MATH 319 - SEC 003, SPRING 2014.
PRACTICE PROBLEMS FOR FINAL EXAM

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Suggestions:

Section 5.2 8,9, 15,16

Section 5.3 7,8, 12,14,

Section 5.4 5,6,7,8, 22,23,24, 37, 38, 40, 49

Section 5.5 6,7,8,9, 11, 13

Section 5.6 6,7,8,9, 16,17, 18, 19, 20

Section 6.1 5, 8,10, 12, 17, 20, 26, 27

Section 6.2 3,4,5,6, 18,19,20, 25,26,27, 28

Section 6.4 3,4,7,8,9,11,12,13

Heat Equation:

Problem: Consider conservation of thermal energy

$$\frac{d}{dt}\int_a^b e dx = \phi(a,t) - \phi(b,t) + \int_a^b Q dx,$$

for any segment of a one-dimensional rod $a \leq x \leq b$. By using the fundamental theorem of calculus,

$$\frac{\partial}{\partial b}\int_a^b f(x)dx = f(b),$$

derive the heat equation

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q.$$

Problem : Consider the diffusion equation for $0 \leq x \leq 2\pi$ with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \\ u(0,t) &= 0 \\ u(2\pi,t) &= 0 \end{cases}$$

where $k > 0$ is a positive constant. Show that $u_{\text{steady}}(x) = 0$ is the only steady-state solution satisfying the boundary conditions above. Find all solutions of the form

$$u(x,t) = \phi(t)\sin(x)$$

Prove that in all cases,

$$\lim_{t\to\infty} u(x,t) = 0 = u_{\text{steady}}(x).$$

Problem: Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

(a) $\frac{Q}{K_0} = x^2$, $u(0) = T$, $\frac{\partial u}{\partial x}(L) = 0$

(b) $Q = 0$, $\frac{\partial u}{\partial x}(0) - [u(0) - T] = 0$, $\frac{\partial u}{\partial x}(L) = \alpha$

Problem : For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

(a)
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+1,\ u(x,0)=f(x),\ \frac{\partial u}{\partial x}(0,t)=1,\ \frac{\partial u}{\partial x}(L,t)=\beta$$
(b)
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ u(x,0)=f(x),\ \frac{\partial u}{\partial x}(0,t)=1,\ \frac{\partial u}{\partial x}(L,t)=\beta$$
(c)
$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+x-\beta,\ u(x,0)=f(x),\ \frac{\partial u}{\partial x}(0,t)=0,\ \frac{\partial u}{\partial x}(L,t)=0.$$

Problem : Consider the heat equation
$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$$
subject to the boundary conditions
$$u(0,t)=0 \text{ and } u(L,t)=0.$$
Solve the initial value problem if the temperature is initially

(b) $u(x,0)=3\sin(\frac{\pi x}{L})-\sin(\frac{3\pi x}{L})$

(c) $u(x,0)=2\cos\frac{3\pi x}{L}$

(d)
$$u(x,0)=\begin{cases}1, & 0<x\le\frac{L}{2}\\ 2, & \frac{L}{2}<x<L\end{cases}$$

Problem : Consider
$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}-\alpha u.$$
This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are
$$u(0,t)=0,\ \text{and } u(L,t)=0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha>0$?

(b) Solve the time-dependent problem $[u(x,0)=f(x)]$ if $\alpha>0$. Analyze the temperature for large time ($t\to\infty$) and compare to part (a).

Problem: Suppose that we did not know equation (6.2.15) in the textbook, but thought it possible to approximate d^2f/dx^2 by an unknown linear combination of there functionns values, $f(x_0-\Delta x), f(x_0), f(x_0+\Delta x)$:
$$\frac{d^2 f}{dx^2}\approx af(x_0-\Delta x)+bf(x_0)+cf(x_0+\Delta x).$$
Determine a, b and c by expanding the right-hand side in a Taylor series around x_0 using (6.210) and (6.2.11) and equating coefficients through d^2f/dx^2.

Problem : Numerically compute solutions to the heat equation
$$\begin{aligned}&u_t=ku_{xx}, 0\le x\le 1,\ k=1\\ &u(0,t)=u(1,t)=0\\ &u(x,0)=f(x)=\begin{cases}0, & 0\le x<0.5\\ 1, & 0.5\le x<1\end{cases}\end{aligned}$$
using the numerical scheme
$$u_j^{(m+1)}=u_j^{(m)}+s\left(u_{j-1}^{(m)}-2u_j^{(m)}+u_{j+1}^{(m)}\right), j=1,\ldots,N-1,\ s=k\Delta t/\Delta x^2$$
with $\Delta x=0.1$ ($N=10$). Do for various s (discuss stability):
$$s=0.49,\ s=0.50,\ s=0.51,\ s=0.52$$