0914.3

Problem 5: Find the separated equations satisfied by $a(x)$ and $b(y)$ for the following PDEs:

(a) $u_{xx} - 2u_{yy} = 0$

Answer: $u_{xx} = a''(x)b(y)$, $u_{yy} = a(x)b''(y)$

$$\Rightarrow a''(x)b(y) - 2a(x)b''(y) = 0$$

$$\Rightarrow \frac{a''(x)}{a(x)} = \frac{2b''(y)}{b(y)} \Rightarrow \begin{cases} \dfrac{a''(x)}{a(x)} = \lambda \\ \dfrac{2b''(y)}{b(y)} = \lambda \end{cases} \quad \lambda = \text{constant.}$$

(b) $u_{xx} + u_{yy} + 2u_x = 0$

Answer: $\left[a''(x)b(y) + a(x)b''(y) + 2a'(x)b(y) = 0\right] / a(x)b(y)$

$$\Rightarrow \frac{a''(x)}{a(x)} + 2\frac{a'(x)}{a(x)} + \frac{b''(y)}{b(y)} = 0$$

$$\Rightarrow \begin{cases} \dfrac{a''(x)}{a(x)} + 2\dfrac{a'(x)}{a(x)} = \lambda \\ \dfrac{b''(y)}{b(y)} = -\lambda \end{cases} \quad \lambda = \text{constant.}$$

(c) $x^2 u_{xx} - 2y u_y = 0$

Answer: $x^2 a''(x)b(y) - 2y\,a(x)b'(y) = 0 \quad / a(x)b(y)$

$$x^2 \frac{a''(x)}{a(x)} - 2y\frac{b'(y)}{b(y)} = 0$$

$$\Rightarrow \begin{cases} x^2 \dfrac{a''(x)}{a(x)} = \lambda \\ 2y\dfrac{b'(y)}{b(y)} = \lambda \end{cases} \quad \lambda = \text{constant.}$$

(d) $u_{xx} + u_x + u_y - u = 0$ $\Rightarrow a''(x)b(y) + a'(x)b(y) + a(x)b'(y) - a(x)b(y) = 0$

$$\Rightarrow \frac{a''(x)}{a(x)} + \frac{a'(x)}{a(x)} + \frac{b'(y)}{b(y)} - 1 = 0 \Rightarrow \begin{cases} \dfrac{a''(x)}{a(x)} + \dfrac{a'(x)}{a(x)} = \lambda \\ \dfrac{b'(y)}{b(y)} - 1 = -\lambda \end{cases} \quad \lambda = \text{constant.}$$

Problem 7: Find the separated solution $u(x,y)$ of Laplace's equation

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < L,\ y > 0$$

that satisfies the b.c.

$$\begin{cases} u(0,y) = 0 \\ u(L,y) = 0 \end{cases}$$

and the boundedness condition $|u(x,y)| \le M$ for $y > 0$, M a constant independent of (x,y).

Answer:

In class we found the separated solutions of the Laplace's equation:

$$u(x,y) = (c_1 e^{\alpha x} + c_2 e^{-\alpha x})(c_3 \cos\alpha y + c_4 \sin\alpha y), \quad \lambda = \alpha^2 > 0$$

$$u(x,y) = (c_1 \cos\alpha x + c_2 \sin\alpha x)(c_3 e^{\alpha y} + c_4 e^{-\alpha y}), \quad \lambda = -\alpha^2 < 0$$

$$u(x,y) = (c_1 + c_2 x)(c_3 x + c_4) \qquad \lambda = 0$$

Since $u(0,y) = 0$, $u(L,y) = 0 \Rightarrow$ The dependance in x has to be sines and cosines.

$\Rightarrow \lambda = -\alpha^2 < 0$, and $u(x,y) = (c_1 \cos\alpha x + c_2 \sin\alpha x)(c_3 e^{\alpha y} + c_4 e^{-\alpha y})$

$$0 = u(0,y) = c_1 \cdot (c_3 e^{\alpha y} + c_4 e^{-\alpha y}) \Rightarrow c_1 = 0$$

$$0 = u(L,y) = c_2 \sin\alpha L (c_3 e^{\alpha y} + c_4 e^{-\alpha y}) \Rightarrow \sin\alpha L = 0$$

$$\Rightarrow \lambda = -\alpha^2 = -\left(\frac{n\pi}{L}\right)^2$$

Sines and cosines oscillate between -1 and 1 and so are bounded. However

$$\lim_{y \to +\infty} e^{\alpha y} = +\infty, \quad \lim_{y \to +\infty} e^{-\alpha y} = 0 \Rightarrow c_3 = 0$$

$$\therefore u(x,y) = c_2 \sin\frac{n\pi x}{L}\, e^{-\frac{n\pi y}{L}}$$

10914.4

Problem 8: Find the separated solutions $u(x,t)$ of the heat equation $u_t - u_{xx} = 0$, $0 \le x \le L$, $t > 0$, with b.c.:

$$\begin{cases} u(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 0. \end{cases}$$

$u = \phi(x) G(t) \Rightarrow \phi G' - \phi'' G = 0 \Rightarrow \frac{\phi''}{\phi} = \frac{G'}{G}$

$$\Rightarrow \begin{cases} \frac{\phi''}{\phi} = -\lambda \\ \frac{G'}{G} = -\lambda \end{cases} \qquad \lambda = \text{const.}$$

$\Rightarrow G(t) = c e^{-\lambda t}$.

$0 = u(0,t) = \phi(0) G(t)$
$\Rightarrow \phi(0) = 0$
$0 = \frac{\partial u}{\partial x}(L,t) = \phi'(L) G(t)$
$\Rightarrow \phi'(L) = 0$

Case 1: $\lambda = -\alpha^2 < 0$

$\Rightarrow \phi(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$$\left.\begin{array}{l} 0 = \phi(0) = c_1 + c_2 \\ 0 = \phi'(L) = c_1 \alpha - c_2 \alpha \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

$\Rightarrow \lambda = -\alpha^2 < 0$ is not an e-value.

Case 2: $\lambda = 0 \Rightarrow \phi(x) = c_1 x + c_2$

$$\left.\begin{array}{l} 0 = \phi(0) = c_2 \\ 0 = \phi'(L) = c_1 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

$\Rightarrow \lambda = 0$ is not an e-value.

Case 3: $\lambda = \alpha^2 > 0$.

$\Rightarrow \phi(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$.

$0 = \phi(0) = c_1 \Rightarrow c_1 = 0$

$0 = \phi'(L) = c_2 \alpha \cos \alpha L \Rightarrow \cos \alpha L = 0 \Rightarrow \alpha L = \frac{\pi}{2} + n\pi$, $n \ge 0$.

$\Rightarrow \lambda = \alpha^2 = \left(\frac{1}{L}\left(\frac{\pi}{2}+n\pi\right) \right)^2$

$\Rightarrow u(x,t) = c \cos \frac{1}{L}\left(\frac{\pi}{2}+n\pi\right)x \; e^{-\left(\frac{1}{L}(\pi/2+n\pi)\right)^2 t}$

Problem 9: (a) Show that $L(u) = \frac{\partial}{\partial x}\left[K_0(x) \frac{\partial u}{\partial x} \right]$ is a linear operator.

Answer: $L(c_1 u_1 + c_2 u_2) = \frac{\partial}{\partial x}\left[k_0(x) \frac{\partial}{\partial x}(c_1 u_1 + c_2 u_2) \right]$

$= \frac{\partial}{\partial x}\left[k_0(x)\left(c_1 \frac{\partial u_1}{\partial x} + c_2 \frac{\partial u_2}{\partial x} \right)\right] = \frac{\partial}{\partial x}\left[k_0 c_1 \frac{\partial u_1}{\partial x} + k_0 c_2 \frac{\partial u_2}{\partial x} \right]$

$= c_1 \frac{\partial}{\partial x}\left[k_0(x) \frac{\partial u_1}{\partial x} \right] + c_2 \frac{\partial}{\partial x}\left[k_0(x) \frac{\partial u_2}{\partial x} \right] = c_1 L(u_1) + c_2 L(u_2).$

(b) Show that usually $L(u) = \frac{\partial}{\partial x}\left[k_0(x,u) \frac{\partial u}{\partial x} \right]$ is not a linear operator.

Answer: It suffices to show a contradiction.

Define $k_0(x,u) = u$

$\Rightarrow L(u) = \frac{\partial}{\partial x}\left[u \frac{\partial u}{\partial x} \right] = \left(\frac{\partial u}{\partial x}\right)^2 + u \frac{\partial^2 u}{\partial x^2}$

If L was linear, then $L(2u) = 2L(u)$

However, $L(2u) = 4L(u)$

Therefore, L is not linear.