bloodstream after a long time?

The equilibrium solution is $Q = \frac{500}{0.4}\,\text{mg} = 1250\,\text{mg}$

(c) Find the amount of drug $Q(t)$ present in the bloodstream at any time assuming $Q(0) = 0$.

$$\frac{dQ}{dt} + 0.4 Q = 500 \qquad \mu = e^{0.4t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{0.4t} Q\right] = 500 e^{0.4t}$$

$$e^{0.4t} Q = \frac{500}{0.4} e^{0.4t} + c$$

$$Q = 1250\,\text{mg} + c e^{-0.4t}$$

$$0 = Q(0) = 1250 + c \qquad c = -1250$$

$$\Rightarrow Q = 1250\,\text{mg}\left(1 - e^{-\frac{0.4}{\text{h}} t}\right).$$

(d) How much time must elapse for the amount of drug to reach 90% of its limiting quantity found in part (b).

Note: Since you can't use your calculator, you can leave your answer implicit.

$1250\text{mg}\left(1-e^{-0.4t}\right) = 0.9 \times 1250\text{mg}$

$e^{-0.4t} = 0.1 \qquad t = -\dfrac{\ln(0.1)}{0.4}\ h$