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A LAMPERTI-TYPE REPRESENTATION OF CONTINUOUS-STATE BRANCHING PROCESSES WITH IMMIGRATION

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Guided by the relationship between the breadth-first walk of a rooted tree and its sequence of generation sizes, we are able to include immigration in the Lamperti representation of continuous-state branching processes. We provide a representation of continuous-state branching processes with immigration by solving a random ordinary differential equation driven by a pair of independent Lévy processes. Stability of the solutions is studied and gives, in particular, limit theorems (of a type previously studied by Grimvall, Kawazu and Watanabe and by Li) and a simulation scheme for continuous-state branching processes with immigration. We further apply our stability analysis to extend Pitman's limit theorem concerning Galton–Watson processes conditioned on total population size to more general offspring laws.

1. Introduction.

1.1. *Motivation.* In this document, we extend the Lamperti representation of continuous state branching processes so that it allows immigration. First, we will see how to find discrete (and simpler) counterparts to our results in terms of the familiar Galton–Watson process with immigration and its representation using two independent random walks.

Consider a genealogical structure with immigration such as the one depicted in Figure 1. When ordering its elements in breadth-first order (with the accounting policy of numbering immigrants after the established population in each generation), χ_i will denote the number of children of individual i. Define a first version of the breadth-first walk $\tilde{x} = (\tilde{x}_i)$ by

$$\tilde{x}_0 = 0 \quad \text{and} \quad \tilde{x}_{i+1} = \tilde{x}_i + \chi_{i+1}.$$

Consider also the immigration process $y = (y_n)_{n \geq 0}$ where y_n is the quantity of immigrants arriving at generations less than or equal to n (not counting the initial

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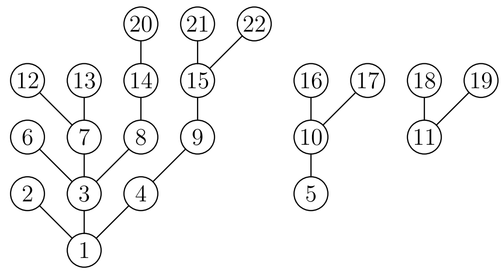

FIG. 1. *A genealogical structure allowing immigration.*

members of the population as immigrants). Finally, suppose the initial population has k members. If c_n denotes the number of individuals of generations 0 to n, c_{n+1} is obtained from c_n by adding the quantity of sons of each member of the nth generation plus the immigrants, leading to

$$c_{n+1} = c_n + (\chi_{c_{n-1}+1} + \cdots + \chi_{c_n}) + (y_{n+1} - y_n).$$

By induction we get

$$c_{n+1} = k + \tilde{x}_{c_n} + y_{n+1}.$$

Let z_n denote the number of individuals of generation n so that $z_0 = c_0 = k$ and for $n \geq 1$

$$z_n = c_n - c_{n-1};$$

if $\eta_i = \chi_i - 1$, we can define a second version of the breadth-first walk of the population by setting

$$x_0 = 0 \quad \text{and} \quad x_i = x_{i-1} + \eta_i$$

(so that $x_i = \tilde{x}_i - i$). We then obtain

$$z_{n+1} = k + x_{c_n} + y_{n+1}. \tag{1}$$

This representation of the sequence of generation sizes z in terms of the breadth-first walk x and the immigration function y can be seen as a *discrete Lamperti transformation*. It is the discrete form of the result we aim at analyzing. However, we wish to consider a random genealogical structure which is not discrete. Randomness will be captured by making the quantity of sons of individuals an i.i.d. sequence independent of the i.i.d. sequence of immigrants per generation, so that the model corresponds to a Galton–Watson with immigration. Hence x and y would become two independent random walks, whose jumps take values in $\{-1, 0, 1, \ldots\}$ and $\{0, 1, \ldots\}$, respectively. Discussion of nondiscreteness in the random genealogy model would take us far apart [we are motivated by Lévy trees with or without immigration, discussed, e.g., by Abraham and Delmas (2009), Duquesne (2009), Duquesne and Le Gall (2002), Lambert (2002)]. We only mention that continuum trees are usually defined through a continuum analogue of the depth-first walk; our point of view is that generation sizes should be obtained in terms of the continuum analogue of the breadth-first walk. Indeed, in analogy with the discrete model, we just take X and Y as independent Lévy processes, the former without negative jumps (a spectrally positive Lévy process) and the latter with increasing sample paths (a subordinator). The discrete Lamperti transformation of (1) then takes the form

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_t. \tag{2}$$

This should be the continuum version of a Galton–Watson process with immigration, namely, the continuous-state branching processes with immigration introduced by Kawazu and Watanabe (1971).

1.2. *Preliminaries.*

1.2.1. (*Possibly killed*) *Lévy processes.* A spectrally positive Lévy process (spLp) is a stochastic process $X = (X_t)_{t\geq 0}$ which starts at zero, takes values on $(-\infty, \infty]$, has independent and stationary increments, càdlàg paths, and no negative jumps. Such a process is characterized by its Laplace exponent Ψ by means of the formula

$$\mathbb{E}(e^{-\lambda X_t}) = e^{t\Psi(\lambda)},$$

where

$$\Psi(\lambda) = -\kappa + a\lambda + \frac{\sigma^2\lambda^2}{2} + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{x\leq 1})\nu(dx)$$

for $\lambda > 0$; here ν is the so-called Lévy measure on $(0, \infty)$ and satisfies

$$\int 1 \wedge x^2 \nu(dx) < \infty.$$

The constant κ will be for us the killing rate; a Lévy process with killing rate κ can be obtained from one with zero killing rate by sending the latter to ∞ at an independent exponential time of parameter κ; σ^2 is called the diffusion coefficient, while a is the drift.

We shall also make use of subordinators, which are spLp with increasing trajectories. The Laplace exponent Φ of a subordinator X is defined as the negative of its Laplace exponent as a spLp, so

$$\mathbb{E}(e^{-\lambda X_t}) = e^{-t\Phi(\lambda)}.$$

Since the Lévy measure ν of a subordinator actually satisfies

$$\int 1 \wedge x \nu(dx) < \infty,$$

and subordinators have no Brownian component (i.e., $\sigma^2 = 0$), we can write

$$\Phi(\lambda) = \kappa + d\lambda + \int (1 - e^{-\lambda x})\nu(dx).$$

So, we have the relationship

$$-d = a + \int_0^1 x\nu(dx)$$

between the parameters of X seen as a spLp and as a subordinator.

1.2.2. *Continuous-state branching processes and the Lamperti representation.* Continuous-state branching (CB) processes are the continuous time and space version of Galton–Watson processes. They were introduced in different levels of generality by Jiřina (1958), Lamperti (1967b) and Silverstein (1967/1968). They are Feller processes with state-space $[0, \infty]$ (with any metric that makes it homeomorphic to $[0, 1]$) satisfying the following branching property: the sum of two independent copies started at x and y has the law of the process started at $x + y$. The states 0 and ∞ are absorbing. The branching property can be recast by stating that the logarithm of the Laplace transform of the transition semigroup is given by a linear transformation of the initial state.

As shown by Silverstein (1967/1968), CB processes are in one to one correspondence with Laplace exponents of (killed) spectrally positive Lévy processes, which are called the branching mechanisms. In short, the logarithmic derivative of the semigroup of a CB process at zero applied to the function $x \mapsto e^{-\lambda x}$ exists and is equal to $x \mapsto x\Psi(\lambda)$. The function Ψ is the called the *branching mechanism* of the CB process and it is the Laplace exponent of a spLp. A probabilistic form of this assertion is given by Lamperti (1967a) who states that if X is a spLp with Laplace exponent Ψ, and for $x \geq 0$, we set T for its hitting time of $-x$,

$$I_t = \int_0^t \frac{1}{x + X_{s\wedge T}}\, ds$$

and C equal to its right-continuous inverse, then

$$Z_t = x + X_{C_{t\wedge T}}$$

is a CB process with branching mechanism Ψ, or CB(Ψ). This does not seem to be directly related to (2). The fact that it is related gives us what we think is the right perspective on the Lamperti transformation and the generalization considered in this work. Indeed, as previously shown in Ethier and Kurtz [(1986), Chapter 6, Section 1], Z is the only process satisfying

$$Z_t = x + X_{\int_0^t Z_s\, ds}, \tag{3}$$

which is absorbed at zero. This is (2) in the absence of immigration. To see that a process satisfying (3) can be obtained as the Lamperti transform of X, note that if

$C_t = \int_0^t Z_s\,ds$, then while Z has not reached zero, C is strictly increasing so that it has an inverse, say I, whose right-hand derivative I'_+ is given by

$$I'_+(t) = \frac{1}{C'_+(I_t)} = \frac{1}{Z_{I_t}} = \frac{1}{x + X_{C\circ I(t)}} = \frac{1}{x + X_t}.$$

1.2.3. *Continuous-state branching processes with immigration.* Continuous-state branching processes with immigration (or CBI processes) are the continuous time and space version of Galton–Watson processes with immigration and were introduced by Kawazu and Watanabe (1971). They are Feller processes with state-space $[0, \infty]$ such that the logarithm of the Laplace of the transition semigroup is given by an affine transformation of the initial state. [They thus form part of the affine processes studied by Dawson and Li (2006).] As shown by Kawazu and Watanabe (1971), they are characterized by the Laplace exponents of a spLp and of a subordinator: the logarithmic derivative of the semigroup of a CB process at zero applied to the function $x \mapsto e^{-\lambda x}$ exists and is equal to the function

$$x \mapsto x\Psi(\lambda) - \Phi(\lambda),$$

where Ψ is the Laplace exponent of a spLp and Φ is the Laplace exponent of a subordinator. They are, respectively, called the *branching and immigration mechanisms* and characterize the process which is therefore named CBI(Ψ, Φ).

We aim at a probabilistic representation of CBI processes in the spirit of the Lamperti representation.

1.3. *Statement of the results.* We propose to construct a CBI(Ψ, Φ) that starts at x by solving the functional equation

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_t. \tag{4}$$

We call such a process Z the *Lamperti transform* of $(X, x + Y)$ and denote it by $Z = L(X, x + Y)$; however, the first thing to do is to show that there exists a unique process which satisfies (4). When Y is zero, a particular solution to (4) is the Lamperti transform of $X + x$ recalled above. Even in this case there could be many solutions to (4), in clear contrast to the discrete case where one can proceed recursively to construct the unique solution. Our stepping stone for the general analysis of (4) is the following partial result concerning existence and uniqueness proved in Section 2.

A pair of càdlàg functions (f, g) such that f has no negative jumps, g is non-decreasing and $f(0) + g(0) \geq 0$ is termed an *admissible breadth-first pair*; f and g will be termed the *reproduction and immigration functions*, respectively. When g is constant, we say that $f + g$ is *absorbed at zero* if $f(x) + g = 0$ implies $f(y) + g = 0$ for all $y > x$.

THEOREM 1. *Let (f, g) be an admissible breadth-first pair. There exists a nonnegative h satisfying the equation*

$$h(t) = f\left(\int_0^t h(s)\,ds\right) + g(t).$$

Furthermore, the solution is unique when g is strictly increasing, when $f + g(0)$ is a strictly positive function or when g is constant and $f + g$ is absorbed at zero.

In the context of Theorem 1, much is gained by introducing the function c given by

$$c(t) = \int_0^t h(s)\,ds,$$

which has a right-hand derivative c'_+ equal to h. This is because the functional equation for h can then be recast as the initial value problem

$$\mathrm{IVP}(f, g) = \begin{cases} c'_+ = f \circ c + g, \\ c(0) = 0. \end{cases}$$

Our forthcoming approximation results for the function h of Theorem 1 rely on the study of a functional inequality. Let (f, g) be an admissible breadth-first pair. We will be interested in functions c which satisfy

$$\int_s^t f_- \circ c(r) + g(r)\,dr \le c(t) - c(s) \le \int_s^t f \circ c(r) + g(r)\,dr \quad \text{for } s \le t. \tag{5}$$

Note that any solution c to IVP(f, g) satisfies (5): the second inequality is actually an equality by definition of IVP(f, g), and since $f \ge f_-$ as f has no negative jumps, we get the first inequality. Hence, the functional inequality (5) admits solutions. Regarding uniqueness, if the solution to (5) is unique, then the solution to IVP(f, g) is unique, and since the latter is nonnegative and nondecreasing, so is the former. Also, similar sufficient conditions for uniqueness of IVP(f, g) of Theorem 1 imply uniqueness of nondecreasing solutions of the functional inequality (5).

PROPOSITION 1. *Let (f, g) be an admissible breadth-first pair. If either g is strictly increasing, $f_- + g(0)$ is strictly positive or g is constant and $f_- + g(0)$ is absorbed at zero, then* (5) *has an unique nondecreasing solution starting at zero.*

However, as is shown in Section 4.1, assuming that (5) admits an unique solution is stronger than just assuming that IVP(f, g) has an unique solution.

As a consequence of the analytic Theorem 1, we solve a probabilistic question raised by Lambert (1999, 2007).

COROLLARY 1. *Let X be a spectrally positive α-stable Lévy process. For any càdlàg and strictly increasing process Y independent of X, there is weak existence and uniqueness for the stochastic differential equation*

$$Z_t = x + \int_0^t |Z_s|^{1/\alpha}\, dX_s + Y_t. \tag{6}$$

When X is twice a Brownian motion and $Y_t = \delta t$ for some $\delta > 0$, this might be one of the simplest proofs available of weak existence and uniqueness of the SDE defining squared Bessel processes, since it makes no mention of the Tanaka formula or local times; it is based on Knight's theorem and Theorem 1. When X is a Brownian motion and $dY_t = b(t)\, dt$ for some Lipschitz and deterministic $b : [0, \infty) \to [0, \infty)$, Le Gall (1983) actually proves pathwise uniqueness through a local time argument. Our result further shows that if b is measurable and strictly positive, then there is weak uniqueness. In the case Y is an $(\alpha - 1)$-stable subordinator independent of X, we quote Lambert (1999, 2007):

> *... whether or not uniqueness holds for* (6) *remains an open question.*

Corollary 1 answers affirmatively. Note that when $Y = 0$, the stated result follows from Zanzotto (2002), and is handled by a time-change akin to the Lamperti transformation. Fu and Li (2010) obtain strong existence and pathwise uniqueness for a different kind of SDE related to CBI processes with stable reproduction and immigration.

Regarding solutions to (4), Theorem 1 is enough to obtain the process Z when the subordinator Y is strictly increasing. When Y is compound Poisson, a solution to (4) can be obtained by pasting together Lamperti transforms. However, further analysis using the pathwise behavior of X when Y is zero or compound Poisson implies the following result.

PROPOSITION 2. *Let $x \geq 0$, X be a spectrally positive Lévy process and Y an independent subordinator. Then there is a unique càdlàg process Z which satisfies*

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t.$$

The above equation is satisfied by any càdlàg process Z satisfying the functional inequality

$$x + X_{\int_0^t Z_s\, ds-} + Y_t \leq Z_t \leq x + X_{\int_0^t Z_s\, ds} + Y_t,$$

which also has a unique solution.

Our main result, a pathwise construction of a $\mathrm{CBI}(\Psi, \Phi)$, is the following.

THEOREM 2. *Let X be a spectrally positive Lévy process with Laplace exponent Ψ and Y an independent subordinator with Laplace exponent Φ. The unique stochastic process Z which solves*

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t$$

is a CBI(Ψ, Φ) *that starts at* x.

We view Theorems 1 and 2 as a first step in the construction of branching processes with immigration where the immigration can depend on the current value of the population. One generalization would be to consider solutions to

$$Z_t = x + X_{\int_0^t a(s,Z_s)\,ds} + Y_{\int_0^t b(s,Z_s)\,ds},$$

where a is interpreted as the breeding rate, and b as the rate at which the arriving immigration is incorporated into the population. For example, Abraham and Delmas (2009) consider a continuous branching process where immigration is proportional to the current state of the population. This could be modeled by the equation

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_{\int_0^t \alpha Z_s\,ds},$$

which, thanks to the particular case of Theorem 2 stated by Lamperti (1967a), has the law of a CB$(\Psi - \alpha\Phi)$ started at x; this is the conclusion of Abraham and Delmas (2009), where they rigorously define the model in terms of a Poissonian construction of a more general class of CBI processes which is inspired in previous work of Pitman and Yor (1982) for CBIs with continuous sample paths. Another representation of CBI processes, this time in terms of solutions to stochastic differential equations was given by Dawson and Li (2006) under moment conditions.

The usefulness of Theorem 2 is two-fold: first, we can use known sample path properties of X and Y to deduce sample-path properties of Z, and second, this representation gives a particular coupling with monotonicity properties which are useful in limit theorems involving Z, as seen in Corollaries 6, 7 and Theorem 4. Simple applications of Theorem 2 include the following.

COROLLARY 2 [Kawazu and Watanabe (1971)]. *If* Ψ *is the Laplace exponent of a spectrally positive Lévy process, and* Φ *is the Laplace exponent of a subordinator, there exists a CBI process with branching mechanism* Ψ *and immigration mechanism* Φ.

COROLLARY 3. *A* CBI(Ψ, Φ) *process does not jump downward.*

Caballero, Lambert and Uribe Bravo (2009) give a direct proof of this when $\Phi = 0$.

COROLLARY 4. *Let* Z *be a* CBI(Ψ, Φ) *that starts at* $x > 0$, *let* $\tilde{\Phi}$ *be the right-continuous inverse of* Ψ, *and define*

$$\alpha(t) = \frac{\log|\log t|}{\tilde{\Phi}(t^{-1}\log|\log t|)}.$$

There exists a constant ζ (*in general nonzero*) *such that*

$$\liminf_{t\to 0} \frac{Z_t - x}{\alpha(xt)} = \zeta.$$

The case $x = 0$ in Corollary 4 is probably very different, as seen when $\Psi(\lambda) = 2\lambda^2$ and $\Phi(\lambda) = d\lambda$, which corresponds to the squared Bessel process of dimension d. Indeed, Itô and McKean [(1974), page 80] show that for a squared Bessel process Z of integer dimension that starts at 0, we have

$$\limsup_{t\to 0} \frac{Z_t}{2t \log|\log t|} = 1.$$

We have not been able to obtain this result using the Lamperti transformation. However, note that starting from positive states, we can obtain the lower growth rate, since it is the reproduction function X that determines it, while starting from 0, it is probably a combination of the local growth of X and Y that drives that of Z.

A solution c to IVP(f, g) is said to *explode* if there exists $t \in (0, \infty)$ such that $c(t) = \infty$. (Demographic) explosion is an unavoidable phenomena of IVP(f, g). When $f > 0$ and $g = 0$, it is known that explosion occurs if and only if

$$\int^{\infty} \frac{1}{f(x)}\, dx < \infty.$$

Actually, even when there is immigration, the main function responsible for explosion is the reproduction function.

PROPOSITION 3. *Let (f, g) be an admissible pair, and let $f^+ = \max(f, 0)$.*

(1) *If $\int^\infty 1/f^+(x)\, dx = \infty$, then no solution to* IVP(f, g) *explodes.*

(2) *If $\int^\infty 1/f^+(x)\, dx < \infty$, $\lim_{x\to\infty} f(x) = \infty$ and $g(\infty)$ exceeds the maximum of $-f$, then any solution to* IVP(f, g) *explodes.*

We call f an *explosive reproduction function* if

$$\int^{\infty} \frac{1}{f^+(x)}\, dx < \infty.$$

Recall that ∞ is an absorbing state for CBI processes; Proposition 3 has immediate implications on how a CBI process might reach it. First of all, CBI processes might jump to ∞, which happens if and only if either the branching or the immigration corresponds to killed Lévy processes. When there is no immigration and the branching mechanism Ψ has no killing rate, the criterion is due to Ogura (1969/1970) and Grey (1974), who assert that the probability that a CB(Ψ) started from $x > 0$ is absorbed at infinity in finite time is positive if and only if

$$\int_{0+} \frac{1}{\Psi(\lambda)}\, d\lambda > -\infty.$$

One can even obtain a formula for the distribution of its explosion time; cf. the proof of Theorem 2.2.3.2 in Lambert (2008), page 95. We call such Ψ an *explosive branching mechanism*. From Proposition 3 and Theorem 2 we get:

COROLLARY 5. *Let $x > 0$.*

(1) *The probability that a* CBI(Ψ, Φ) Z *that starts at* x *jumps to* ∞ *is positive if and only if* $\Psi(0)$ *or* $\Phi(0)$ *are nonzero.*

(2) *The probability that* Z *reaches* ∞ *continuously is positive if and only if* $\Psi(0) = 0$ *and* Ψ *is an explosive branching mechanism.*

(3) *The probability that* Z *reaches* ∞ *continuously is equal to* 1 *if* $\Psi(0) = \Phi(0) = 0$, Φ *is not zero and* Ψ *is explosive.*

We mainly use stochastic integration by parts in our proof of Theorem 2; however, a weak convergence type of proof, following the case $\Phi = 0$ presented by Caballero, Lambert and Uribe Bravo (2009), could also be achieved in conjunction with a stability result, based on the forthcoming Theorem 3.

The following result deals with stability of IVP(f, g) under changes in f and g and even includes a discretization of the initial value problem, itself. Indeed, consider the following approximation procedure: given $\sigma > 0$, called the *span*, consider the partition

$$t_i = i\sigma, \qquad i = 0, 1, 2, \dots,$$

and construct a function c^σ by the recursion

$$c^\sigma(0) = 0$$

and for $t \in [t_{i-1}, t_i)$,

$$c^\sigma(t) = c^\sigma(t_{i-1}) + (t - t_{i-1})\big[f \circ c^\sigma(t_{i-1}) + g(t_{i-1})\big]^+.$$

Equivalently, the function c^σ is the unique solution to the equation

$$\mathrm{IVP}_\sigma(f, g) : c^\sigma(t) = \int_0^t \big[f \circ c^\sigma\big(\lfloor s/\sigma \rfloor \sigma\big) + g\big(\lfloor s/\sigma \rfloor \sigma\big)\big]^+ \, ds.$$

We will write IVP$_0(f, g)$ to mean IVP(f, g). Let D_+ denote the right-hand derivative.

The stability result is stated in terms of the usual *Skorohod* J_1 *topology* for càdlàg functions: a sequence f_n converges to f if there exist a sequence of homeomorphisms of $[0, \infty)$ into itself such that

$$f_n - f \circ \lambda_n \quad \text{and} \quad \lambda_n - \mathrm{Id} \qquad \text{converge to zero uniformly on compact sets}$$

(where Id denotes the identity function on $[0, \infty)$). However, part of the theorem uses another topology on nonnegative càdlàg functions introduced by Caballero, Lambert and Uribe Bravo (2009), which we propose to call the *uniform* J_1 *topology*. Consider a distance d on $[0, \infty]$ which makes it homeomorphic to $[0, 1]$. Then the uniform J_1 topology is characterized by the following: a sequence f_n converges to f if there exist a sequence of homeomorphisms of $[0, \infty)$ into itself such that

$$d(f_n, f \circ \lambda_n) \to 0 \quad \text{and} \quad \lambda_n - \mathrm{Id} \to 0 \qquad \text{uniformly on } [0, \infty).$$

THEOREM 3. *Let (f, g) be an admissible breadth-first pair and suppose there is a unique nondecreasing function c which satisfies $c(0) = 0$ and* (5) [*and is therefore the unique solution to* $\mathrm{IVP}(f, g)$]; *define its explosion time by*

$$\tau = \inf\{t \geq 0 : c(t) = \infty\} \in (0, \infty].$$

Let (f_n, g_n) be admissible breadth-first pairs. Suppose $f_n \to f$ and $g_n \to g$ in the Skorohod J_1 topology and that σ_n is a sequence of nonnegarive real numbers which tend to zero. Let c_n be the unique solution to $\mathrm{IVP}_{\sigma_n}(f_n, g_n)$ *when $\sigma_n > 0$ and any solution to* $\mathrm{IVP}(f_n, g_n)$ *when $\sigma_n = 0$. Then $c_n \to c$ pointwise and uniformly on compact sets of* $[0, \tau)$.

Furthermore, if $f \circ c$ and g do not jump at the same time, then $D_+ c_n \to D_+ c$:

(1) *in the Skorohod J_1 topology if $\tau = \infty$, and*

(2) *in the uniform J_1 topology if $\tau < \infty$ if we additionally assume that $f_n(x), f(x) \to \infty$ as $x \to \infty$ uniformly in n.*

It is not very hard to show that the jumping condition of Theorem 3 holds in a stochastic setting.

PROPOSITION 4. *Let X be a spLp, Y an independent subordinator with Laplace exponents Ψ and Φ and, for $x \geq 0$, let Z the unique process such that*

$$Z_t = x + X_{C_t} + Y_t \qquad \text{where } C_t = \int_0^t Z_s \, ds.$$

Almost surely, the processes $X \circ C$ and Y do not jump at the same time.

From Theorem 3 and Propositions 2 and 4, we deduce the following weak continuity result.

COROLLARY 6. *Let Ψ_n, Ψ be Laplace exponents of spLps and Φ_n, Φ be Laplace exponents of subordinators and suppose that $\Psi_n \to \Psi$ and $\Phi_n \to \Phi$ pointwise. If (x_n) is a sequence in* $[0, \infty]$ *converging to x and Z_n* (*resp.*, Z) *are CBIs with branching and immigration mechanisms Ψ_n and Φ_n* (*resp.*, Ψ *and* Φ) *and starting at x_n* (*resp.*, x) *then $Z_n \to Z$ in the Skorohod J_1 topology on càdlàg paths on* $[0, \infty]$ *if Ψ is nonexplosive and in the uniform J_1 topology if Ψ is explosive.*

Theorem 3 also allows us to simulate CBI processes. Indeed, if we can simulate random variables with distribution X_t and Y_t for every $t > 0$, we can then approximately simulate the process Z as the right-hand derivative of the solution to $\mathrm{IVP}_\sigma(X, x + Y)$. (Alternatively, if we can approximate X and Y, e.g., by compound Poisson processes with drift, we can also apply IVP_σ to approximate the paths of Z.) The procedure $\mathrm{IVP}_\sigma(X, x + Y)$ actually corresponds to an Euler method of span σ to solve IVP$(X, x+Y)$. Theorem 3 implies the convergence of the Euler method as the span goes to zero when applied to IVP$(X, x+Y)$, even with the discontinuous driving functions X and Y!

We also give an application of Theorem 3 to limits of Galton–Watson processes with immigration. Let X^n and Y^n be independent random walks with step distributions μ_n and ν_n supported on $\{-1, 0, 1, \ldots\}$ and $\{0, 1, 2, \ldots\}$, and for any $k_n \geq 0$, define recursively the sequences C^n and Z^n by setting

$$C_0^n = Z_0^n = k_n, \qquad Z_{m+1}^n = k_n + X_{C_m^n}^n + Y_{m+1}^n \quad \text{and} \quad C_{m+1}^n = C_m^n + Z_{m+1}^n.$$

As discussed in Section 1.1, the sequence Z^n is a Galton–Watson process with immigration with offspring and immigration distributions μ_n and ν_n. However, if X^n and Y^n are extended by constancy on $[m, m+1)$ for $m \geq 0$ (keeping the same notation), then C^n is the approximation of the Lamperti transformation with span 1 applied to X^n and Y^n and Z^n is the right-hand derivative of C^n. In order to apply Theorem 3 to these processes, define the scaling operators S_a^b by

$$S_a^b f(t) = \frac{1}{b} f(at).$$

COROLLARY 7. *Suppose the existence of sequences a_n, b_n such that*

$$X_{a_n}^n/n \quad \textit{and} \quad Y_{b_n}^n/n$$

converge weakly to the infinitely divisible distributions μ and ν corresponding to a spectrally positive Lévy process and a subordinator; denote by Ψ and Φ their Laplace exponents. Suppose that $b_n \to \infty$ and, for any $\alpha > 0$, $a_{\lfloor \alpha n \rfloor}/n \to \infty$. Let $k_n \to \infty$, and suppose that either

$$\frac{k_n b_{\lfloor k_n/x \rfloor}}{x a_{\lfloor k_n/x \rfloor}} \to c \in [0, \infty) \quad \textit{or} \quad \frac{x a_{\lfloor k_n/x \rfloor}}{k_n b_{\lfloor k_n/x \rfloor}} \to 0$$

as $n \to \infty$. Setting $e_n = b_{\lfloor k_n/x \rfloor}$ in the first case and $e_n = x a_{\lfloor k_n/x \rfloor}/k_n$ in the second, we have that

$$S_{e_n}^{k_n/x} Z^n$$

converges in distribution, toward a CBI$(c\Psi, \Phi)$ *in the first case and toward a* CB(Ψ) *in the second. The convergence takes place in the Skorohod J_1 topology if Ψ is nonexplosive and in the uniform J_1 topology, otherwise.*

When Ψ is nonexplosive and $\Phi = 0$, the above theorem was proved by Grimvall (1974). He also proved the convergence of finite-dimensional distributions in the explosive case, which we complement with a limit theorem. For general Φ, but nonexplosive Ψ, a similar result was proven by Li (2006). However, as will be seen in the proof (which relies on the stability of the Lamperti transformation stated in Theorem 3), if the convergence of $S_{a_n}^n X^n$ and $S_{b_n}^n Y^n$ takes place almost surely, then $S_{e_n}^n Z^n$ also converges almost surely.

The stability result of Theorem 3 applies not only in the Markovian case of CBI processes. As an example, we generalize work of Pitman (1999) who considers the scaling limits of conditioned Galton–Watson processes in the case of the Poisson offspring distribution. Let μ be an offspring distribution with mean 1 and suppose that Z^n is a Galton–Watson process started at k_n and conditioned on

$$\sum_{i=0}^{\infty} Z_i^n = n.$$

We shall consider the scaling limit of Z^n as $n \to \infty$ whenever the shifted reproduction law $\tilde{\mu}_k = \mu_{k+1}$ is in the domain of attraction of a stable law without the need of centering. The scaling limit of a random walk with step distribution $\tilde{\mu}$ is then a spectrally positive stable law of index $\alpha \in (1, 2]$ with which one can define, for every $l > 0$ the first passage bridge F^l starting at l and ending at 0 of length 1 of the associated Lévy process. Informally this is the stable process started at l, conditioned to be above 0 on [0, 1] and conditioned to end at 0 at time 1. This intuitive notion was formalized by Chaumont and Pardo (2009). The Lamperti transform of F^l will be the right-hand derivative of the unique solution to IVP$(F^l, 0)$.

THEOREM 4. *Let Z^n be a Galton–Watson process with critical offspring law μ which starts at k_n and is conditioned on $\sum_{i=1}^{\infty} Z_i^n = n$. Let S be a random walk with step distribution μ and suppose there exist constants $a_n \to \infty$ such that $(S_n - n)/a_n$ converges in law to a spectrally positive stable distribution with Laplace exponent Ψ. Let X be a Lévy process with Laplace exponent Ψ and F^l its first passage bridge from $l > 0$ to 0 of length 1. If $k_n/a_n \to l$, then the sequence*

$$S_{n/a_n}^{a_n} Z^n$$

converges in law to the Lamperti transform of F^l in the Skorohod J_1 topology.

When $\alpha = 2$, the process F^l is a Bessel bridge of dimension 3 between l and 0 of length 1, up to a normalization factor. In this case, Pitman [(1999), Lemma 14] tells us that the Lamperti transform Z^l of F^l satisfies the SDE

$$\begin{cases} dZ_v^l = 2\sqrt{Z_v^l}\, dB_v + \left[4 - \dfrac{(Z_v^l)^2}{1 - \int_0^v Z_u^l\, du}\right] dv, \\ Z_0^l = l, \end{cases}$$

driven by a Brownian motion B, and it is through stability theory for SDEs that Pitman (1999) obtains Theorem 4 when μ is a Poisson distribution with mean 1. Theorem 4 is a complement to the convergence of Galton–Watson forests conditioned on their total size and number of trees given in Chaumont and Pardo (2009). When $l = 0$, our techniques cease to work. Indeed, the corresponding process F^0 would be a normalized Brownian excursion above zero, and the problem

IVP($F^0, 0$) does not have a unique solution, as discussed at the beginning of Section 2. Hence, even if our techniques yield tightness in the corresponding limit theorem with $l = 0$, we would have to give further arguments to prove that any subsequential limit is the correct solution IVP($F^0, 0$). The limit theorem when $l = 0$ and $\alpha = 2$ was conjectured by Aldous (1991), and proved by Drmota and Gittenberger (1997) by analytic methods. For any $\alpha \in (1, 2]$, the corresponding statement was stated and proved by Kersting (1998) by working with the usual Lamperti transformation, which chooses a particular solution to IVP($F^0, 0$).

The paper is organized as follows. Theorem 1, Proposition 2 and Corollary 1 are proved in Section 2 which focuses on the analytic aspects of the Lamperti transformation and its basic probabilistic implications. The representation CBI processes of Theorem 2 is then proved in Section 3, together with Proposition 4, Corollaries 4 and 5. Finally, Section 4 is devoted to the stability of the Lamperti transformation with a proof of Theorem 3, Proposition 1, Corollaries 6, 7 and Theorem 4. (Corollaries 2 and 3 are considered to follow immediately from Theorem 2; proofs have been omitted.)

2. The generalized Lamperti transformation as an initial value problem. Let (f, g) be an admissible breadth-first pair, meaning that f and g are càdlàg functions with g increasing, f without negative jumps and $f(0) + g(0) \geq 0$. We begin by studying the existence of a nonnegative càdlàg function h which satisfies

$$h(t) = f\left(\int_0^t h(s)\, ds\right) + g(t); \tag{7}$$

a priori there might be many solutions.

When g is identically equal to zero, a solution is found by the method of time-changes: let τ be the first hitting time of zero by f, let

$$i_t = \int_0^t \frac{1}{f(s \wedge \tau)}\, ds$$

and consider its right-continuous inverse c so that

$$h = f \circ c$$

satisfies (7) with $g = 0$, and it is the only solution for which zero is absorbing. A generalization of this argument is found in Ethier and Kurtz (1986), Chapter 6, Section 1. In this case the transformation which takes f to h is called the Lamperti transformation, introduced by Lamperti (1967a). There is a slight catch: if f is never zero and goes to infinity, then h exists up to a given time (which might be infinite) when it also goes to infinity. After this time, which we call the explosion time, we set $h = \infty$. With this definition, note that c and h become infinite at the same time.

Solutions to (7) are not unique even when $g = 0$ as the next example shows: take $f(x) = \sqrt{|1-x|}$, $l > 0$, and consider

$$h_1(t) = \frac{(2-t)^+}{2} \quad \text{and} \quad h_2(t) = \begin{cases} \dfrac{2-t}{2}, & \text{if } t \le 2, \\ 0, & \text{if } 2 \le t \le 2+l, \\ \dfrac{t-2-l}{2}, & \text{if } t \ge 2+l. \end{cases}$$

Then h_1 and h_2 are both solutions to (7). As we discussed in the Introduction, a probabilistically relevant example of nonuniqueness is obtained when $g = 0$ and f is the typical sample path of a normalized Brownian excursion $e = (e_t, t \ge 1)$. [See Chapter 11, Section 3 of Revuz and Yor (1999) for its definition as a 3-dimensional Bessel bridge.] Indeed, with probability 1, e has a continuous trajectory which is positive exactly on $(0,1)$. Hence, 0 is a solution to IVP$(e, 0)$. However, its link with the 3-dimensional Bessel process (and time reversal) allows one to prove that $\sqrt{s} = o(e_s)$ as $s \to 0+$ (and a corresponding statement as $s \to 1-$) so that almost surely

$$\int_0^1 \frac{1}{e_s}\, ds < \infty.$$

Hence, one can define the Lamperti transform of e, which is a nontrivial solution to IVP$(e, 0)$. The Lamperti transformation is well defined under more general excursion laws as discussed by Miermont (2003).

We propose to prove Theorem 1 by the following method: we first use the solution for the case $g = 0$ to establish the theorem when g is piecewise constant. When g is strictly increasing, we approximate it by a strictly decreasing sequence of piecewise constant functions $g_n > g$ and let h_n be the solution to (7) which uses g_n. We then consider the primitive c_n of h_n starting at zero, show that it converges, and this is enough to prove the existence of a function whose right-continuous derivative exists and solves (7). Actually, it is by using primitives that one can compare the different solutions to (7) (and study uniqueness), and this is the point of view adopted in what follows. To this end, we generalize (7) into an initial value problem for the function c.

$$\text{IVP}(f, g, x) = \begin{cases} c'_+(t) = f \circ c(t) + g(t), \\ c(0) = x. \end{cases}$$

[The most important case for us is $x = 0$, and we will write IVP(f, g) when referring to it.] We shall term:

- f the *reproduction function*,
- g the *immigration function*,
- x the *initial cumulative population*,
- c the *cumulative population*, and
- c'_+ the *population profile*.

- A solution c to $\mathrm{IVP}(f, g, x)$ is said to have *no spontaneous generation* if the condition $c'_+(t) = 0$ implies that $c(t+s) = c(t)$ as long as $g(t+s) = g(t)$.

In the setting of Theorem 1, spontaneous generation is only relevant when g is piecewise constant, and it will be the guiding principle to chose solutions in this case.

A solution to $\mathrm{IVP}(f, g, x)$ without spontaneous generation when g is a constant γ is obtained by setting $f_x(s) = f(x+s) + \gamma$, calling h_x the Lamperti transform of f_x and setting

$$c_t = x + \int_0^t h_x(s)\, ds.$$

We then have

$$c'_+(t) = h_x(t) = f_x\left(\int_0^t h_x(s)\, ds\right) = f\left(x + \int_0^t h_x(s)\, ds\right) + \gamma = f\bigl(c(t)\bigr) + g(t).$$

Let g be piecewise constant, say

$$g = \sum_{i=1}^n \gamma_i \mathbf{1}_{[t_{i-1}, t_i)}$$

with $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $0 = t_0 < t_1 < \cdots < t_n$. Let us solve (7) by pasting the solutions on each interval: let ψ_1 solve $\mathrm{IVP}(f, \gamma_1, 0)$ on $[0, t_1]$ without spontaneous generation. Let c equal ψ_1 on $[0, t_1]$. Now, let ψ_2 solve $\mathrm{IVP}(f, \gamma_2, c(t_1))$ without spontaneous generation. [If $c(t_1) = \infty$, we set $\psi_2 = \infty$.] Set $c(t) = \psi_2(t - t_1)$ for $t \in [t_1, t_2]$ so that c is continuous. Also, for $t \in [t_1, t_2]$, we have

$$c'_+(t) = \psi'_{2+}(t - t_1) = f\bigl(\psi_2(t - t_1)\bigr) + \gamma_2 = f\bigl(c(t)\bigr) + g(t).$$

We continue in this manner. Note that if c'_+ reaches zero in $[t_{i-1}, t_i)$, say at t, then c is constant on $[t, t_i)$ and that c'_+ solves (7) when g is piecewise constant. By uniqueness of solutions to (7) which are absorbing at zero when $g = 0$, we deduce the uniqueness of solutions to $\mathrm{IVP}(f, g, 0)$ without spontaneous generation when the immigration is piecewise constant.

We first tackle the nonnegativity assertion of Theorem 1. Since f is only defined on $[0, \infty)$, negative values of c do not make sense in equation (7). One possible solution is to extend f to $\mathbb{R}$ by setting $f(x) = f(0)$ for $x \leq 0$.

LEMMA 1. *Any solution h to (7) is nonnegative.*

PROOF. Let h solve (7) where f is extended by constancy on $(-\infty, 0]$, and define

$$c(t) = \int_0^t h(s)\, ds,$$

so that c solves IVP(f, g). We prove that $h \geq 0$ by contradiction. Assume there exists $t \geq 0$ such that $h(t) < 0$. Note that since h has no negative jumps, h can only reach negative values continuously, and, since h is right-continuous, if it is negative at a given t, then there exists $t' > t$ such that h is negative on $[t, t')$. Hence there exists $\varepsilon > 0$ such that

$$\{t \geq 0 : h(t) = 0 \text{ and } h < 0 \text{ on } (t, t+\varepsilon)\} \neq \varnothing.$$

Let τ be its infimum. We assert that $\tau > 0$ and $c(\tau) > 0$. Indeed, if $\tau = 0$, then c would be strictly decreasing and negative on $(0, \varepsilon)$, which would imply that

$$h(t) = f \circ c(t) + g(t) = f(0) + g(t) \geq f(0) + g(0) \geq 0 \qquad \text{for } t \in (0, \varepsilon),$$

a contradiction. A similar argument tells us that $c(\tau) > 0$. We finish the proof by showing the existence of $t_1 \leq \tau$ and $t_2 \in (\tau, \tau + \varepsilon)$ such that $h(t_1) > 0$ and $c(t_1) = c(t_2)$, implying the contradiction

$$0 < h(t_1) = f \circ c(t_1) + g(t_1) = f \circ c(t_2) + g(t_1) \leq f \circ c(t_2) + g(t_2) \leq 0.$$

Indeed, given that $c(\tau) > 0$ we can assume that $c(\tau + \varepsilon) > 0$ by choosing a smaller ε, and then let τ_1 be the last time before τ that c is below $c(\tau + \varepsilon)$ and τ_2 the first instant after τ_1 that c equals $c(\tau)$. Note that $\tau_2 \leq \tau$. Since

$$\int_{\tau_1}^{\tau_2} h(r)\, dr = c(\tau_2) - c(\tau_1) = c(\tau) - c(\tau + \varepsilon) > 0,$$

there exists $r \in (\tau_1, \tau_2)$ such that $h(r) > 0$ and by construction $c(r) \in c((\tau, \tau + \varepsilon))$. □

2.1. *Monotonicity and existence.* We now establish a basic comparison lemma for solutions to IVP(f, g) which will lead to the existence assertion of Theorem 1.

LEMMA 2. *Let c and $\tilde{c}$ solve* IVP(f, g) *and* IVP$(\tilde{f}, \tilde{g})$. *If*

$$g(0) + f(0) < \tilde{g}(0) + \tilde{f}(0), \qquad f \leq \tilde{f}, g \leq \tilde{g}$$

and either $g_- < \tilde{g}_-$ or $f_- < \tilde{f}_-$, then $c_t < \tilde{c}_t$ for every t that is strictly positive and strictly smaller than the explosion time of c.

It is important to note that the inequality $c \leq \tilde{c}$ cannot be obtained from the hypothesis $g \leq \tilde{g}$ using the same reproduction function f. Indeed, we would otherwise have uniqueness for IVP(f, g) which, as we have seen, is not the case even when $g = 0$. Also, since both c and $\tilde{c}$ begin at 0 and equal ∞ after their explosion time, we always have the inequality $c \leq \tilde{c}$ under the conditions of Lemma 2.

PROOF OF LEMMA 2. Let $\tau = \inf\{t > 0 : c(t) = \tilde{c}(t)\}$. Since

$$c'_+(0) = f(0) + g(0) < f(0) + \tilde{g}(0) = \tilde{c}'_+(0),$$

and the right-hand derivatives of c and $\tilde{c}$ are right-continuous, then $\tau > 0$ and $c < \tilde{c}$ on $(0, \tau)$. Note then that the explosion time of c cannot be smaller than τ, since this would force $\tilde{c}$ to explode before τ and so c would equal $\tilde{c}$ before τ.

We now argue by contradiction. If τ were finite, we know that

$$c(\tau) = \tilde{c}(\tau),$$

leaving us with two cases,

$$c(\tau) = \tilde{c}(\tau) = \infty \quad \text{and} \quad c(\tau) = \tilde{c}(\tau) < \infty.$$

In the former, we see that τ is the explosion time of c and so the statement of Lemma 2 holds. In the latter case,

$$\begin{aligned} c'_-(\tau) &= f\bigl(c(\tau)-\bigr) + g(\tau-) = f\bigl(\tilde{c}(\tau)-\bigr) + g(\tau-) \\ &< \tilde{f}\bigl(\tilde{c}(\tau)-\bigr) + \tilde{g}(\tau-) = \tilde{c}'_-(\tau). \end{aligned}$$

It follows that $c'_- < \tilde{c}'_-$ in some interval $(\tau - \varepsilon, \tau)$. However, for $0 < t < \tau$, we have $c(t) < \tilde{c}(t)$, and this implies the contradiction

$$c(\tau) < \tilde{c}(\tau). \qquad \square$$

PROOF OF THEOREM 1, EXISTENCE. Consider a sequence of piecewise constant càdlàg functions g_n satisfying $g_{n+1}(0) < g_n(0)$, $g_{n+1-} < g_{n-}$ and such that $g_n \to g$ pointwise. Let c_n solve IVP(f, g_n) with no spontaneous generation. By Lemma 2, the sequence of nonnegative functions c_n is decreasing, so that it converges to a limit c. Let

$$\tau = \inf\{t \geq 0 : c(t) = \infty\} = \liminf_{n\to\infty}\{t \geq 0 : c_n(t) = \infty\}.$$

Since f is right-continuous and $c < c_n$, $f \circ c_n + g_n$ converges pointwise to $f \circ c + g$ on $[0, \tau)$. By bounded convergence, for $t \in [0, \tau)$,

$$c(t) = \lim_{n\to\infty} c_n(t) = \lim_{n\to\infty} \int_0^t f \circ c_n(s) + g_n(s)\, ds = \int_0^t f \circ c(s) + g(s)\, ds.$$

Hence, $h = c'_+$ proves the existence part of Theorem 1. $\square$

2.2. *Uniqueness.* To study uniqueness of IVP(f, g), we use the following lemma.

LEMMA 3. *If g is strictly increasing, and c solves* IVP(f, g), *then c is strictly increasing.*

PROOF. Note that by Lemma 1, the right-hand derivative of c is nonnegative, so that c is nonnegative and nondecreasing. By contradiction, if c had an interval of constancy $[s, t]$, with $t > s$, then

$$\begin{aligned}0 &= c'_+\left(\frac{t+s}{2}\right)\\ &= f \circ c\left(\frac{t+s}{2}\right) + g\left(\frac{t+s}{2}\right)\\ &> f \circ c(s) + g(s)\\ &= 0.\end{aligned}$$ □

REMARK. As we shall see in the proof of the uniqueness assertion of Theorem 1, if we can guarantee that all solutions to IVP(f, g) are strictly increasing, then uniqueness holds for IVP(f, g). Note that if $f + g(0)$ is strictly positive, then $f(x) + g(t) > 0$ for all $x \geq 0$ and $t \geq 0$, so that all solutions to IVP(f, g) are strictly increasing.

PROOF OF THEOREM 1, UNIQUENESS. Let c and $\tilde{c}$ solve IVP(f, g). To show that $c = \tilde{c}$, we argue by contradiction by studying their inverses i and $\tilde{i}$.

Suppose that c and $\tilde{c}$ are strictly increasing. Then i and $\tilde{i}$ are continuous. If $c \neq \tilde{c}$, then $i \neq \tilde{i}$, and we might without loss of generality suppose there is x_1 such that $i(x_1) < \tilde{i}(x_1)$. Let

$$x_0 = \sup\{x \leq x_1 : i(x) \geq \tilde{i}(x)\},$$

and note that, by continuity of i and $\tilde{i}$, $x_0 < x_1$ and $i \leq \tilde{i}$ on $(x_0, x_1]$. Since i and $\tilde{i}$ are continuous, they satisfy

$$i(y) = \int_0^y \frac{1}{f(x) + g \circ i(x)}\, dx.$$

There must exist $x \in [x_0, x_1]$ such that $i'(x)$ and $\tilde{i}'(x)$ both exist, and the former is strictly smaller since otherwise the inequality $\tilde{i} \leq i$ would hold on $[x_0, x_1]$. For this value of x,

$$f(x) = \frac{1}{\tilde{i}'(x)} - g \circ \tilde{i}(x) < \frac{1}{i'(x)} - g \circ i(x) = f(x),$$

which is a contradiction.

Note that all solutions to IVP(f, g) are strictly increasing whenever g is strictly increasing (by Lemma 3) or f is strictly positive, which implies uniqueness to IVP(f, g) in these cases.

When g is constant, and $f + g$ is absorbed at 0, meaning that if $f(s) + g(0) = 0$, then $f(t) + g(0) = 0$ for all $t \geq s$, we can directly use the Lamperti transformation to obtain uniqueness. Indeed, solutions to IVP(f, g) do not have spontaneous generation and, as stated in the introduction to Section 2 (cf. page 1598), there is an unique solution to IVP$(f + g(0), 0)$ without spontaneous generation. □

2.3. *Uniqueness in the stochastic setting*. We now verify that solutions to (4) are unique even if the subordinator Y is compound Poisson.

PROOF OF PROPOSITION 2. Let X be a spLp and Y an independent subordinator. We first prove that there is an unique process Z which satisfies

$$Z_t = x + X\left(\int_0^t Z_s\,ds\right) + Y_t.$$

When Y is an infinite activity subordinator (its Lévy measure is infinite or equivalently it has jumps in any nonempty open interval) or it has positive drift, then its trajectories are strictly increasing, and so uniqueness holds, thanks to Theorem 1.

It then suffices to consider the case when Y is a compound Poisson process. There is a simple case we can establish: if X is also a subordinator, and $x > 0$, then all solutions to $\mathrm{IVP}(X, x + Y)$ are strictly increasing, and so uniqueness holds (again by Theorem 1). It remains to consider two cases: when X is a subordinator and $x = 0$ and when X is not a subordinator. In the first, note that zero solves $\mathrm{IVP}(X, 0)$, and since every solution is nonnegative, zero is the smallest one. To prove uniqueness, let C^x be the (unique) solution to $\mathrm{IVP}(X, x)$, so that C^x is greater than any solution to $\mathrm{IVP}(X, 0)$ by Lemma 2. If we prove that as $x \to 0$, $C^x \to 0$, then all solutions to $\mathrm{IVP}(X, 0)$ are zero, and so uniqueness holds. For this, use the fact that as $t \to 0$, X_t/t converges almost to the drift coefficient of X, say $d \in [0, \infty)$ [cf. Bertoin (1996), Chapter III, Proposition 8, page 84] so that

$$\int_{0+} \frac{1}{X_s}\,ds = \infty.$$

Let I^x be the (continuous) inverse of C^x (note that C^x is strictly increasing). Since

$$I^x(t) = \int_0^t \frac{1}{x + X_s}\,ds,$$

we see, by Fatou's lemma, that $I^x \to \infty$ as $x \to 0$, so that $C^x \to 0$. Now with X still a subordinator and Y compound Poisson, the preceding case implies that the solution to $\mathrm{IVP}(X, Y)$ is unique until the first jump time of Y; after this jump time, all solutions are strictly increasing, and hence uniqueness holds.

The only remaining case is when Y is compound Poisson and X is not a subordinator. The last hypothesis implies that 0 is regular for $(-\infty, 0)$, meaning that on every interval $[0, \varepsilon)$, X visits $(-\infty, 0)$; cf. Bertoin (1996), Chapter VII, Theorem 1, page 189. From this, it follows that if T is any stopping time with respect to the filtration $\sigma(X_s, s \le t) \vee \sigma(Y), t \ge 0$, then X visits $(-\infty, X_T)$ on any interval to the right of T. Let C be any solution to $\mathrm{IVP}(X, x + Y)$; we will show that it has no spontaneous generation. Since there is an unique solution without spontaneous generation when Y is piecewise constant (as discussed in the introduction to Section 2), we get uniqueness. Let

$$[T_{i-1}, T_i), \qquad i = 1, 2, \ldots,$$

be the intervals of constancy of Y; if C has spontaneous generation on one of these, say $[T_{i-1}, T_i)$, then X reaches the level $-Y_{T_{i-1}}$ and then increases, which we know does not happen since the hitting time of $\{-Y_{T_{i-1}}\}$ by the process X is a stopping time with respect to the filtration $\sigma(X_s, s \leq t) \vee \sigma(Y), t \geq 0$.

We end the proof by showing that any càdlàg process Z satisfying

$$x + X_-\left(\int_0^t Z_s\,ds\right) + Y_t \leq Z_t \leq x + X\left(\int_0^t Z_s\,ds\right) + Y_t \tag{8}$$

actually satisfies

$$Z_t = x + X\left(\int_0^t Z_s\,ds\right) + Y_t.$$

Let

$$C_t = \int_0^t Z_s\,ds.$$

When Y is strictly increasing, an argument similar to the proof of the Monotonicity lemma (Lemma 2) tells us that C is strictly increasing, so that C actually satisfies IVP$(X, x + Y)$.

When $Y = 0$, the previous argument shows that, as long as Z has not reached 0, C coincides with the solution to IVP(X, x). If Z is such that

$$\inf\{t \geq 0 : Z_t = 0\} = \inf\{t \geq 0 : Z_{t-} = 0\},$$

then C solves IVP(X, x), which has an unique solution, so that (8) has an unique solution. We then see that the only way in which Z can cease to solve IVP(X, x) is if X is such that

$$T_{0+} = \inf\{t \geq 0 : x + X_{t-} = 0\} < \inf\{t \geq 0 : x + X_t = 0\} = T_0,$$

which is ruled out almost surely by quasi left-continuity of X. Indeed, T_{0+} is the increasing limit of the stopping times

$$T_\varepsilon = \inf\{t \geq 0 : x + X_t < \varepsilon\},$$

which satisfy $T_\varepsilon < T_{\varepsilon'}$ if $\varepsilon < \varepsilon'$ since X has no negative jumps. Hence X is almost surely continuous at T_{0+} which says that $x + X_{T_{0+}} = 0$ almost surely. In the remaining case when Y is a (nonzero) compound Poisson process, we condition on Y and argue similarly on constancy intervals of Y. $\square$

2.4. *Explosion.* We now turn to the explosion criteria of solutions of IVP(f, g) of Proposition 3.

PROOF OF PROPOSITION 3. (1) If $\int^\infty 1/f^+(x) = \infty$, let c be any solution to IVP(f, g). We show that c is finite at every $t > 0$. Indeed, using the arguments of Lemma 2, we see that c is bounded by any solution to IVP$(f, 1 + g(t))$ on the interval $[0,t]$. A particular solution to $\mathrm{IVP}(f, 1+g(t))$ is obtained by taking the right-continuous inverse of

$$y \mapsto \int_0^y \frac{1}{f(x)+1+g(t)}\,dx.$$

Since

$$\int_0^\infty \frac{1}{f^+(x)+1+g(t)}\,dx = \infty,$$

the particular solution we have considered is everywhere finite.

(2) Let c be a solution to $\mathrm{IVP}(f, g)$ where f is an explosive reproduction function, $\lim_{x\to\infty} f(x) = \infty$ and $g(\infty)$ exceeds the maximum of $-f$. To prove that c explodes, choose $T > 0$ such that $f(x) + g(t) > 0$ for all $x \geq 0$ and $t \geq T$. Then $f \circ c + g > 0$ on $[T, \infty)$. Let $M = c(T)$. We then consider the right-continuous inverse i of c (which is actually an inverse on $[M, \infty)$) and note that for $y > M$,

$$i(y) - i(M) = \int_M^y \frac{1}{f(x) + g \circ i(x)}\,dx \leq \int_M^y \frac{1}{f(x)}\,dx.$$

Hence, $i(y)$ converges to a finite limit as $y \to \infty$ so that c explodes. $\square$

2.5. *Application of the analytic theory.* We now pass to a probabilistic application of Theorem 1.

PROOF OF COROLLARY 1. We consider first the case where Y is deterministic. Since Y is assumed to be strictly increasing, we can consider the unique nonnegative stochastic process Z which satisfies

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_t.$$

(The reader can be reassured by Lemma 5 regarding any qualms on measurability issues.) Since Z is nonnegative, Theorems 4.1 and 4.2 of Kallenberg (1992) imply the existence of a stochastic process $\tilde{X}$ with the same law as X such that

$$Z_t = x + \int_0^t Z_s^{1/\alpha}\,d\tilde{X}_s + Y_t.$$

Hence Z is a weak solution to (6).

Conversely, if Z is a solution to (6), we apply Theorems 4.1 and 4.2 of Kallenberg (1992) to deduce the existence of a stochastic process $\tilde{X}$ with the same law as X such that

$$Z_t = x + \tilde{X}_{\int_0^t Z_s\,ds} + Y_t.$$

Considering the mapping $(f, g) \mapsto F(f, g)$ that associates to every admissible breadth-first pair the solution h to (7), we see that Z has the law of $F(\tilde{X}, x + Y)$. Hence, weak uniqueness holds for (6).

When Y is not deterministic but independent of X, we just reduce to the previous case by conditioning on Y [or by augmenting the filtration with the σ-field $\sigma(Y_t : t \geq 0)$]. $\square$

3. CBI processes as Lamperti transforms. We now move on to the analysis of Theorem 2. Let X and Y be independent Lévy processes such that X is spectrally positive and Y is a subordinator under the probability measure $\mathbb{P}$. Call Ψ and Φ their Laplace exponents (taking care to have $\Phi \geq 0$ as for subordinators). Note that the trajectories of Y are either zero, piecewise constant (in the compound Poisson case), or strictly increasing.

Let Z be the stochastic process that solves

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_t$$

and has no spontaneous generation (when Y is compound Poisson). To prove that Z is a CBI(Ψ, Φ), we should see that it is a càdlàg and homogeneous Markov process and that there exist functions $u_t : (0, \infty) \to (0, \infty)$ and $v_t : (0, \infty) \to (0, \infty)$, satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} u_t(\lambda) = -\Psi \circ u_t(\lambda), \\ u_0(\lambda) = \lambda, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial}{\partial t} v_t(\lambda) = \Phi\bigl(u_t(\lambda)\bigr), \\ v_0(\lambda) = 0, \end{cases} \tag{9}$$

and such that for all $\lambda, t \geq 0$,

$$\mathbb{E}\bigl(e^{-\lambda Z_t}\bigr) = e^{-x u_t(\lambda) - v_t(\lambda)}.$$

[At this point it should be clear that the equation for u characterizes it and that, actually, for fixed $\lambda > 0$, $t \mapsto u_t(\lambda)$ is the inverse function to

$$x \mapsto \int_x^\lambda \frac{1}{\Psi(y)}\,dy.\Bigr]$$

3.1. *A characterization lemma and a short proof of Lamperti's theorem.* The way to compute the Laplace transform of Z is by showing, with martingale arguments to be discussed promptly, that

$$\mathbb{E}\bigl(e^{-\lambda Z_t}\bigr) = \int_0^t \mathbb{E}\bigl(\bigl[\Psi(\lambda) Z_s - \Phi(\lambda)\bigr] e^{-\lambda Z_s}\bigr)\,ds. \tag{10}$$

We are then in a position to apply the following result.

LEMMA 4 (Characterization lemma). *If Z is a nonnegative homogeneous Markov process with càdlàg paths starting at x and satisfying* (10) *for all $\lambda > 0$, then Z is a* CBI(Ψ, Φ) *that starts at x.*

REMARK. Note that the hypotheses on the process Z of Lemma 4 do not allow us to use generator arguments which would shorten the proof, for example, by using the characterization of the infinitesimal generator of a CBI process through exponential functions.

PROOF OF LEMMA 4. Let us prove that the function

$$G(s) = \mathbb{E}\big(e^{-u_{t-s}(\lambda)Z_s - v_{t-s}(\lambda)}\big)$$

satisfies $G'(s) = 0$ for $s \in [0, t]$, so that it is constant on $[0, t]$, implying the equality

$$\mathbb{E}\big(e^{-\lambda Z_t}\big) = G(t) = G(0) = e^{-xu_t(\lambda) - v_t(\lambda)}.$$

We then see that Z_t has the same one-dimensional distributions as a CBI(Ψ, Φ) that starts at x, so that by the Markov property, Z is actually a CBI(Ψ, Φ).

To see that $G' = 0$, we first write

$$\begin{aligned} G(s+h) - G(s) = {}& \big(G(s+h) - \mathbb{E}\big(e^{-u_{t-s-h}(\lambda)Z_s - v_{t-s-h}(\lambda)}\big)\big) \\ & + \big(\mathbb{E}\big(e^{-u_{t-s-h}(\lambda)Z_s - v_{t-s-h}(\lambda)}\big) - G(s)\big). \end{aligned} \tag{11}$$

We now analyze both summands to later divide by h and let $h \to 0$.

For the first summand, use (10) to get

$$\begin{aligned} & G(s+h) - \mathbb{E}\big(e^{-Z_s u_{t-s-h}(\lambda) - v_{t-s-h}(\lambda)}\big) \\ & \quad = e^{-v_{t-s-h}(\lambda)} \int_s^{s+h} \mathbb{E}\big(e^{-Z_r u_{t-s-h}(\lambda)}\big[Z_r \Psi \circ u_{t-s-h}(\lambda) - \Phi \circ u_{t-s-h}(\lambda)\big]\big)\, dr, \end{aligned}$$

so that, since Z has càdlàg paths, we get

$$\begin{aligned} & \lim_{h\to 0} \frac{1}{h}\big[G(s+h) - \mathbb{E}\big(e^{-Z_s u_{t-s-h}(\lambda) - v_{t-s-h}(\lambda)}\big)\big] \\ & \quad = \mathbb{E}\big(e^{-u_{t-s}(\lambda)Z_s - v_{t-s}(\lambda)}\big[Z_s \Psi \circ u_{t-s}(\lambda) - \Phi \circ u_{t-s}(\lambda)\big]\big). \end{aligned}$$

For the second summand in the right-hand side of (11), we differentiate under the expectation to obtain

$$\begin{aligned} & \lim_{h\to 0} \frac{1}{h}\mathbb{E}\big(e^{-u_{t-s-h}(\lambda)Z_s - v_{t-s-h}(\lambda)} - e^{-u_{t-s}(\lambda)Z_s - v_{t-s}(\lambda)}\big) \\ & \quad = \mathbb{E}\bigg(e^{-u_{t-s}(\lambda)Z_s - v_{t-s}(\lambda)}\bigg[Z_s \frac{\partial u_{t-s}(\lambda)}{\partial s} + \frac{\partial v_{t-s}(\lambda)}{\partial s}\bigg]\bigg). \end{aligned}$$

We conclude that $G'(s) = 0$ for all $s \in [0, t]$, using (9). □

A simple case of our proof of Theorem 2 arises when $Y = 0$. Recall from Proposition 4 the notation

$$C_t = \int_0^t Z_s\, ds.$$

PROOF OF THEOREM 2 WHEN $\Phi = 0$. This is exactly the setting of Lamperti's theorem stated by Lamperti (1967a).

When $\Phi = 0$ (or equivalently, Y is zero), then C_t is a stopping time for X [since the inverse of C can be obtained by integrating $1/(x + X)$]. Since Z is the time-change of X using the inverse of an additive functional, Z is a homogeneous Markov process. [Another proof of the Markov property of Z, based on properties of IVP$(X, x + Y)$ is given in (3) of Lemma 5.] Also, we can transform the martingale

$$e^{-\lambda X_t} - \Psi(\lambda) \int_0^t e^{-\lambda X_s} \, ds$$

by optional sampling into the martingale

$$e^{-\lambda Z_t} - \Psi(\lambda) \int_0^t e^{-\lambda Z_s} Z_s \, ds.$$

We then take expectations and apply Lemma 4. □

3.2. *The general case.* For all other cases, we need the following measurability details. Consider the mapping F_t which takes a càdlàg function f with nonnegative jumps and starting at zero, a càdlàg g starting at zero (either piecewise constant or strictly increasing), and a nonnegative real x to $c'_+(t)$ where c solves IVP$(f, x + g)$ and has no spontaneous generation (if g is piecewise constant). [Note that these conditions uniquely specify a solution to IVP$(f, x + g)$.] Then

$$Z_{t+s} = F_t(X_{C_s+\cdot} - X_{C_s}, Y_{s+\cdot} - Y_s, Z_s). \tag{12}$$

The mapping F_t is measurable. Indeed, we can view it as the composition of three measurable mappings. The first one is the mapping that associates to $(f, g + x)$ the unique solution to IVP(f, g) (without spontaneous generation), from the space of admissible breadth-first pairs equipped with the σ-fields generated by the projections $(f, g) \mapsto f(t)$ and $(f, g) \mapsto g(t)$ for any $t \geq 0$ to the space of nondecreasing continuous functions with càdlàg derivative (equipped also with the σ-field generated by projections). This mapping is measurable when $g = 0$ by measurability of the Lamperti transformation. Next, when g is piecewise constant this follows by concatenation of Lamperti transforms as in the introduction to Section 2, and for strictly increasing g, this follows since the unique solution to IVP(f, g) is the limit of solutions to IVP(f, g_n) with piecewise constant functions g_n, as seen in the proof of Theorem 1. The second mapping sends a continuous function with càdlàg derivative to its derivative, which is measurable by approximation of the derivative by a sequence of differential quotients. The third mapping is simply the projection of a càdlàg function to its value at time t; its measurability is proved in Theorem 12.5, page 134 of Billingsley (1999).

We suppose that our probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete, and let $\mathscr{T}$ stand for the sets in $\mathscr{F}$ of probability zero. For fixed $y, t \in [0, \infty]$, let $\mathscr{G}^t_y = \mathscr{F}^X_y \vee \mathscr{F}^Y_t \vee \mathscr{T}$.

LEMMA 5 (Measurability details). (1) *The filtration* $(\mathscr{G}^t_y, y \geq 0)$ *satisfies the usual hypotheses*.

(2) C_t *is a stopping time for the filtration* $(\mathscr{G}^t_y, y \geq 0)$, *and we can therefore define the* σ*-field*

$$\mathscr{G}^t_{C_t} = \{A \in \mathscr{F} : A \cap \{C_t \leq y\} \in \mathscr{G}^t_y\}.$$

(3) Z *is a homogeneous Markov process with respect to the filtration* $(\mathscr{G}^t_{C_t}, y \geq 0)$.

PROOF. (1) We just need to be careful to avoid *one of the worst traps involving* σ*-fields* by using independence; cf. Chaumont and Yor (2003), Example 25, page 29.

(2) We are reduced to verifying

$$\{C_t < y\} \in \mathscr{G}^t_y. \tag{13}$$

We prove (13) in two steps, first when Y is piecewise constant, then when Y is strictly increasing.

Let Y be piecewise constant, jumping at the stopping times $T_1 < T_2 < \cdots$, and set $T_0 = 0$. We first prove that

$$\{C_{T_n} < y\} \cap \{T_n \leq t\} \in \mathscr{G}^t_y \tag{14}$$

and this result and a similar argument will yield (13). The membership in (14) is proved by induction using the fact that C can be written down as a Lamperti transform on each interval of constancy of Y. Let I_t be the functional on the subspace of Skorohod space consisting of functions with nonnegative jumps that aids in defining the Lamperti transformation: when applied to a given function f, we first define

$$T_0(f) = \inf\{t \geq 0 : f(t) = 0\}$$

and then

$$I_t(f) = \int_0^{t \wedge T_0(f)} \frac{1}{f(s)}\, ds.$$

We then have

$$\{C_{T_1} < y\} \cap \{T_1 \leq t\} = \{I_y(X + Y_0) > T_1 \wedge t\} \cap \{T_1 \leq t\} \in \mathscr{G}^t_y.$$

If we suppose that

$$\{C_{T_n} < y\} \cap \{T_n \leq t\} \in \mathscr{G}^t_y,$$

then the decomposition

$$
\begin{aligned}
&\{C_{T_{n+1}} < y\} \cap \{T_{n+1} \le t\} \\
&\quad = \bigcup_{q \in (0,y)\cap\mathbb{Q}} \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{2^{-m}\lfloor 2^m(y-q)\rfloor} \left\{\frac{k}{2^m} \le C_{T_n} < \frac{k+1}{2^m}\right\} \cap \{T_{n+1} \le t\} \\
&\qquad\qquad \cap \left\{I_q(x + X_{(k/2^m)+\cdot} + Y_{T_n}) > T_{n+1} - T_n\right\}
\end{aligned}
$$

allows us to obtain (14). Then the decomposition

$$
\begin{aligned}
\{C_t < y\} = \bigcup_{n=0}^{\infty} \bigcup_{q \in (0,y)\cap\mathbb{Q}} \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{2^{-m}\lfloor 2^m(y-q)\rfloor} &\{T_n \le t < T_{n+1}\} \\
&\cap \left\{\frac{k}{2^m} \le C_{T_n} < \frac{k+1}{2^m}\right\} \\
&\cap \left\{I_q(x + X_{(k/2^m)+\cdot} + Y_{T_n}) > t - T_n\right\}
\end{aligned}
$$

gives (13) when Y is piecewise constant.

When Y is strictly increasing, consider a sequence ε_n decreasing strictly to zero and a decreasing sequence (π_n) of partitions of $[0,t]$ whose norms tend to zero, with

$$
\pi_n = \{t_0^n = 0 < t_1^n < \cdots < t_{k_n}^n = t\}.
$$

Consider the process $Y^n = (Y_s^n)_{s\in[0,t]}$ defined by

$$
Y_s^n = \varepsilon_n + \sum_{i=1}^{k_n} Y_{t_i^n} \mathbf{1}_{[t_{i-1}^n, t_i^n)}(s) + Y_t \mathbf{1}_{s=t}.
$$

Since π^n is contained in π^{n+1} and $\varepsilon_n > \varepsilon_{n+1}$, $Y^n > Y^{n+1}$. If C^n is the solution to IVP$(X, x + Y^n)$ with no spontaneous generation (defined only on $[0,t]$), then Lemma 2 gives $C^n > C^{n+1}$. Hence, (C^n) converges as $n \to \infty$, and since the limit is easily seen to be a solution to IVP$(X, x + Y)$, the limit must equal C by the uniqueness statement in Theorem 1. To obtain (13), we note that

$$
\{C_t^n < y\} \in \mathscr{F}_y^X \vee \mathscr{F}_t^{Y^n} \subset \mathscr{F}_y^X \vee \mathscr{F}_t^Y
$$

and

$$
\{C_t < y\} = \bigcup_n \{C_t^n < y\}.
$$

(3) Mimicking the proof of the Strong Markov Property for Brownian motion [as in Kallenberg (2002), Theorem 13.11] and using (13), one proves that the process

$$
(X_{C_t+s} - X_{C_t}, Y_{t+s} - Y_t)_{s\ge 0}
$$

has the same law as (X, Y) and is independent of $\mathscr{G}^t_{C_t}$, which we can restate as

> $(X_{C_t+s} - X_{C_t}, Y_{t+s} - Y_t)_{s\geq 0}$ has the same law as (X, Y) and is independent of (X^{C_t}, Y^t) where $X^{C_t}_s = X_{C_t\wedge s}$ and $Y^t_s = Y_{t\wedge s}$.

Equation (12) implies that the conditional law of Z_{t+s} given $\mathscr{G}^s_{C_s}$ is actually Z_s measurable, implying the Markov property. The transition semigroup is homogeneous and in t units of time is given by the law $P_t(x,\cdot)$ of $F_t(X, Y, x)$ under $\mathbb{P}$. Note that this semigroup is conservative on $[0,\infty]$. □

We will need Proposition 4 for our proof of Theorem 2.

PROOF OF PROPOSITION 4. Consider the filtration $\mathscr{G} = (\mathscr{G}_t)$ given by

$$\mathscr{G}_t = \sigma(X_s : s \leq t) \vee \sigma(Y_s : s \geq 0) \vee \mathscr{T}.$$

If Y is strictly increasing, then C is strictly increasing and continuous. For fixed $\varepsilon > 0$, let $T_1 < T_2 < \cdots$ be the jumps of Y of magnitude greater than ε. Arguing as in Lemma 5, we see that C_{T_i} is a $\mathscr{G}$-stopping time which is the almost sure limit of the $\mathscr{G}$-stopping times $C_{(T_i-1/n)^+}$ as $n \to \infty$. Since X is a $\mathscr{G}$-Lévy process and $C_{(T_i-1/n)^+} < C_{T_i}$ for all n, quasi left-continuity of X implies that $X \circ C$ does not jump at T_i almost surely. Since this is true for any $\varepsilon > 0$, then $X \circ C$ and Y do not jump at the same time.

If Y is compound Poisson, we argue on its constancy intervals, denoted $[T_{i-1}, T_i)$, $i = 1, 2, \ldots$. On the set $\{C_s < C_{T_i}$ for all $s < T_i\}$, we can argue as above, using quasi left-continuity. On the set $\{C_s = C_{T_i}$ for some $s < T_i\}$, we note that X reaches $-Y_{T_i}$ for the first time at C_{T_i}. The hitting time of $-Y_{T_i}$ by X is a $\mathscr{G}$-stopping time which is the almost sure limit of the hitting times of $-Y_{T_i} + 1/n$ as $n \to \infty$. The latter are strictly smaller than the former since X has no negative jumps. Hence, by quasi left-continuity, X is almost surely continuous at C_{T_i}. □

PROOF OF THEOREM 2. Since

$$\left(e^{-\lambda X_y} - \Psi(\lambda)\int_0^y e^{-\lambda X_s}\,ds\right)_{t\geq 0}$$

is a $(\mathscr{G}^t_y)_{y\geq 0}$-martingale, it follows that $M = (M_t)_{t\geq 0}$, given by

$$M_t = e^{-\lambda X_{C_t}} - \Psi(\lambda)\int_0^t e^{-\lambda X_{C_s}} Z_s\,ds,$$

is a $(\mathscr{G}^t_{C_t})_{t\geq 0}$-local martingale. With respect to the latter filtration, the stochastic process $N = (N_t)_{t\geq 0}$ given by

$$N_t = e^{-\lambda Y_t} + \Phi(\lambda)\int_0^t e^{-\lambda Y_s}\,ds$$

is a martingale. Hence $e^{-\lambda X\circ C}$ and $e^{-\lambda(x+Y)}$ are semimartingales to which we may apply integration by parts to get

$$e^{-\lambda Z_t} = \text{local martingale} + \int_0^t e^{-\lambda Z_s}\big[\Psi(\lambda)Z_s - \Phi(\lambda)\big]\,ds \\ + \big[e^{-\lambda X\circ C}, e^{-\lambda x-\lambda Y}\big]_t,$$

where the local martingale part is

$$t \mapsto \int_0^t e^{-\lambda(x+Y_s)}\,dM_s + \int_0^t e^{-\lambda X\circ C_s}\,dN_s.$$

Since $X\circ C$ and Y do not jump at the same time by Proposition 4 and Y is of finite variation, we see that

$$\big[e^{-\lambda X\circ C}, e^{-\lambda x-\lambda Y}\big] = 0;$$

cf. Kallenberg (2002), Theorem 26.6(vii).

We deduce that

$$e^{-\lambda Z_t} - \int_0^t e^{-\lambda Z_s}\big[\Psi(\lambda)Z_s - \Phi(\lambda)\big]\,ds$$

is a martingale, since it is a local martingale whose sample paths are uniformly bounded on compacts thanks to the nonnegativity of Z. Taking expectations, we get (10), and we conclude by applying Lemma 4 since Z is a Markov process thanks to Lemma 5. □

3.3. *Translating a law of the iterated logarithm.*

PROOF OF COROLLARY 4. Let X be a spLp with Laplace exponent Ψ, $\tilde{\Phi}$ be the right-continuous inverse of Ψ and

$$\alpha(t) = \frac{\log|\log t|}{\tilde{\Phi}(t^{-1}\log|\log t|)}.$$

Recall that $\tilde{\Phi}$ is the Laplace exponent of the subordinator $T = (T_t, t \geq 0)$ where

$$T_t = \inf\{s \geq 0 : X_s \leq -t\};$$

cf. Bertoin (1996), Chapter VII, Theorem 1. If $\tilde{d}$ is the drift coefficient of $\tilde{\Phi}$, then Proposition 1 of Bertoin [(1996), Chapter III] gives

$$\lim_{\lambda\to\infty} \frac{\tilde{\Phi}(\lambda)}{\lambda} = \tilde{d}.$$

Hence,

$$\text{as } t \to 0+ \qquad \begin{cases} \alpha(t) \sim t/\tilde{d}, & \text{if } \tilde{d} > 0,\\ t = o\big(\alpha(t)\big), & \text{if } \tilde{d} = 0.\end{cases}$$

We now assert that if $a_t \to 1$ as $t \to 0$, then

$$\lim_{t\to 0} \frac{\alpha(a_t t)}{\alpha(t)} = 1.$$

This is clear when $\tilde{d} > 0$, so suppose that $\tilde{d} = 0$. Since $t \mapsto \log|\log t|$ is slowly varying at zero, it suffices to show that if $b_\lambda \to 1$ as $\lambda \to \infty$, then

$$\lim_{\lambda\to\infty} \frac{\tilde{\Phi}(b_\lambda \lambda)}{\tilde{\Phi}(\lambda)} = 1. \tag{15}$$

However, concavity of $\tilde{\Phi}$, increasingness and nonnegativity give (if $b > 1$)

$$\tilde{\Phi}(b\lambda)/b \le \tilde{\Phi}(\lambda) \le \tilde{\Phi}(b\lambda),$$

which implies (15).

As noted by Bertoin (1995), Fristedt and Pruitt (1971) prove the existence of a constant $\zeta \neq 0$ such that

$$\liminf_{t\to 0} \frac{X_t}{\alpha(t)} = \zeta.$$

Let Z be the unique solution to

$$Z_t = x + X_{\int_0^t Z_s\,ds} + Y_t$$

with $x > 0$, where X and Y are independent Lévy processes, with X spectrally positive of Laplace exponent Ψ and Y a subordinator with Laplace exponent Φ. Since $Z_0 = x$, and Z is right-continuous, then

$$\lim_{t\to 0+} \frac{1}{t}\int_0^t Z_s\,ds = x$$

almost surely. Hence

$$\lim_{t\to 0+} \frac{\alpha(\int_0^t Z_s\,ds)}{\alpha(xt)} = 1$$

and so

$$\liminf_{t\to 0+} \frac{X_{\int_0^t Z_s\,ds}}{\alpha(xt)} = \zeta.$$

On the other hand, if d is the drift of Φ, then

$$\lim_{t\to 0} \frac{Y_t}{t} = d$$

[cf. Bertoin (1996), Chapter III, Proposition 8] so that if $\tilde{d} = 0$, $Y_t = o(\alpha(t))$ and

$$\liminf_{t\to 0+} \frac{Z_t - x}{\alpha(xt)} = \zeta.$$

If $\tilde{d} > 0$, then by Proposition 8 of Bertoin [(1996), Chapter III], we actually have

$$\liminf_{t \to 0} \frac{X_t}{\alpha(t)} = -1$$

so that

$$\liminf_{t \to 0+} \frac{Z_t - x}{\alpha(xt)} = -1 + \frac{d\tilde{d}}{x}. \qquad \square$$

3.4. *Explosion criteria for CBI.* As a probabilistic application of the deterministic explosion criteria of Proposition 3, we prove Corollary 5.

PROOF OF COROLLARY 5. Let $x > 0$, and consider a spectrally positive Lévy process X with Laplace exponent Ψ independent of a subordinator Y with Laplace exponent Φ. Let Z be the unique solution to

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t,$$

which is a CBI(Ψ, Φ) that starts at x. Also, let

$$C_t = \int_0^t Z_s\, ds.$$

(1) Let Y be a nonzero subordinator. Path by path, we see that Z jumps to infinity if and only if either X jumps to infinity or Y does. However, the probability that either X or Y jumps to infinity is positive if and only if either $\Psi(0) > 0$ or $\Phi(0) > 0$. When Y is zero, Z jumps to infinity if X jumps to infinity and never reaches $-x$, which has positive probability.

(2) The Ogura–Grey explosion criterion for continuous state branching processes (as stated just before Corollary 5) can be restated as follows: a CBI($\Psi, 0$) started at x reaches ∞ continuously at a finite time with positive probability if and only if $\Psi(0) = 0$, and Ψ is an explosive branching mechanism. It is also simple to see that a CBI($\Psi, 0$) jumps to ∞ at a finite time with positive probability if and only if $\Psi(0) > 0$.

Path by path, we see that if Z reaches ∞ continuously (say at time τ), then Y does not jump to infinity on $[0, \tau)$. Also, if we let $\tilde{C}$ be the unique solution to IVP($x + Y_{\tau-} + \varepsilon + X, 0$) and $\tilde{Z}$ as the right-hand derivative of $\tilde{C}$, where $\varepsilon > 0$, then $C < \tilde{C}$ on $(0, \tau)$ (as follows from the argument proving Lemma 2). Hence $\tilde{C}$ explodes on $[0, \tau)$. We conclude that the branching mechanism of $\tilde{Z}$ is explosive by the Ogura–Grey explosion criterion. Hence, the assumption $\mathbb{P}(Z \text{ reaches } \infty \text{ continuously}) > 0$ implies that $\Psi(0) = 0$ and that Ψ is an explosive branching mechanism.

On the other hand, if $\Psi(0) = 0$ and Ψ is explosive, let $\tilde{\Phi} = \Phi - \Phi(0)$, Y be a subordinator independent of X with Laplace exponent $\tilde{\Phi}$, so that sending Y

to infinity at an exponential random variable with parameter $\Phi(0)$ (independent of both X and Y) leaves us with a subordinator with Laplace exponent Φ independent of X. Let C^1 be a solution to IVP$(x/2 + X, 0)$ and C^2 be a solution to IVP$(x + X, Y)$ so that $C^1 \le C^2$, and hence C^2 explodes if C^1 does. Let Z^i be the right-hand derivative of C^i. Z^1 is a CBI$(\Psi, 0)$ starting at $x/2$ while Z^2 is a CBI$(\Psi, \tilde{\Phi})$ started at x; notice that the process Z obtained by sending Z^2 to infinity at the same exponential as Y leaves us with a CBI(Ψ, Φ). By assumption, X cannot jump to infinity and Z^1 explodes with positive probability. Hence, Z^2 explodes with positive probability and can only do so continuously. Hence,

$$\mathbb{P}(Z \text{ reaches } \infty \text{ continuously}) \ge e^{-t\Phi(0)}\mathbb{P}\big(Z^2 \text{ reaches } \infty \text{ continuously by time } t\big)$$

and the right-hand side is positive for t large enough.

(3) We also deduce that

$$\mathbb{P}(Z \text{ reaches } \infty \text{ continuously}) = 1$$

if and only if $\Phi(0) = 0$ and $\mathbb{P}(Z^2 \text{ reaches } \infty \text{ continuously}) = 1$. A necessary and sufficient condition for the latter is that, additionally, Φ is not zero. Indeed, when Φ is not zero, then $Y_t \to \infty$ as $t \to \infty$. Since Ψ is explosive and $\Phi(0) = 0$, then $\lim_{t\to\infty} X_t = \infty$ and so Proposition 3 implies that the solution to IVP$(X, x + Y)$ explodes. If $\Phi = 0$, then Z^2 is a CBI$(\Psi, 0)$, which cannot explode continuously almost surely since the probability that Z^2 is absorbed at zero is the probability that X goes below $-x$, which is positive. $\square$

4. Stability of the generalized Lamperti transformation.

We now turn to the proof of Theorems 3 and 4, and of Corollaries 6 and 7, which summarize the stability theory for IVP(f, g).

4.1. *Proof of the analytic assertions.*

In order to compare the initial value problem IVP(f, g) with the functional inequality (5), we now construct an example of an admissible breadth-first pair (f, g) such that IVP(f, g) has an unique solution, but (5) has at least two. Indeed, consider $g = 0$, and take

$$f(x) = \begin{cases} \sqrt{1-x}, & \text{if } x < 1, \\ 1, & \text{if } x \ge 1. \end{cases}$$

Then IVP(f, g) has a unique solution, by Theorem 1, since f is strictly positive. The solution is the function c given by

$$c(t) = \begin{cases} t - t^2/4, & \text{if } t \le 2, \\ c(2) + t - 2, & \text{if } t \ge 2. \end{cases}$$

Since c is strictly increasing, it also solves (5). However, the function

$$\tilde{c}(t) = \begin{cases} c(t), & \text{if } t \le 2, \\ c(2), & \text{if } t \ge 2, \end{cases}$$

is also a solution to (5). Hence, the assumption of Theorem 3 is stronger than just uniqueness of IVP(f, g) although related (as seen by comparing Theorem 1 and Proposition 1).

We start with a proof of Proposition 1.

PROOF OF PROPOSITION 1. Let c be any nondecreasing solution to

$$\int_s^t f_- \circ c(r) + g(r)\, dr \le c(t) - c(s) \le \int_s^t f \circ c(r) + g(r)\, dr \qquad \text{for } s \le t$$

such that $c(0) = 0$. This automatically implies continuity of c and so $f \circ c + g$ is càdlàg and does not jump downwards.

Note that c is strictly increasing if $f_- + g(0)$ is strictly positive or g is strictly increasing, we have equalities in (5), implying that c solves IVP(f, g) which has a unique solution with these hypotheses. Indeed, if $f_- + g(0)$ is a positive function, then the lower bound integrand is strictly positive, and so c cannot have a constancy interval. If on the other hand g is strictly increasing, note first that the nondecreasing character of c implies, through (5), that $f \circ c + g$ is nonnegative (first almost everywhere, but then everywhere since it is càdlàg). Also, $f \circ c + g$ can only reach zero continuously since it lacks negative jumps. If c had a constancy interval $[s, t]$ with $s < t$, there would exist $r \in (s, t)$ such that

$$f_- \circ c(s) + g(s) = f_- \circ c(s) + g(r) = 0,$$

which implies that g has a constancy interval on $[0, t]$, a contradiction. Hence, c has no constancy intervals.

When g is a constant and $f_- + g$ is absorbed at zero, then also $f + g$ is absorbed at zero and at the same time. Hence, c is strictly increasing until it is absorbed, so that again both bounds for the increments of c are equal. Then c solves IVP(f, g) which has a unique solution under this hypothesis. □

We now continue with a proof of Theorem 3. It is divided in two parts: convergence of the cumulative population which is then used to prove convergence of population profiles. The strategy is simple: we first use the functional equations satisfied by (c_n) to prove that $c_n \wedge K$ is uniformly bounded and equicontinuous. Then, we pass to the limit in the functional equations satisfied by c_n to see that any subsequential limit of $c_n \wedge K$ equals $c \wedge K$. [This is where the assumption that (5) has an unique solution comes into play.] Having established convergence of c_n to c, we then verify some technical hypotheses enabling us to apply results on continuity of composition and addition on adequate subspaces of Skorohod space and deduce that $f_n \circ c_n + g_n$ converges to $f \circ c + g$.

PROOF OF THEOREM 3, CONVERGENCE OF CUMULATIVE POPULATIONS. Let $K, \varepsilon > 0$ and consider the sequence $c_n \wedge K$ consisting of nondecreasing functions with càdlàg right-hand derivatives. Since

$$\begin{aligned}
0 &\le D_+c_n \wedge K(t) \\
&= \mathbf{1}_{c_n(t)\le K} \times \begin{cases} \bigl[f_n \circ c_n\bigl(\lfloor t/\sigma_n\rfloor\sigma_n\bigr) + g_n(t)\bigr]^+, & \text{if } \sigma_n > 0, \\ f_n \circ c_n(t) + g_n(t), & \text{if } \sigma_n = 0, \end{cases} \\
&\le \sup_{y\le K} f(y) + g(t) + \varepsilon
\end{aligned}$$

for n large enough (by the convergence of $f_n \to f$ on $[0, K]$ with the J_1 topology and $g_n \to g$ on $[0, t]$ with the J_1 topology), we see that the sequence $c_n \wedge K$ is uniformly bounded and equicontinuous on compacts. To prove convergence of $c_n \wedge K$ (uniformly on compact sets), it is enough to prove by Arzelà–Ascoli that any subsequential limit is the same. Let $t > 0$ and $c_{n_k} \wedge K$ be a uniformly convergent subsequence on $[0, t]$. Denote by $\tilde{c}$ its uniform limit, which is then nondecreasing. If $s \in [0, t]$ is such that $\tilde{c}(s) < x$, then $c_{n_k}(s) < x$ for k large enough. Since f has no negative jumps, then

$$\liminf_{x\to y} f(x) = f_-(y) \quad \text{and} \quad \limsup_{x\to y} f(x) = f(y)$$

so that

$$f_- \circ \tilde{c} \le \liminf f_{n_k} \circ c_{n_k} \le \limsup f_{n_k} \circ c_{n_k} \le f \circ \tilde{c}.$$

By Fatou's lemma, for any $s_1 \le s_2 \le s$,

$$\int_{s_1}^{s_2} \bigl[f_- \circ \tilde{c}(r) + g(r)\bigr]^+ \, dr \le \tilde{c}(s_2) - \tilde{c}(s_1) \le \int_{s_1}^{s_2} \bigl[f \circ \tilde{c}(r) + g(r)\bigr]^+ \, dr.$$

As $\tilde{c}$ is nondecreasing, we might remove the positive parts in the above display and conclude, from uniqueness to (5), that $\tilde{c} = c$ on $[0, s]$. If, on the other hand, $\tilde{c}(s) = K$, then $c_{n_k} \wedge K(s) \to K$ which implies that $c_{n_k} \wedge K \to c \wedge K$ uniformly on compact sets.

Let τ be the explosion time of c. If $t < \tau$, then $c(t) < \infty$, and so [choosing $K > c(t)$ in the paragraph above] we see that $c_n \to c$ uniformly on $[0, t]$. If $t \ge \tau$, then $c(t) = \infty$, and so $c_n(t) \wedge K \to K$ for any $K > 0$. Hence $c_n(t) \to \infty = c(t)$. □

PROOF OF THEOREM 3, CONVERGENCE OF POPULATION PROFILES. Let

$$h_n = D_+c_n \quad \text{and} \quad h = D_+c.$$

We now prove that $h_n \to h$ in the Skorohod J_1 topology if the explosion time τ is infinite. Recall that $h = f \circ c + g$ and that

$$h_n = \begin{cases} f_n \circ c_n + g_n, & \text{if } \sigma_n = 0, \\ \bigl[f_n \circ c_n\bigl(\lfloor t/\sigma_n\rfloor\sigma_n\bigr) + g_n\bigl(\lfloor t/\sigma_n\rfloor\sigma_n\bigr)\bigr]^+, & \text{if } \sigma_n > 0. \end{cases}$$

Assume that $\sigma_n = 0$ for all n, the case $\sigma_n > 0$ being analogous. Then the assertion $h_n \to h$ is reduced to proving that $f_n \circ c_n \to f \circ c$, which is related to the continuity of the composition mapping on (adequate subspaces of) Skorohod space, and then deducing that $f_n \circ c_n + g_n \to f \circ c + g$, which is related to continuity of addition on (adequate subspaces of) Skorohod space. Both continuity assertions require conditions to hold: the convergence $f_n \circ c_n \to f \circ c$ can be deduced from Wu [(2008), Theorem 1.2] if we prove that f is continuous at every point at which c^{-1} is discontinuous, and then the convergence of $f_n \circ c_n + g_n$ will hold because of Whitt [(1980), Theorem 4.1] since we assumed that $f \circ c$ and g do not jump at the same time. Hence, the convergence $h_n \to h$ is reduced to proving that f is continuous at discontinuities of c^{-1}.

If c is strictly increasing [which happens when g is strictly increasing or $f + g(0) > 0$], then c^{-1} is continuous. (This is the most important case in the stochastic setting, since otherwise immigration is compound Poisson, therefore piecewise constant, and one might argue by pasting together Lamperti transforms.)

Suppose that c is not strictly increasing, and let x be a discontinuity of c^{-1}. Let

$$s = c^{-1}(x-) < c^{-1}(x) = t,$$

so that $c = x$ on $[s, t]$ while $c < x$ on $[0, s)$ and $c > x$ on (t, ∞). Since $D_+c = f \circ c + g = 0$ on $[s, t)$, we see that g is constant on $[s, t)$. We assert that

$$\inf\{y \geq 0 : f(y) = -g(s)\} = x.$$

Indeed, if f reached $-g(s)$ at $x' < x$, there would exist $s' < s$ such that

$$0 = f \circ c(s') + g(s) \geq f \circ c(s') + g(s') \geq 0,$$

so that actually g is constant on $[s', t)$. Hence, c has spontaneous generation which implies there are at least two solutions to IVP(f, g): one that is constant on $(s', t]$, and c. This contradicts the assumed uniqueness to (5). Since f has no negative jumps and reaches the level $-g(s)$ at time x, then f is continuous at x.

Finally, we assume that the explosion time τ is finite but that $f_n(x), f(x) \to \infty$ as $x \to \infty$ uniformly in n and prove that $h_n \to h$ in the uniform J_1 topology. Let $\varepsilon > 0$, d be a bounded metric on $[0, \infty]$ that makes it homeomorphic to $[0, 1]$, and consider $M > 0$ such that $d(x, y) < \varepsilon$ if $x, y \geq M$. Let $K > 0$ be such that $f(x), f_n(x) > M$ if $x > K$ and n is large enough. Let $T < \tau$ be such that f is continuous at $c(T)$ and $K < c(T)$. Then $f_n \to f$ in the usual J_1 topology on $[0, c(T)]$ and, arguing as in the nonexplosive case, we see that

$$h_n = f_n \circ c_n + g_n \to f \circ c + g = h$$

in the usual J_1 topology on $[0, T]$. Hence, there exists a sequence (λ_n) of increasing homeomorphisms of $[0, T]$ into itself such that $h_n - h \circ \tilde{\lambda}_n \to 0$ uniformly on $[0, T]$. Define now λ_n to equal $\tilde{\lambda}_n$ on $[0, T]$ and the identity on $[T, \infty)$. Then (λ_n) is a sequence of homeomorphisms of $[0, \infty)$ into itself which converges uniformly to the identity, and since $K < c(T)$, then $K < c_n(T)$ eventually and so $M < h_n, h$ eventually thanks to the choice of K, so that $d(h_n(t), h(t)) < \varepsilon$ on $[T, \infty)$ eventually. Hence, $h_n \to h$ in the uniform J_1 topology. $\square$

In order to apply Theorem 3 to Galton–Watson-type processes, we need a lemma relating the discretization of the Lamperti transformation and scaling. Define the scaling operators S_a^b by

$$S_a^b f(t) = \frac{1}{b} f(at).$$

Let also c^σ be the approximation of span σ to IVP(f, g), which is the unique function satisfying

$$c^\sigma(t) = \int_0^t \left[f \circ c^\sigma\big(\sigma \lfloor s/\sigma \rfloor\big) + g\big(\sigma \lfloor s/\sigma \rfloor\big)\right]^+ ds.$$

We shall denote $c^\sigma(f, g)$ to make the dependence on f and g explicit in the following lemma and denote by $h^\sigma(f, g)$ the right-hand derivative of $c^\sigma(f, g)$.

LEMMA 6. *We have*

$$S_a^b c^\sigma(f, g) = c^{\sigma/a}\big(S_b^{b/a} f, S_a^{b/a} g\big) \quad \textit{and} \quad S_a^{b/a} h^\sigma(f, g) = h^{\sigma/a}\big(S_b^{b/a} f, S_a^{b/a} g\big).$$

The proof is an elementary change of variables.

4.2. *Weak continuity of CBI laws.*

PROOF OF COROLLARY 6. Let X_n and X be spLps with Laplace exponents Ψ_n and Ψ and Y_n and Y be subordinators with Laplace exponents Φ_n and Φ such that X_n (resp., X) is independent of Y_n (resp., Y).

The hypotheses $\Psi_n \to \Psi$ and $\Phi_n \to \Phi$ imply that (X_n, Y_n) converges weakly to (X, Y) in the Skorohod J_1 topology. By Skorohod's representation theorem, we can assume that the convergence takes place almost surely on an adequate probability space.

Let Z_n (resp., Z) be the Lamperti transform of $(X_n, x_n + Y_n)$ [resp., $(X, x + Y)$]. When X is nonexplosive, Propositions 2 and 4 and Theorem 3 then imply that Z_n converges almost surely to Z, which is a CBI(Ψ, Φ), thanks to Theorem 2.

When X is explosive, let ρ be a distance on $[0, \infty]$ which makes it homeomorphic to $[0, 1]$ and, for any $\varepsilon > 0$, choose M_ε such that $\rho(x, y) < \varepsilon$ if $x, y \geq M_\varepsilon$. Recall that d_∞ stands for the uniform J_1 topology. Since the $X^n \to X$ and $Y^n \to Y$ in the usual Skorohod topology as $n \to \infty$ almost surely, then reasoning as in the proof of uniform J_1 convergence of Theorem 3, we see that, for any $\varepsilon > 0$,

$$\mathbb{P}\big(d_\infty(Z^n, Z) > \varepsilon, X_s^n, X_s > M_\varepsilon \text{ for all } s \geq t\big) \to 0 \qquad \text{as } n \to \infty.$$

However, choosing t and M big enough, we can make

$$\mathbb{P}\big(X_s^n \leq M \text{ for some } s \geq t\big)$$

arbitrarily small for all n large enough, so that $d_\infty(Z^n, Z) \to 0$ in probability, which is enough to guarantee that $Z^n \to Z$ weakly in the uniform J_1 topology. Indeed, since X is explosive, we have that $\Psi'(0+) = -\infty$ [cf. Lambert (2008), proof of Theorem 2.2.3.2, page 95] which means that X drifts to ∞; cf. Bertoin (1996), Chapter VII, Corollary 2.ii. Since the latter result implies that the negative of the infimum of X has an exponential distribution of parameter η, where

$$\eta = \inf\{\lambda > 0 : \Psi(\lambda) = 0\},$$

we see that

$$\begin{aligned}
&\mathbb{P}(X_s \leq M \text{ for some } s \geq t) \\
&\quad \leq \mathbb{P}(X_t \leq 2M) + \mathbb{P}(X_t > 2M \text{ and } X_s \leq M \text{ for some } s \geq t) \\
&\quad \leq \mathbb{P}(X_t \leq 2M) + e^{-\eta M}.
\end{aligned}$$

Since X drifts to infinity, the term $\mathbb{P}(X_t \leq 2M)$ goes to zero as $t \to \infty$. Asymptotically, the same bounds hold for X^n since $\Psi^n \to \Psi$ and hence, by convexity of Ψ,

$$\lim_{n\to\infty} \big(\inf\{\lambda > 0 : \Psi^n(\lambda) = 0\}\big) = \inf\{\lambda > 0 : \Psi(\lambda) = 0\} = \eta. \qquad \square$$

4.3. *A limit theorem for Galton–Watson processes with immigration.*

PROOF OF COROLLARY 7. By Skorohod's theorem, if X and Y are Lévy processes whose distributions at time 1 are μ and ν, then

$$S_{a_n}^n X^n \to X \quad \text{and} \quad S_{b_n}^n Y^n \to Y,$$

where the convergence is in the J_1 topology. Assume first that X is nonexplosive.

We can apply Lemma 6 to get either

$$S_{b_{\lfloor k_n/x \rfloor}}^{k_n/x} Z^n = h^{1/b_{\lfloor k_n/x \rfloor}} \big(S_{k_n b_{\lfloor k_n/x \rfloor}/x}^{k_n/x} X^n, x + S_{b_{\lfloor k_n/x \rfloor}}^{k_n/x} Y^n\big)$$

or

$$S_{x a_{\lfloor k_n/x \rfloor}/k_n}^{k_n/x} Z^n = h^{k_n/(x a_{\lfloor k_n/x \rfloor})} \big(x + S_{a_{\lfloor k_n/x \rfloor}}^{k_n/x} X^n, S_{x a_{\lfloor k_n/x \rfloor}/k_n}^{k_n/x} Y^n\big).$$

Let Z be the unique process satisfying

$$Z_t = x + X_{c \int_0^t Z_s \, ds} + Y_t$$

as in Proposition 2. If $\frac{k_n}{x} b_{\lfloor k_n/x \rfloor} / a_{\lfloor k_n/x \rfloor} \to c \in [0, \infty)$, we see that

$$S_{b_{\lfloor k_n/x \rfloor}}^{k_n/x} Z^n \to Z,$$

thanks to Propositions 2 and 4 and Theorems 2 and 3.

When $\frac{k_n}{x} b_{\lfloor k_n/x \rfloor}/a_{\lfloor k_n/x \rfloor} \to \infty$, let Z instead be the unique solution to

$$Z_t = x + X_{\int_0^t Z_s\,ds}.$$

Then

$$S^{k_n/x}_{x a_{\lfloor k_n/x \rfloor}/k_n} Z^n \to Z.$$

When X is explosive, the arguments in the proof of Corollary 6 show that, in order to obtain the stated convergence in the uniform J_1 topology, it is enough to prove that for all $M > 0$,

$$\lim_{M\to\infty} \lim_{t\to\infty} \limsup_n \mathbb{P}\left(\frac{1}{n} X^n_{\lfloor s a_n \rfloor} \le M \text{ for some } s \ge t\right) = 0.$$

Since X drifts to infinity if it is explosive, Ψ has an unique positive root which we denote η.

Let

$$G_n(\lambda) = \mathbb{E}\left(e^{-\lambda X^n_1}\right).$$

Recall that since the increments of X^n are bounded below by -1, minus the random variable

$$I_n = \min_{m \ge 0} X^n_m$$

has a geometric distribution with parameter $e^{-\eta_n}$ where η_n is the greatest nonnegative real number at which G_n achieves the value 1; cf. Asmussen [(2003), Part B, Chapter VIII, Section 5, Corollary 5.5, page 235] or the forthcoming Lemma 7. By log-convexity of G_n, $\eta_n = \inf\{\lambda > 0 : G_n(\lambda) > 1\}$. If we assume the convergence of $n\eta_n$ to η as $n \to \infty$, we see that

$$\limsup_n \mathbb{P}\left(-\frac{1}{n} I_n \ge M\right) = e^{-\eta M}.$$

We now use the Markov property to conclude that if the distribution of X_1 is continuous at M, then

$$\limsup_n \mathbb{P}\left(\frac{1}{n} X^n_{\lfloor s a_n \rfloor} \le M \text{ for some } s \ge t\right) \le \mathbb{P}(X_t \le 2M) + \mathbb{P}(X_t \ge 2M) e^{-\eta M}.$$

To conclude, we should prove that $n\eta_n \to \eta$. This, however, is implied by the following convergence of Laplace transforms:

$$\mathbb{E}\left(e^{-\lambda/n X^n_{a_n}}\right) \to \mathbb{E}\left(e^{-\lambda X_1}\right) = e^{\Psi(\lambda)}.$$

Indeed, recall that $\mathbb{E}(e^{-\lambda X_1}) < 1$ exactly on $(0, \eta)$ and that $\mathbb{E}(e^{-\lambda/n X^n_{a_n}}) < 1$ exactly on $(0, n\eta_n)$. If we consider $\lambda < \eta$ then $\mathbb{E}(e^{-\lambda/n X^n_{a_n}}) < 1$ for large enough n, so that $\lambda \le n\eta_n$ for large enough n. This implies $\eta \le \liminf_n n\eta_n$; the upper bound is proved similarly. Convergence of Laplace transforms is actually the condition imposed by Li (2006) to prove limit theorems for Galton–Watson processes with immigration. That this already follows from our hypotheses is the content of the following lemma, which concludes the proof of Corollary 7. □

LEMMA 7. *Let X^n be a sequence of random walks with jumps in $\{-1, 0, 1, \dots\}$ satisfying the conditions of Corollary 7, and suppose that X is not a subordinator. Then*

$$\mathbb{E}\big(e^{-\lambda/n X^n_{a_n}}\big) \to e^{\Psi(\lambda)}$$

for all $\lambda > 0$.

This is the content of Theorem 2.1 of Grimvall (1974); we present a proof using basic fluctuation theory for independent increment processes.

PROOF OF LEMMA 7. Using Skorohod's theorem again, we assume that $X^n_{\lfloor a_n \cdot \rfloor}/n$ converges almost surely to X in the Skorohod J_1 topology. Also, enlarge the probability space so that it admits an exponential random variable R_λ of parameter λ which is independent of X and X^n.

Let

$$G_n(\lambda) = \mathbb{E}\big(e^{-\lambda X^n_1}\big).$$

Since X is not a subordinator, then $\mathbb{P}(X^n_1 = -1) > 0$ for large enough n, and we can assume that this happens for every n. Hence, $G_n(\lambda) \to \infty$ as $\lambda \to \infty$, and we can define

$$F_n(s) = \inf\{\lambda > 0 : G_n(\lambda) > 1/s\} \qquad \text{for } s \in (0, 1].$$

Using optional sampling at the first time T^n_k at which X^n reaches $-k$ for the first time, applied to the martingale

$$e^{-\lambda X^n_m} G_n(\lambda)^{-m},$$

we obtain

$$\mathbb{E}\big(s^{T^n_k}\big) = e^{-k F_n(s)}$$

for $s \in (0, 1]$. Define the random variables

$$I_\lambda = \min_{s \le R_\lambda} X_s \quad \text{and} \quad I^n_\lambda = \min_{s \le R_\lambda} \frac{1}{n} X^n_{\lfloor a_n s \rfloor}.$$

Since $\lfloor a_n R_\lambda \rfloor$ has a geometric distribution of parameter $e^{-\lambda/a_n}$, it follows that

$$\mathbb{P}\big(-n I^n_\lambda \ge k\big) = \mathbb{P}\big(T^n_k < \lfloor a_n R_\lambda \rfloor\big) = \mathbb{E}\big(e^{-\lambda/a_n T^n_k}\big) = e^{-k F_n(e^{-\lambda/a_n})}$$

so that $-nI^n_\lambda$ has a geometric distribution. Also, from Corollary 2 in Bertoin [(1996), Chapter VII], I_λ has an exponential distribution of parameter $\tilde{\Phi}(\lambda)$ where

$$\tilde{\Phi}(\lambda) = \inf\{\tilde{\lambda} > 0 : \Psi(\tilde{\lambda}) > \lambda\}.$$

However, since X does not jump almost surely at R_λ and the minimum is a continuous functional on Skorohod space (on the interval $[0, R_\lambda]$), we see that I^n_λ converges weakly to I_λ. This implies

$$nF_n(e^{-\lambda/a_n}) \to \tilde{\Phi}(\lambda),$$

and by passing to inverses, we get

$$G_n(\lambda/n)^{a_n} \to e^{\Psi(\lambda)}$$

for $\lambda > \tilde{\Phi}(0)$.

Finally, if $\lambda \in (0, \tilde{\Phi}(0)]$, pick $p > 1$ such that $p\lambda > \tilde{\Phi}(0)$; we have just proved that the sequence

$$G_n(p\lambda/n)^{a_n}, \qquad n \geq 1,$$

and being convergent, it is bounded. Hence the sequence

$$e^{-\lambda/nX^n_{a_n}}, \qquad n \geq 1,$$

is bounded in L_p and converges weakly to $e^{-\lambda X_1}$. We then get

$$G_n(\lambda/n)^{a_n} = \mathbb{E}(e^{-\lambda/nX^n_{a_n}}) \to \mathbb{E}(e^{-\lambda X_1}) = e^{\Psi(\lambda)}. \qquad \square$$

4.4. *A limit theorem for conditioned Galton–Watson processes.*

PROOF OF THEOREM 4. Let Z^n be a Galton–Watson process with critical offspring law μ such that $Z^n_0 = k_n$ and is conditioned on $\sum_{i=1}^\infty Z^n_i = n$. Then, Z^n has the law of the discrete Lamperti transformation of the n steps of a random walk with jump distribution $\tilde{\mu}$ (the shifted reproduction law) which starts at 0 and is conditioned to reach $-k_n$ in n steps; call the latter process X^n, so that

$$Z^n = h^1(k_n + X^n, 0).$$

Thanks to Chaumont and Pardo (2009), if $k_n/a_n \to l$, then

$$S^{a_n}_n X^n \to F^l.$$

Thanks to Lemma 6, we see that

$$S^{a_n}_{n/a_n} Z^n = h^{a_n/n}(S^{a_n}_n X^n, 0).$$

Let $\alpha \in (1, 2]$ be the index of the stable process in the statement of Theorem 4, and recall that a_n is of the form $n^{1/\alpha}L(n)$ where L is a slowly varying function, so that $a_n = o(n)$. Since F^l is absorbed at zero [as is easily seen by the pathwise construction of F^l by Chaumont and Pardo (2009), Theorem 4.3], then Proposition 1 guarantees that the Lamperti transform Z of F^l is the unique process which satisfies

$$\int_s^t F^l_{\int_0^r Z_u\,du-} \leq \int_s^t Z_r\,dr \leq \int_s^t F^l_{\int_0^r Z_u\,du}.$$

Theorem 3 implies that

$$S^{a_n}_{n/a_n} Z^n \to Z. \qquad \square$$

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