Math 319 : Midterm 2

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* PLEASE WRITE YOUR NAME ON EVERY PAGE
* SHOW ALL OF YOUR WORK FOR FULL CREDIT

TOTAL NUMBER OF PAGES: 8

YOUR NAME:

Solutions

Prob 1 /20	
Prob 2 /15	
Prob 3 /25	
Prob 4 /20	
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TOTAL /100	

Good luck!

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Problem 1 An equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, t > 0, \tag{1}$$

where α and β are real constants, is called an Euler equation

(a) **5 Points.** Let $x = \ln t$ and calculate $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t} = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(e^{-x}\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right)\frac{dx}{dt} = e^{-x}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right)$$

$$= e^{-x}\left(e^{-x}\frac{d^2y}{dx^2} - e^{-x}\frac{dy}{dx}\right) = e^{-2x}\frac{d^2y}{dx^2} - e^{-2x}\frac{dy}{dx}$$

(b) **5 Points** Use the results of part (a) to transform equation (1) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{2}$$

Observe that equation (2) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of equation (2), $y_1(\ln t)$ and $y_2(\ln t)$ form a set of solutions of equation (1).

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta t = e^{2x}\left(e^{-2x}\frac{d^2y}{dx^2} - e^{-2x}\frac{dy}{dx}\right)$$

$$+ \alpha e^{x} e^{-x}\frac{dy}{dx} + \beta y$$

$$= \frac{d^2y}{dx^2} + (\alpha-1)\frac{dy}{dx} + \beta y = 0.$$

Continuation of Problem 1...

(c) **10 Points.** Apply the method from parts (a) and (b) to solve the following equation:

$$t^2\frac{d^2y}{dt^2} + 7t\,\frac{dy}{dt} + 10y = 0, t > 0.$$

Here $\alpha = 7$, $\beta = 10$

$\Rightarrow$ In the x variable the equation becomes:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$$

Char polynomial $r^2 + 6r + 10$

Roots: $r = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 10}}{2} = \dfrac{-6 \pm 2i}{2} = -3 \pm i$

The two fundamental solutions in the x variable are:

$y_1(x) = e^{-3x}\cos x$, $y_2(x) = e^{-3x}\sin x$

In the variable t, the ~~equa~~ fundamental solution is:

$y_1(t) = e^{-3\ln t}\cos \ln t = t^{-3}\cos \ln t$

$y_2(t) = e^{-3\ln t}\sin \ln t = t^{-3}\sin \ln t$

The general solution is $y = C_1\dfrac{\cos \ln t}{t^3} + C_2\dfrac{\sin \ln t}{t^3}$

Problem 2. 15 Points. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of turbulence flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

$y_1 = e^{-\delta x^2/2}$ $\quad y_1' = -\delta x\, e^{-\delta x^2/2}$

$y_2'' = -\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2}$

$\Rightarrow y_2'' + \delta(xy' + y) = -\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2} + \delta x(-\delta x e^{-\delta x^2/2}) + \delta e^{-\delta x^2/2}$

$= \cancel{-\delta e^{-\delta x^2/2}} \cancel{+\delta^2 x^2 e^{-\delta x^2/2}} \cancel{-\delta^2 x^2 e^{-\delta x^2/2}} \cancel{+\delta e^{-\delta x^2/2}} = 0.$

Method of reduction of order:

$y_2 = v y_1$ $\quad y_2' = v' y_1 + v y_1'$ $\quad y_2'' = v'' y_1 + 2v' y_1' + v y_1''$

$\Rightarrow 0 = v'' y_1 + \underline{2v' y_1'} + v y_1'' + \underline{\delta x v' y_1} + \delta x v y_1' + \delta v y_1$

$= v'' y_1 + (2y_1' + \delta x y_1) v' + v(\underbrace{\cancel{y_1'' + \delta x y_1' + \delta y_1}}_{0})$

$w := v' \Rightarrow y_1 w' + (2y_1' + \delta x y_1) w = 0$

$\Rightarrow e^{-\delta x^2/2} w' + (-2\delta x e^{-\delta x^2/2} + \delta x e^{-\delta x^2/2}) w = 0$

$\Rightarrow w' - \delta x w = 0$ $\quad \ln|w| = +\delta x^2/2 \Rightarrow w = e^{+\delta x^2/2}$ $\quad (c_1 = 1)$

$\Rightarrow v = \int_0^x e^{+\delta z^2/2}\, dz \Rightarrow y_2 = e^{-\delta x^2/2} \int_0^x e^{\delta z^2/2}\, dz$

General solution: $y = c_1 e^{-\delta x^2/2} + c_2 e^{-\delta x^2/2} \int_0^x e^{\delta z^2/2}\, dz.$

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Problem 3. Consider the following differential equation

$$(1+x^2)y'' - xy' + 2y = 0.$$

(a) **10 Points.** Seek power series solutions of the given differential equation about the point $x_0 = 0$; find the recurrence relation

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = (1+x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0} n a_n x^{n-1} + 2\sum_{n=0} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n(n^2-n-n+2)x^n$$

$$= \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n(n^2-2n+2)\right]x^n$$

Recurrence relation $a_{n+2} = -\dfrac{n^2-2n+2}{(n+2)(n+1)}a_n$

(b) **10 Points.** Find the first four terms in each of the two fundamental solutions y_1, y_2

More space in the next page...

a_0, a_1 arbitrary.

$n=0$: $a_2 = -\dfrac{2}{2}a_0 = -a_0$

$n=1$: $a_3 = -\dfrac{1-2+2}{3\cdot 2}a_1 = -\dfrac{1}{6}a_1$

$n=2$: $a_4 = -\dfrac{4-4+2}{4\cdot 3}a_2 = -\dfrac{1}{6}(-a_0) = \dfrac{1}{6}a_0$

Continuation of Problem 3...

$n=3:\quad a_5 = -\dfrac{9-6+2}{5\cdot 4} a_3 = -\dfrac{1}{4}\left(-\dfrac{1}{6}a_1\right) = \dfrac{1}{24}a_1$

$n=4:\quad a_6 = -\dfrac{16-8+2}{6\cdot 5} a_4 = -\dfrac{10}{6\cdot 5}\dfrac{1}{6}a_0 = -\dfrac{1}{18}a_0$

$n=5\quad a_7 = -\dfrac{25-10+2}{7\cdot 6} a_5 = -\dfrac{17}{7\cdot 6}\dfrac{1}{24}a_1 = -\dfrac{17}{1008}a_1$

$y = a_0 + a_1 x + (-a_0)x^2 + \left(-\dfrac{1}{6}a_1\right)x^3 + \dfrac{1}{6}a_0 x^4 + \dfrac{1}{24}a_1 x^5 - \dfrac{1}{18}a_0 x^6 - \dfrac{17}{1008}a_1 x^7 + \dots$

$= a_0\left(1 - x^2 + \dfrac{1}{6}x^4 - \dfrac{1}{18}x^6 + \dots\right) + a_1\left(x - \dfrac{1}{6}x^3 + \dfrac{1}{24}x^5 - \dfrac{17}{1008}x^7\right)$

$\Rightarrow y_1 = 1 - x^2 + \dfrac{1}{6}x^4 - \dfrac{1}{18}x^6 + \dots$

$y_2 = x - \dfrac{1}{6}x^3 + \dfrac{1}{24}x^5 - \dfrac{17}{1008}x^7 + \dots$

(c) **5 Points.** By evaluating the Wronskian $W(y_1, y_1)(x_0)$, confirm that y_1 and y_2 form a fundamental set of solutions.

Notice that $y_1(0) = 1,\ y_1'(0) = 0$

$y_2(0) = 0,\ y_2'(0) = 1$

$\Rightarrow W(y_1, y_2)(0) = 1 \Rightarrow$ The form a fundamental set of solutions.

Problem 4. Consider the following differential equation

$$x^2y'' - 3xy' + 4y = x^2.$$

(a) **5 Points.** Show that $y_1 = x^2$ and $y_2 = x^2 \ln|x|$ satisfy the corresponding **homogeneous** equation.

$y_1 = x^2 \quad y_1' = 2x \quad y_1'' = 2$

$x^2 y_1'' - 3xy_1' + 4y_1 = x^2 \cdot 2 - 3x\, 2x + 4x^2 = 2x^2 - 6x^2 + 4x^2 \equiv 0.$

$y_2 = x^2 \ln|x| \qquad y_2' = 2x \ln|x| + x^2 \frac{1}{\cancel{x}} = 2x \ln|x| + x.$

$y_2'' = 2\ln|x| + 2x\frac{1}{x} + 1 = 2\ln|x| + 3$

$\Rightarrow x^2 y_2'' - 3xy_2' + 4y_2 = x^2(2\ln|x| + 3) - 3x(2x\ln|x| + x) + 4x^2\ln|x|$

$= \cancel{2x^2\ln|x|} + \cancel{3x^2} - \cancel{6x^2\ln|x|} - \cancel{3x^2} + \cancel{4x^2\ln|x|} = 0.$

(b) **15 Points.** Find a particular solution of the non-homogeneous equation.

Method of variation of parameters:

$y = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

$p(x) = -\frac{3}{x}, \quad q(x) = \frac{4}{x^2}, \quad g(x) = 1$

$W(y_1, y_2)(x) = y_1 y_2' - y_1' y_2 = x^2(2x\cancel{\ln|x|} + x) - 2x\,\cancel{x^2\ln|x|}$

$= x^3.$

$u_1 = -\int \frac{y_2(x)g(x)}{W}\,dx = -\int \frac{x^2 \ln x \cdot 1}{x^3}\,dx = -\int \frac{\ln x}{x}\,dx = -\frac{1}{2}\ln^2|x|$

$u_2 = \int \frac{y_1(x)g(x)}{W}\,dx = \int \frac{x^2 \cdot 1}{x^3}\,dx = \ln|x|$

Then a particular solution is $y = -\frac{1}{2}(\ln^2|x|)\, x^2 + (\ln|x|)\, x^2 \ln|x|$

$= \frac{1}{2} x^2 \ln^2|x|.$

Problem 5. 20 Points. Find a general solution of the following differential equation

$$y'' - y' - 2y = e^{-t}$$

Method of undetermined coefficients:

Characteristic Polynomial $r^2 - r - 2$

Roots: $r = \frac{1 \pm \sqrt{1 - 4\cdot(-2)}}{2} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = -1, 2$

Since e^{-t} is a solution to the homogeneous equation, our guess is: $y_p = Ate^{-t}$

$\Rightarrow y_p' = Ae^{-t} - Ate^{-t}$

$y_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$

$\Rightarrow y_p'' - y_p' - 2y_p = -2Ae^{-t} + \cancel{Ate^{-t}} - Ae^{-t} + \cancel{Ate^{-t}} - 2\cancel{Ate^{-t}}$

$= -3Ae^{-t} = e^{-t}$

$\Rightarrow A = -\frac{1}{3} \Rightarrow y_p = -\frac{1}{3}te^{-t}$

General solution: $y = C_1 e^{-t} + C_2 e^{2t} - \frac{1}{3}te^{-t}$