Homework 2.

Problem 1 If the laplace's equation is satisfied in three dimensions, show that

$$\oiint \nabla u \cdot \hat{n}\, ds = 0.$$

for any closed surface. Give a physical interpretation of this result (in the context of heat flow).

Answer:

Using the divergence theorem, we get:

$$\oiint \nabla u \cdot \hat{n}\, ds = \iiint_R \nabla \cdot \nabla u \, dv \qquad \text{where } S = \partial R.$$

$$= \iiint_R \nabla^2 u \, dv = 0$$

Therefore $\oiint \nabla u \cdot \hat{n}\, ds = 0.$

Equilibrium solutions of the heat equation

$$\frac{\partial u}{\partial t} = k \nabla^2 u \qquad \text{satisfy}$$

$$\nabla^2 u = 0.$$

The heat flux is given by $\phi = -K_0 \nabla u$. (K_0 constant here)

$\Rightarrow \oiint \phi \cdot \hat{n}\, ds.$

The normal component $\phi \cdot \hat{n}$ of the flux only accounts for the ~~energy~~ heat flowing through the boundary. Then, the results above show that the total energy flowing through the boundary is zero for an equilibrium temperature distribution.

Problem 2: Assume that the temperature is circularly symmetric: $u = u(r, t)$, where $r^2 = x^2 + y^2$. Consider any circular annulus $a \le r \le b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho u r\,dr$.

Answer:

The Jacobian of the transformation $x = r\cos\theta$, $y = r\sin\theta$ is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

$\Rightarrow$ The total energy is;

$$\iint_{a \le \sqrt{x^2+y^2} \le b} c\rho u\,dx\,dy = \int_a^b \int_0^{2\pi} c\rho u J\,dr\,d\theta = \int_a^b \int_0^{2\pi} c\rho u(r) r\,dr$$

$$= 2\pi \int_a^b c\rho u(r) r\,dr.$$

(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is

$$-2\pi b K_0 \frac{\partial u}{\partial r}\Big|_{r=b}.$$

Answer: The normal vector $\hat{n}$ pointing outwards at $r = b$ is $\hat{n} = \frac{(x,y)}{\|x,y\|} = \frac{(x,y)}{r}$

$\Rightarrow$ The flow of heat energy at $r = b$ is

$$\int_{r=b} -K_0 \nabla u \cdot \hat{n}\,ds = \int_{r=b} -K_0 \nabla u \cdot \frac{(x,y)}{r}\,ds = \int_{r=b} -K_0 \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) ds$$

$$= \int_{r=b} -K_0 \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) ds = \int_{r=b} -K_0 \frac{\partial u}{\partial r}\,dr$$

$$= -K_0 \frac{\partial u}{\partial r}\Big|_{r=b} 2\pi b$$ because $\frac{\partial u}{\partial r}(r)$ is constant at $r = b$ and the perimeter is $2\pi b$.

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Similarly the flow of heat energy at $r=a$ is $\cancel{2\pi}$ $2\pi a k_0 \frac{\partial u}{\partial r}\Big|_{r=a}$

The opposite sign is because $\hat{n} = -\frac{(x,y)}{\|x,y\|} = -\frac{(x,y)}{r}$ here.

(c) Use part (a) and (b) to derive the circulary symmetric heat equation without sources

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Answer: Using parts (a) and (b), the rate of change of heat energy is:

$$2\pi\frac{d}{dt}\int_a^b c\rho u r\,dr = 2\pi b K_0 \frac{\partial u}{\partial r}\Big|_{r=b} - 2\pi a K_0 \frac{\partial u}{\partial r}\Big|_{r=a}$$

Taking the derivative w.r.t. b we get:

$$\frac{\partial}{\partial t}(c\rho u b) = K_0 \frac{\partial}{\partial b}\left(b\frac{\partial u}{\partial r}\Big|_{r=b}\right)$$

Changing b by r, and dividing by $c\rho r$ we get:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Problem 3: Determine the equilibrium solution (circ. symm.) temperature distribution inside a circular annulus

(a) if the outer radius is at temp. T_2 and the inner at T_1

Answer: $\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0 \Rightarrow r\frac{\partial u}{\partial r} = C_1$

$\Rightarrow \frac{\partial u}{\partial r} = \frac{C_1}{r} \Rightarrow u = C_1 \ln r + C_2$

$T_2 = u(r_2) = c_1 \ln r_2 + c_2$

$T_1 = u(r_1) = c_1 \ln r_1 + c_2$

$\Rightarrow c_1 \ln\left(\frac{r_2}{r_1}\right) = T_2 - T_1 \Rightarrow c_1 = \dfrac{T_2 - T_1}{\ln(r_2/r_1)}$

$c_2 = T_2 - \ln r_2 \dfrac{T_2 - T_1}{\ln(r_2/r_1)}$.

$\Rightarrow u(r) = \dfrac{T_2 - T_1}{\ln(r_2/r_1)} \ln r + T_2 - \ln r_2 \dfrac{T_2 - T_1}{\ln(r_2/r_1)}$.

(b) If the outer radius is insulated and the inner radius is ~~tent~~ at temp. T_1

Answer:

$T_1 = u(r_1) = c_1 \ln r_1 + c_2 \qquad \dfrac{\partial u}{\partial r} = \dfrac{c_1}{r}$

$\Rightarrow 0 = \dfrac{c_1}{r_2} \Rightarrow c_1 = 0$

$\Rightarrow u(r) = c_2 =$ constant. and $c_2 = T_1$

$\Rightarrow u(r) = T_1$.

Problem 4: Consider $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$, $a \le r \le b$.

subject to $u(r,0) = f(r)$, $\dfrac{\partial u}{\partial r}(a,t) = \beta$, and $\dfrac{\partial u}{\partial r}(b,t) = 1$.

Using physical reasoning for what value(s) of β does an equilibrium temp. distribution exist?

Answer:

For an equilibrium temp. distribution to exist, the total energy flowing through the boundary $r=a$ has to be the same (with opposite sign) to that flowing through the boundary $r=b$. This happens if $\beta = \dfrac{b}{a}$ (see problem 3).