Bernoulli **20**(1), 2014, 190–206
DOI: 10.3150/12-BEJ481

Bridges of Lévy processes conditioned to stay positive

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We consider Kallenberg's hypothesis on the characteristic function of a Lévy process and show that it allows the construction of weakly continuous bridges of the Lévy process conditioned to stay positive. We therefore provide a notion of normalized excursions Lévy processes above their cumulative minimum. Our main contribution is the construction of a continuous version of the transition density of the Lévy process conditioned to stay positive by using the weakly continuous bridges of the Lévy process itself. For this, we rely on a method due to Hunt which had only been shown to provide upper semi-continuous versions. Using the bridges of the conditioned Lévy process, the Durrett–Iglehart theorem stating that the Brownian bridge from 0 to 0 conditioned to remain above $-\varepsilon$ converges weakly to the Brownian excursion as $\varepsilon \to 0$, is extended to Lévy processes. We also extend the Denisov decomposition of Brownian motion to Lévy processes and their bridges, as well as Vervaat's classical result stating the equivalence in law of the Vervaat transform of a Brownian bridge and the normalized Brownian excursion.

Keywords: Lévy processes; Markovian bridges; Vervaat transformation

1. Introduction and statement of the results

Our discussion will use the canonical setup: $X = (X_t)_{t \geq 0}$ denotes the canonical process on the Skorohod space of càdlàg trajectories, $\mathscr{F}$ denotes σ-field generated by X (also written $\sigma(X_s, s \geq 0)$), $(\mathscr{F}_t, t \geq 0)$ is the canonical filtration, where $\mathscr{F}_t = \sigma(X_s, s \leq t)$ and $\theta_t, t \geq 0$, are the shift operators given by $\theta_t(\omega)_s = \omega_{t+s}$. Emphasis is placed on the various probability measures considered.

Focus will be placed on two special (Markovian) families of probability measures, denoted $(\mathbb{P}_x, x \in \mathbb{R})$ and $(\mathbb{P}^{\uparrow}_x, x \geq 0)$. The probability measure $\mathbb{P}_0$ corresponds to the law of a Lévy process: under $\mathbb{P}_0$ the canonical process has independent and stationary increments and starts at 0. Then $\mathbb{P}_x$ is simply the law of $x + X$ under $\mathbb{P}_0$, and under each $\mathbb{P}_x$ the canonical process is Markov and the conditional law of $(X_{t+s}, s \geq 0)$ given $\mathscr{F}_t$ is $\mathbb{P}_{X_t}$. (Collections of probability measures on Skorohod space satisfying the latter property are termed Markovian families.) We also make use of the dual Lévy process by letting $\hat{\mathbb{P}}_x$ denote the law of $x - X$ under $\mathbb{P}_0$. Associated to $\mathbb{P}_x, x \in \mathbb{R}$, $\mathbb{P}^{\uparrow}_x$ can be interpreted as the law of the Lévy process conditioned to stay positive; as this event can have probability zero, the precise definition of $\mathbb{P}^{\uparrow}_x$ can be described as follows: a Lévy process conditioned to stay positive is the (weak) limit of X conditioned to stay positive until an independent exponential T_α of parameter α as $\alpha \to 0$ (cf. Chaumont and Doney [5], Proposition 1). It is actually simpler to actually construct Lévy processes conditioned to stay

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positive by a Doob transformation and justifying this passage to the limit afterwards, as recalled in Section 4.

Under very general conditions, given a Markovian family of probability laws like $(\mathbb{P}_x, x \in \mathbb{R})$, one can construct weakly continuous versions of the conditional laws of $(X_s, s \le t)$ under $\mathbb{P}_x$ given $X_t = y$. They are termed bridges of $\mathbb{P}_x$ between x and y of length t and usually denoted $\mathbb{P}^t_{x,y}$. In Section 2, we review the construction of these bridges from Chaumont and Uribe Bravo [9]. Our first result is to show that one can apply this general recipe to the laws $\mathbb{P}^{\uparrow}_x$. To this end, we impose two conditions on the Lévy process.

(K) Under $\mathbb{P}_0$ and for any $t > 0$, $\int |\mathbb{E}_0(\mathrm{e}^{\mathrm{i}uX_t})|\, \mathrm{d}u < \infty$.
(R) 0 is regular for both half-lines $(-\infty, 0)$ and $(0, \infty)$.

Assumption (K) was introduced by Kallenberg as a means of imposing the existence of densities for the law of X_t for any $t > 0$ which posses good properties (in particular continuity). A construction of Lévy process bridges under hypothesis (K) was first accomplished in Kallenberg [16] by means of convergence criteria for processes with exchangeable increments. This construction is retaken as an example of the general construction of Markovian bridges in Chaumont and Uribe Bravo [9].

Theorem 1. *Under* (K) *and* (R), *we can construct bridges* $\mathbb{P}^{\uparrow,t}_{x,y}$ *of* $\mathbb{P}^{\uparrow}_x$ *for any* $x, y \ge 0$ *and* $t > 0$ *and that they are weakly continuous as functions of* x *and* y.

Theorem 1 presents another example of the applicability of Theorem 1 in Chaumont and Uribe Bravo [9], and the proof of the former consists on verifying the technical hypotheses in the latter. These technical hypotheses are basically: the existence of a continuous and positive version of the densities of X_t under $\mathbb{P}^{\uparrow}_x$. For nonzero starting states, we will inherit absolute continuity from that of the Lévy process killed upon becoming negative (in Lemma 3) and the later can be studied by a technique inspired from Hunt [15] for the Brownian case, in which a transition density for the killed Lévy process is obtained from a transition density of the Lévy process using its bridges. (Cf. Equation (5.1) and Lemma 2.) Hunt's technique has typically allowed only the construction of lower semicontinuous versions of the density, but with weakly continuous bridges one can show that Hunt's density is actually continuous. This is one possible application of the existence of weakly continuous Markovian bridges. Another problem is then to characterize the points at which the density is positive. Hunt does this for Brownian motion and the result has been extended to (multidimensional) stable Lévy processes in the symmetric case by Chen and Song [10], Theorem 2.4 and in the asymmetric case by Vondraček [24], Theorem 3.2. We study positivity of the density by exploiting the cyclic exchangeability property of Lévy processes, following Knight [17].

Recall that when $\mathbb{P}_0$ is the law of Brownian motion, $\mathbb{P}^t_{x,y}$ is the law of the Brownian bridge between x and y of length t and the corresponding law $\mathbb{P}^{\uparrow,t}_{0,0}$ is the law of a Brownian excursion of length t. In this context, our next result is an extension to Lévy processes of the classical result of Durrett, Iglehart and Miller [13], which covers the Brownian case.

Corollary 1. *The conditional law of* $(X_s, s \leq t)$ *under* $\mathbb{P}^t_{0,0}$ *given* $\underline{X}_t > -\varepsilon$, *where*

$$\underline{X}_t = \inf_{s \leq t} X_s,$$

converges weakly, as $\varepsilon \to 0$ *to* $\mathbb{P}^{\uparrow,t}_{0,0}$.

The Brownian case of Corollary 1 was first proved by Durrett, Iglehart and Miller [13] by showing the convergence of finite-dimensional distributions and then tightness, which follows from explicit computations with Brownian densities. Another proof for the Brownian case was given by Blumenthal [3] this time using rescaling, random time change and simple infinitesimal generator computations. For us, Corollary 1 is a simple consequence of Theorem 1.

We now present a generalization of a decomposition of the Brownian trajectory at the time it reaches its minimum on a given interval due to Denisov [11].

Let ρ_t be the first time that X reaches its minimum on the interval $[0, t]$. Consider the pre- and post-minimum processes on the interval $[0, t]$ given by:

$$X^{\leftarrow}_s = X_{(\rho_t - s)^+ -} - \underline{X}_t \quad \text{and} \quad X^{\rightarrow}_s = X_{(\rho_t + s) \wedge t} - \underline{X}_t$$

defined for $s \geq 0$, where X_{s-} is the left limit of X at s.

A Lévy meander of length t (following Chaumont and Doney [7]) is the weak limit as $\varepsilon \to 0$ of X conditioned to remain above $-\varepsilon$ on $[0, t]$ under $\mathbb{P}_0$. Lévy meanders can also be characterized by an absolute continuity relationship with Lévy processes conditioned to stay positive as recalled in Section 4. Denote by $\mathbb{P}^{\mathrm{me},t}$ the law of a meander of length t and by $\hat{\mathbb{P}}^{\mathrm{me},t}$ the meander of the dual Lévy process.

Theorem 2. *Assume conditions* (K) *and* (R). *Under* $\mathbb{P}_0$, *the conditional law of* $(X^{\leftarrow}, X^{\rightarrow})$ *given* ρ_t *is* $\hat{\mathbb{P}}^{\mathrm{me},\rho_t} \otimes \mathbb{P}^{\mathrm{me},t-\rho_t}$.

The previous result is a consequence of results in Chaumont and Doney [5]. It is our stepping stone on the way to our generalization of Vervaat's relationship between the Brownian bridge and the normalized Brownian excursion. This extension requires the following conditioned version of Theorem 2.

Theorem 3. *A regular conditional distribution of* $(X^{\leftarrow}, X^{\rightarrow})$ *given* $\rho_t = s$ *and* $-\underline{X}_t = y$ *under* $\mathbb{P}^t_{0,0}$ *is* $\hat{\mathbb{P}}^{\uparrow,s}_{0,y} \otimes \mathbb{P}^{\uparrow,t-s}_{0,y}$.

We finally turn to an extension of the classical relationship between the Brownian bridge between 0 and 0 and the Brownian excursion of the same length.

Theorem 4. *Define the Vervaat transformation* V *of* X *on* $[0, t]$ *by*

$$V_s = X_{(\rho_t + s) \bmod t} - \underline{X}_t.$$

Under (K) *and* (R), *the law of* V *under* $\mathbb{P}^t_{0,0}$ *is* $\mathbb{P}^{\uparrow,t}_{0,0}$.

Theorem 4 was found by Vervaat [23] for Brownian motion and proved there using approximation by a simple random walk. Biane [2] gives a proof using excursion theory for Brownian motion. Then Chaumont [4] gave a definition of normalized stable excursion and proved Theorem 4 in the case of stable Lévy processes, again using excursion theory. This extension of Vervaat's theorem is the closest to the one in this work. Miermont [18] gives a version of Theorem 4 for spectrally positive Lévy processes in the context of the intensity measures for excursions above the cumulative minimum, with an explicit link with the Lévy process conditioned to stay positive. Finally, Fourati [14] gives an abstract version of Theorem 4 for Lévy processes, again as a relation between two σ-finite measures which can be though of as bridges of random length, although there is no explicit link with Lévy processes conditioned to stay positive. After establishing this link, Theorem 4 would follow from the theory developed in Fourati [14] using regularity results for bridges (like weak continuity) in order to condition by the length. Instead of that, we propose a direct proof.

The paper is organized as follows. In Sections 2 and 3, we review the construction of Markovian and Lévy bridges of Chaumont and Uribe Bravo [9]. In Section 4, we define, following Chaumont and Doney [5], Lévy processes conditioned to stay positive and meanders. Section 5 is devoted to the construction of bridges of Lévy processes conditioned to stay positive, where we prove Theorem 1 and Corollary 1. In Section 6, we consider extensions and consequences of Denisov's theorem, proving in particular Theorems 2 and 3. Finally, in Section 7, we prove our extension of Vervaat's theorem, which is Theorem 4.

2. Weakly continuous bridges of Markov processes

Let $\mathbf{P}_x$ be the law of a Feller process which starts at x which is an element of a polish space S (for us either $\mathbb{R}$, $(0,\infty)$, or $[0,\infty)$). Suppose P is its semigroup and assume that:

(AC) There is a σ-finite measure μ and a function $h_t(x,\cdot)$ such that

$$P_t f(x) = \int f(y) h_t(x,y) \mu(\mathrm{d}y).$$

(C) The function $(s,x,y) \mapsto h_s(x,y)$ is continuous.

(CK) The Chapman–Kolmogorov equations

$$h_{s+t}(x,z) = \int h_s(x,y) h_t(y,z) \mu(\mathrm{d}y)$$

are satisfied.

Let us denote by $B_\delta(y)$ the ball of radius δ centered at y and

$$\mathscr{P}_{x,t} = \{ y \in S: \; h_t(x,y) > 0 \}.$$

Theorem 5 (Chaumont and Uribe Bravo [9]). *Under* (AC), (C) *and* (CK), *the law of* X *on* $[0,t]$ *under* $\mathbf{P}_x$ *given* $X_t \in B_\delta(y)$ *converges weakly in the Skorohod* J_1 *topology to a measure* $\mathbf{P}^t_{x,y}$ *for every* $y \in \mathscr{P}_{x,t}$. *Furthermore*:

1. *The family* $\{\mathbf{P}^t_{x,y}\colon\ y \in \mathscr{P}_{x,t}\}$ *is a regular conditional distribution for* X *on* $[0,t]$ *given* X_t *under* $\mathbf{P}_x$.
2. *The finite-dimensional distributions of* $\mathbf{P}^t_{x,y}$ *are given by*

$$\mathbf{P}^t_{x,y}(X_{t_1} \in \mathrm{d}x_1, \dots, X_{t_n} \in \mathrm{d}x_n)$$
$$= h_{t_1}(x,x_1)h_{t_2}(x_1,x_2)\cdots h_{t_n - t_{n-1}}(x_{n-1},x_n)\frac{h_{t-t_n}(x_n,y)}{h_t(x,y)}\,\mathrm{d}x_1\cdots \mathrm{d}x_n.$$

3. *As* $y' \to y$ *and* $x' \to x$, $\mathbf{P}^t_{x',y'}$ *converges weakly to* $\mathbf{P}^t_{x,y}$.

Remark. The finite-dimensional distributions of the bridge laws can be written succinctly using the following local absolute continuity condition valid for $s < t$:

$$\mathbf{P}^t_{x,t}|_{\mathscr{F}_s} = \frac{h_{t-s}(X_s,y)}{h_t(x,y)}\cdot \mathbf{P}_x|_{\mathscr{F}_s}. \tag{2.1}$$

The reasoning in Revuz and Yor [21], Chapter VIII, implies that for any stopping time T taking values in $[0,t)$:

$$\mathbf{P}^t_{x,t}|_{\mathscr{F}_T} = \frac{h_{t-T}(X_T,y)}{h_t(x,y)}\cdot \mathbf{P}_x|_{\mathscr{F}_T}. \tag{2.2}$$

3. Lévy processes and their bridges

Let $\mathbb{P}_x$ be the Markovian family of a Lévy process which satisfies assumptions (K) and (R). As argued by Kallenberg [16], Fourier inversion implies that X_t possesses continuous and bounded densities which vanish at infinity (by the Riemann-Lebesgue lemma) for all $t > 0$. Actually, (K) also implies that the continuous version f_t of the density of X_t under $\mathbb{P}_0$ satisfies a form of the Chapman–Kolmogorov equations:

$$f_t(x) = \int f_s(y) f_{t-s}(x-y)\,\mathrm{d}y \qquad \text{for } 0 < s < t. \tag{3.1}$$

From f_t one can build a bi-continuous transition density p_t by means of $p_t(x,y) = f_t(y-x)$ which satisfies (AC), (C) and (CK).

Under hypotheses (K) and (R), Sharpe [22] shows that f_t is strictly positive for all $t > 0$, which implies that $p_t > 0$.

From Theorem 5, we see that under (K) and (R), we can consider the bridges $\mathbb{P}^t_{x,y}$ from x to y of length t for any $x, y \in \mathbb{R}$ and any $t > 0$, and that these are jointly weakly continuous in x and y.

4. Lévy processes conditioned to stay positive and meanders

The most general construction for Lévy processes conditioned to stay positive, now recalled, is from Chaumont and Doney [5] (see Chaumont and Doney [6] for a correction and Doney [12] for a lecture note presentation). When the initial state is positive, it is a Doob transformation of $\mathbb{P}_x$ by a procedure we now detail. Let

$$\underline{X}_t = \min_{s \le t} X_s$$

and consider the Markov process $R = X - \underline{X}$. Under (R), 0 is regular state of R for itself and so we can consider the local time at zero L of R. We can then define the downwards ladder height process H of X by

$$H = -X \circ L,$$

which is a (possibly killed) subordinator (cf. Bertoin [1] or Doney [12]). Let h be the renewal function of H given by

$$h(x) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{H_s \le x} \, \mathrm{d}s\right).$$

For $x > 0$, let $\mathbb{Q}_x$ be the law of $x + X$ under $\mathbb{P}$ killed when it leaves $(0, \infty)$, which is a Markov process on $(0, \infty)$ whose semigroup is denoted $Q = (Q_t, t \ge 0)$. Chaumont and Doney [5] prove that if X drifts to $-\infty$ ($\lim_{t\to\infty} X_t = -\infty$ almost surely) then h is excessive and otherwise h is invariant for Q_t and proceed to define the semigroup $P_t^{\uparrow}$ by

$$P_t^{\uparrow}(x, \mathrm{d}y) = \frac{h(y)}{h(x)} Q_t(x, \mathrm{d}y) \qquad \text{for } x > 0.$$

The Markovian laws $\mathbb{P}_x^{\uparrow}$, $x > 0$, define the Lévy process conditioned to stay positive. Note that X has finite lifetime under $\mathbb{P}_x^{\uparrow}$ if and only if X drifts to $-\infty$ under $\mathbb{P}_x$. Under hypothesis (R), Chaumont and Doney [5] prove that $\mathbb{P}_x^{\uparrow}$ has a weak limit (in the Skorhod J_1 topology) as $x \to 0$, denoted $\mathbb{P}_0^{\uparrow}$, and that $(\mathbb{P}_x^{\uparrow})_{x \ge 0}$ is Markovian and has the Feller property.

We now give an alternate definition of the meander, from which one can justify the weak limit construction we have alluded to (cf. Chaumont and Doney [7], Lemma 4). A Lévy meander is a stochastic process whose law $\mathbb{P}^{\mathrm{me},t}$ satisfies the following absolute continuity relationship with respect to the law $\mathbb{P}_0^{\uparrow}$ on $\mathscr{F}_t = \sigma(X_s \colon s \le t)$:

$$\mathbb{P}^{\mathrm{me},t}|_{\mathscr{F}_t} = \frac{1}{\beta_t h(X_t^{\uparrow})} \cdot \mathbb{P}_0^{\uparrow}|_{\mathscr{F}_t} \qquad \text{with } \beta_t = \mathbb{E}_0^{\uparrow}\left(\frac{1}{h(X_t^{\uparrow})}\right).$$

5. Bridges of Lévy processes conditioned to stay positive

We now construct the bridges of a Lévy process conditioned to stay positive under hypotheses (K) and (R). This is done through Theorem 5 by verifying the existence of a continuous version of their densities (cf. Lemma 3). For positive arguments, the density is constructed from the density of the killed Lévy process, a continuous version of which is constructed using bridges of the Lévy process itself in Lemma 2. Then, a delicate point is to study the densities at 0; this requires the following duality lemma. Let $\hat{\mathbb{P}}_x$ be the law of $x - X$ and let $\hat{P}_t$ be its semigroup. We can also consider the objects $\hat{h}$, $\hat{\mathbb{P}}_x^{\uparrow}$, etc... associated with $-X$ instead of X as well as $\hat{p}$.

Lemma 1. *The semigroups* $P_t^{\uparrow}$ *and* $\hat{P}_t^{\uparrow}$ *are in duality with respect to the measure* $\lambda^{\uparrow}$ *given by*

$$\lambda^{\uparrow}(\mathrm{d}x) = h(x)\hat{h}(x)\,\mathrm{d}x.$$

Proof. Since P_t and $\hat{P}_t$ are in duality with respect to Lebesgue measure λ, it follows that Q_t and $\hat{Q}_t$ are also in duality with respect to λ.

Hence, we get

$$\int f P_t^{\uparrow}(g)\,\mathrm{d}\lambda^{\uparrow} = \int f \frac{Q_t(gh)}{h} h\hat{h}\,\mathrm{d}\lambda = \int \hat{Q}_t(f\hat{h})gh\,\mathrm{d}\lambda = \int \hat{P}_t^{\uparrow}(f)gh\hat{h}\,\mathrm{d}\lambda$$
$$= \int \hat{P}_t^{\uparrow}(f)g\,\mathrm{d}\lambda^{\uparrow}.$$
□

We now consider the absolute continuity of the semigroup of X killed when it becomes negative.

Lemma 2. *Under* (K) *and* (R), *let*

$$q_t(x,y) = \mathbb{E}^t_{x,y}(\underline{X}_t > 0)p_t(x,y) \qquad \text{for } x, y > 0. \tag{5.1}$$

Then q_t *is a transition density for* Q_t *with respect to Lebesgue measure which is continuous, strictly positive, bounded by* p, *satisfies the Chapman–Kolmogorov equations, and which satisfies the following duality formula*:

$$q_t(x,y) = \hat{q}_t(y,x).$$

Remark. It is simple to see that the absolute continuity of $P_t(x,\cdot)$ translates into absolute continuity of $Q_t(x,\cdot)$ since if A has Lebesgue measure zero then

$$Q_t(x,A) = \mathbb{P}_x(X_t \in A, \underline{X}_t > 0) \le \mathbb{P}_x(X_t \in A) = 0.$$

What is more difficult, is to see that the q is strictly positive; similar results have been obtained in the literature for killed (multidimensional) Brownian motion and stable Lévy processes in Hunt [15], Chen and Song [10], Vondraček [24]. Our proof of uses the weakly continuous Markovian bridges provided by Theorem 5. The almost sure positivity of q can also be obtained from Theorem 4 of Pitman and Uribe Bravo [20].

Proof of Lemma 2. Conditioning on X_t, we see that

$$\mathbb{E}_x\big(\mathbf{1}_{\underline{X}_t>0} f(X_t)\big) = \mathbb{E}_x\big[\mathbb{P}^t_{x,X_t}(\underline{X}_t > 0) f(X_t)\big]$$
$$= \int \mathbb{P}^t_{x,y}(\underline{X}_t > 0) f(y) p_t(x,y)\lambda(\mathrm{d}y)$$

for measurable and bounded f. On the other hand, the definition of the law $\mathbb{Q}_t$ gives

$$\mathbb{E}_x\big(\mathbf{1}_{\underline{X}_t>0} f(X_t)\big) = \mathbb{Q}_x\big(f(X_t)\big) = \int f(y) Q_t(x,\mathrm{d}y)$$

so that q is a transition density of Q with respect to Lebesgue measure.

We know that p is continuous. To see that q is continuous, it suffices to apply the portemanteau theorem. Note that the boundary $\partial\{\underline{X}_t > 0\}$ of $\{\underline{X}_t > 0\}$ has $\mathbb{P}^t_{x,y}$-measure zero. Indeed, since the minimum on $[0,t]$ is a continuous functional on Skorohod space (cf. Whitt [25], Section 13.4):

$$\partial\{\underline{X}_t > 0\} \subset \big\{X_s \geq 0 \text{ for all } s \in [0,t] \text{ and there exists } s \in [0,t] \text{ such that } X_s = 0\big\}.$$

Since $x, y > 0$ and under $\mathbb{P}^t_{x,y}$ we have $X_{0+} = x$ and $X_{t-} = y$ almost surely, we see that the process cannot touch zero at times 0 or t. However, using the local absolute continuity relationship (2.2) at the first time T such that $X_T = 0$, we see that $\mathbb{P}^t_{x,y}(\partial\{\underline{X}_t > 0\} = 0)$ as soon as

$$\mathbb{P}_x\big(\text{Touching zero on } (0,t) \text{ and staying nonnegative}\big) = 0,$$

which is true since 0 is regular for $(-\infty, 0)$.

To prove the duality formula for q, we first Proposition II.1 of Bertoin [1], which proves that $p_t(x,y) = \hat{p}_t(y,x)$ for almost all x and y and remove the almost all qualifier by continuity. Next, Corollary II.3 of Bertoin [1] proves that for almost all x and y the image of $\mathbb{P}^t_{x,y}$ under the time reversal operator is $\hat{\mathbb{P}}^t_{y,x}$, which by weak continuity of bridge laws can be extended to every x and y. Since the event $\underline{X}_t$ is invariant under time reversal, we see that

$$q_t(x,y) = \mathbb{E}^t_{x,y}(\underline{X}_t > 0) p_t(x,y) = \hat{\mathbb{E}}^t_{y,x}(\underline{X}_t > 0) \hat{p}_t(y,x) = \hat{q}_t(y,x).$$

By definition, we see that $q \leq p$ almost everywhere, and so continuity implies that q is bounded by p everywhere; this will help us prove that q satisfies the (CK) equations. Indeed, the Markov property implies that

$$q_{t+s}(x,z) = \int q_s(x,y) q_t(y,z) \lambda(\mathrm{d}y) \qquad \text{for } \lambda\text{-almost all } z. \tag{5.2}$$

Since

$$0 \leq q_s(x,y) q_t(y,z) \leq p_s(x,y) p_t(y,z)$$

and

$$\int p_s(x,y) p_t(y,z) \lambda(\mathrm{d}z) = p_{t+s}(x,z),$$

which is continuous in z, the generalized dominated convergence theorem tells us that

$$z \mapsto \int q_s(x,y) q_t(y,z) \lambda(\mathrm{d}y)$$

is continuous (on $(0,\infty)$). Because both sides of (5.2) are continuous, we can change the almost sure qualifier to for all z.

It remains to see that $q_t(x,y) > 0$ if $x, y, t > 0$. We first prove that for any $x, y, t > 0$, if $\delta > 0$ is such that $B_\delta(y) \subset (0,\infty)$, then

$$Q_t\big(x, B_\delta(y)\big) > 0. \tag{5.3}$$

This is done by employing a technique of Knight [17]. For any $s \in (0,t)$, consider the process $(X^s_r, r \le t)$ given by

$$X^s_r = X_0 + X_{(r+s) \bmod t} - X_s.$$

Since X has independent and stationary increments, then, for any fixed s, the laws of X^s and $(X_r, r \le t)$ coincide under $\mathbb{P}_x$ for any $x \in \mathbb{R}$; this is referred to as the cyclic exchangeability property in Chaumont, Hobson and Yor [8]. Note that $X^s_t = X_t$; if s is close to the place where X reaches its minimum on $(0,t)$, then the minimum $\underline{X}^s_t$ of X^s on the interval $[0,t]$ is positive. Hence, the random variable

$$I = \int_0^t \mathbf{1}_{\underline{X}^s_t > 0, X_t \in B_\delta(y)} \, \mathrm{d}s$$

is positive on $\{X_t \in B_\delta(y)\}$ which has positive probability since p_t is strictly positive. On the other hand, from cyclic exchangeability, we can compute:

$$0 < \mathbb{E}_x(I) = \int_0^t \mathbb{P}_x\big[\underline{X}^s_t > 0, X^s_t \in B_\delta(y)\big] \, \mathrm{d}s = t\mathbb{P}_x\big[\underline{X}_t > 0, X_t \in B_\delta(y)\big] = tQ_t\big(x, B_\delta(y)\big),$$

which proves (5.3). To prove positivity of q_t, first note that since $\hat{\mathbb{P}}_y$ almost surely $X_{0+} = y$, then for s small enough:

$$\hat{Q}_s\big(y, B_\delta(y)\big) = \hat{\mathbb{P}}_y\big(X_s \in B_\delta(y), \underline{X}_s > 0\big) \ge \hat{\mathbb{P}}_y\big(X_r \in B_\delta(y) \text{ for all } r \in [0,s]\big) > 0,$$

so that, by continuity of q_s, there exists an open subset U_s of $B_\delta(y)$ such that $q_s(\cdot, y) = \hat{q}_s(y, \cdot) > 0$ on U_s. By Chapman–Kolmogorov and (5.3), we see that

$$q_t(x,y) \ge \int_{U_s} Q_{t-s}(x, \mathrm{d}z) q_s(z,y) > 0. \qquad \square$$

We now turn to a similar result for Lévy processes conditioned to stay positive.

Lemma 3. *Under* (K) *and* (R), $P^\uparrow_t(x,\cdot)$ *is equivalent to Lebesgue measure for all* $t > 0$ *and* $x \ge 0$. *Furthermore, there exists a version of the transition density* $p^\uparrow$ *which is continuous, strictly positive, and satisfies the Chapman–Kolmogorov equations.*

Therefore, the density $p^\uparrow$ satisfies the assumptions (AC), (C) and (CK) of Theorem 5.

Proof of Lemma 3. Since the renewal function of a subordinator is positive, continuous and finite, we deduce by h-transforms and Lemma 2 that the function

$$p^\uparrow_t(x,y) = \frac{q_t(x,y)}{h(x)\hat{h}(y)}, \qquad x > 0, y > 0, t > 0,$$

is a transition density for $P^\uparrow$ starting at positive states:

$$P^\uparrow_t f(x) = \int p^\uparrow_t(x,y) f(y) \lambda^\uparrow(\mathrm{d}y) \qquad \text{for } x > 0.$$

Notice that $p^{\uparrow}$ so defined is strictly positive, continuous, and satisfies the Chapman–Kolmogorov equations.

For $0 < s < t$, consider the function

$$p_t^{\uparrow s}(y) = \int P_s^{\uparrow}(0, \mathrm{d}x) p_{t-s}^{\uparrow}(x, y) > 0 \qquad \text{for } y > 0.$$

On one hand, Chapman–Kolmogorov implies that for any bounded measurable f:

$$\int p_t^{\uparrow s}(y) f(y) \lambda^{\uparrow}(\mathrm{d}y) = \iint P_s^{\uparrow}(0, \mathrm{d}x) p_{t-s}^{\uparrow}(x, y) f(y) \lambda^{\uparrow}(\mathrm{d}y) = \int P_t^{\uparrow}(0, \mathrm{d}y) f(y),$$

so that $p_t^{\uparrow s}$ is a version of the density of $P_t^{\uparrow}(0, \cdot)$ with respect to $\lambda^{\uparrow}$ and so if $0 < s < s' < t$ then $p_t^{\uparrow s}(y) = p_t^{\uparrow s'}(y)$ for λ-almost all y. On the other hand, we now see that $p_t^{\uparrow s}(y)$ is a continuous function of y, so that actually, the almost sure qualifier can be dropped. Indeed, since

$$M_{t-s} := \sup_{x,y} p_{t-s}(x, y) < \infty$$

and from Chaumont and Doney [5]

$$\beta_s := \mathbb{P}_0^{\uparrow}\big(1/h(X_s)\big) < \infty,$$

continuity of $p_t^{\uparrow s}$ follows from the dominated convergence theorem.

We can now define

$$p_t^{\uparrow}(0, y) = p_t^{\uparrow s}(y), \qquad y > 0,$$

for any $s \in (0, t)$. Since $p_t^{\uparrow}(0, y)$ is continuous for $y \in (0, \infty)$, and is a version of the density of $P_t^{\uparrow}(0, \cdot)$, the Markov property implies:

$$p_{t+s}^{\uparrow}(0, y) = \int p_t^{\uparrow}(0, x) p_s^{\uparrow}(x, y) \lambda^{\uparrow}(\mathrm{d}x).$$

Furthermore, we have the bound

$$p_t^{\uparrow}(0, y) \le \beta_s M_{t-s} / \hat{h}(y) \qquad \text{for } y > 0.$$

We now provide an uniform bound for the transition density in the initial state. Recall that $p_t(x, y) \to 0$ as $x \to \infty$. Since $q_t \le p_t$, then $p_t^{\uparrow}(x, y) \to 0$ as $x \to \infty$ for any $t > 0$ and $y > 0$. Choose now any $s \in (0, t)$. By Chapman–Kolmogorov:

$$\begin{aligned}
p_t^{\uparrow}(x, z) &= \int p_s^{\uparrow}(x, y) p_{t-s}^{\uparrow}(y, z) \lambda^{\uparrow}(\mathrm{d}y) \\
&\le \int p_s^{\uparrow}(x, y) \frac{M_{t-s}}{h(y)\hat{h}(z)} \lambda^{\uparrow}(\mathrm{d}y) \\
&\le \mathbb{E}_x^{\uparrow}\left(\frac{1}{h(X_s)}\right) \frac{M_{t-s}}{\hat{h}(z)}.
\end{aligned}$$

Note that $x \mapsto \mathbb{E}_x^{\uparrow}(\frac{1}{h(X_s)})$ is continuous on $(0, \infty)$, hence bounded on compact subsets of $(0, \infty)$. Continuity at zero is proved in Corollary 1 of Chaumont and Doney [6]. Hence, we obtain

$$\sup_{x \geq 0} p_t^{\uparrow}(x, y) < \infty \qquad \text{for all } y > 0 \text{ and } t > 0.$$

We now prove that

$$\lim_{x \to 0} p_t^{\uparrow}(x, y) = p_t^{\uparrow}(0, y) \qquad \text{for } y > 0. \tag{5.4}$$

Indeed, from the Chapman–Kolmogorov equations

$$p_t^{\uparrow}(x, z) = \int p_{t-s}^{\uparrow}(y, z) P_s^{\uparrow}(x, \mathrm{d}y).$$

Note that $P_s^{\uparrow}(x, \cdot)$ converges weakly to $P_s^{\uparrow}(0, \cdot)$ as $x \to 0$ and that $p_{t-s}^{\uparrow}(\cdot, z)$ is continuous and bounded on $(0, \infty)$, which is the support of $P_s^{\uparrow}(0, \cdot)$.

By applying the above arguments to the dual process, we can define $p_t^{\uparrow}(x, 0)$ as $\hat{p}_t^{\uparrow}(0, x)$ and note that

$$\lim_{y \to 0} p_t^{\uparrow}(x, y) = p_t^{\uparrow}(x, 0) \qquad \text{for } x > 0.$$

We can now define

$$p_t^{\uparrow, s}(0, 0) = \int p_s^{\uparrow}(0, y) p_{t-s}^{\uparrow}(y, 0) \lambda^{\uparrow}(\mathrm{d}y).$$

To show that the above definition does not depend on s, we now show that $\lim_{z \to 0} p_t^{\uparrow}(0, z) = p_t^{\uparrow, s}(0, 0)$. By Chapman–Kolmogorov, we get

$$p_t^{\uparrow}(0, z) = \int p_s^{\uparrow}(0, y) p_{t-s}^{\uparrow}(y, z) \lambda^{\uparrow}(\mathrm{d}y).$$

We know that $p_{t-s}^{\uparrow}(y, z)$ converges to $p_{t-s}^{\uparrow}(y, 0)$ as $z \to 0$. Dominated convergence, which applies because of the bound

$$p_{t-s}^{\uparrow}(y, z) \leq C / h(y),$$

then implies

$$\lim_{z \to 0} p_t^{\uparrow}(0, z) = \int p_s^{\uparrow}(0, y) p_{t-s}^{\uparrow}(y, 0) \lambda^{\uparrow}(\mathrm{d}y),$$

which shows that we can define $p_t^{\uparrow}(0, 0) = p_t^{\uparrow, s}(0, 0)$, and we have

$$\lim_{z \to 0} p_t^{\uparrow}(0, z) = p_t^{\uparrow}(0, 0) \qquad \text{and by duality} \quad \lim_{x \to 0} p_t^{\uparrow}(x, 0) = p_t^{\uparrow}(0, 0).$$

Finally, we will prove that

$$\lim_{x,z\to 0} p_t^{\uparrow}(x,z) = p_t(0,0).$$

Take $x_n, z_n \to 0$ and write

$$\begin{aligned}
&\limsup_n \left| p_t^{\uparrow}(x_n,z_n) - p_t^{\uparrow}(0,0) \right| \\
&\le \limsup_n \left| p_t^{\uparrow}(x_n,z_n) - p_t^{\uparrow}(x_n,0) \right| + \limsup_n \left| p_t^{\uparrow}(x_n,0) - p_t^{\uparrow}(0,0) \right| \\
&\le \limsup_n \int P_s^{\uparrow}(x_n,\mathrm{d}y) \left| p_{t-s}^{\uparrow}(y,z_n) - p_{t-s}^{\uparrow}(y,0) \right|.
\end{aligned}$$

Since $P_s^{\uparrow}(x_n,\cdot)$ weakly to $P_s^{\uparrow}(0,\cdot)$ and

$$\left| p_{t-s}^{\uparrow}(y,z_n) - p_{t-s}^{\uparrow}(y,0) \right| \le C/h(y),$$

where C is a finite constant, we obtain the desired result. □

The main result of this section is the construction of weakly continuous bridges for the Lévy process conditioned to stay positive. Indeed, by applying Theorem 5 and Lemma 3, we obtain Theorem 1.

The proof of Corollary 1 is simple from Theorem 1 and the following remarks. First, we note that the finite-dimensional distributions of the bridges $\mathbb{P}_{x,y}^{\uparrow,t}$ and $\mathbb{Q}_{x,y}^{t}$ are identical if $x, y, t > 0$ (because we have an h-transform relationship between $\mathbb{Q}_x$ and $\mathbb{P}_x^{\uparrow}$ for $x > 0$). Next, note that the law of $X - \varepsilon$ under $\mathbb{Q}_{\varepsilon,\varepsilon}^{t} = \mathbb{P}_{\varepsilon,\varepsilon}^{\uparrow,t}$ is precisely that of $\mathbb{P}_{0,0}^{t}$ conditioned on $\underline{X}_t > -\varepsilon$. Finally, since the laws $\mathbb{P}_{x,y}^{\uparrow,t}$ are weakly continuous, Corollary 1 is established.

6. An extension of the Denisov decomposition of the Brownian trajectory

We now turn to the extension of the Denisov decomposition of the Brownian trajectory of Theorem 2.

Proof of Theorem 2. We will use Lemma 4 in Chaumont and Doney [7], which states that if $x_n \to 0$ and $t_n \to t > 0$ then the law of $(X_s, s \le t_n)$ conditionally on $\underline{X}_{t_n} > 0$ under $\mathbb{P}_{x_n}$ converges as $n \to \infty$ in the sense of finite-dimensional distributions to $\mathbb{P}^{\mathrm{me},t}$ when 0 is regular for $(0,\infty)$. (This was only stated in Chaumont and Doney [7] for fixed t and follows from Corollary 2 in Chaumont and Doney [5]. However, the arguments, which are actually found in Chaumont and Doney [6], also apply in our setting.)

Fix $t > 0$. Since 0 is regular for both half-lines, X reaches its minimum $\underline{X}_t$ on the interval $[0,t]$ continuously at an unique place ρ_t, as proved in Propositions 2.2 and 2.4 of Millar [19].

Let

$$\rho_t^n = \lfloor \rho_t 2^n \rfloor / 2^n$$

and note that

$$\underline{X}_t = \min_{s \in [\rho_t^n, \rho_t^n + 1/2^n]} X_s.$$

For continuous and bounded $f\colon \mathbb{R} \to \mathbb{R}$ and functions F of G of the form $h(X_{t_1}, \ldots, X_{t_m})$ for some $t_1, \ldots, t_m \geq 0$ and continuous and bounded h, we will compute the quantity

$$\mathbb{E}_0\big(F(X_{\cdot \wedge \rho_n}) f(\rho_n) G(X_{(\rho_n + 1/2^n + \cdot) \wedge t} - X_{\rho_n})\big).$$

This is done by noting the decomposition

$$\big\{\rho_t^n = k/2^n\big\} = A_{k,n} \cap B_{k,n},$$

where

$$\begin{aligned} A_{k,n} &= \big\{m_{k,n} \leq X_s \text{ for } s \leq k/2^n\big\}, \\ B_{k,n} &= \big\{m_{k,n} \leq X_s \text{ for } s \in \big[(k+1)/2^n, t\big]\big\} \end{aligned}$$

and

$$m_{k,n} = \inf_{r \in [k/2^n, (k+1)/2^n]} X_r.$$

Applying the Markov property at time $(k+1)/2^n$ we obtain

$$\begin{aligned} &\mathbb{E}_0\big(F(X_{\cdot \wedge \rho_n}) f(\rho_n) G(X_{(\rho_n + 1/2^n + \cdot) \wedge t} - m_{k,n}) \mathbf{1}_{\rho_n = k/2^n}\big) \\ &\quad = \mathbb{E}_0\big(F(X_{\cdot \wedge \rho_n}) f(\rho_n) H\big(t - (k+1)/2^n, X_{(k+1)/2^n}, \underline{X}_{(k+1)/2^n}\big) \mathbf{1}_{A_{k,n}}\big), \end{aligned}$$

where

$$H(s, x, y) = \mathbb{E}_x\big(G\big(X^s - y\big) \mathbf{1}_{\underline{X}_s > y}\big) = \mathbb{E}_{x-y}\big(G\big(X^s\big) \mathbf{1}_{\underline{X}_s > 0}\big).$$

By reversing our steps, we obtain

$$\begin{aligned} &\mathbb{E}_0\big(F(X_{\cdot \wedge \rho_n}) f(\rho_n) H\big(t - (k+1)/2^n, X_{(k+1)/2^n}, \underline{X}_{(k+1)/2^n}\big) \mathbf{1}_{A_{k,n}}\big) \\ &\quad = \mathbb{E}_0\big(F(X_{\cdot \wedge \rho_n}) f(\rho_n) \tilde{H}\big(t - (k+1)/2^n, X_{(k+1)/2^n}, \underline{X}_{(k+1)/2^n}\big) \mathbf{1}_{\rho_t^n = k/2^n}\big), \end{aligned}$$

where

$$\tilde{H}(s, x, y) = \mathbb{E}_{x-y}\big(G\big(X^s\big) \mid \underline{X}_s > 0\big).$$

By the continuity assumptions of f, F and G we can pass to the limit using the Chaumont–Doney lemma to get

$$\mathbb{E}_0\big(F(X_{\cdot \wedge \rho_t}) f(\rho_t) G\big(X^{\rightarrow}\big)\big) = \mathbb{E}_0\big[F(X_{\cdot \wedge \rho_t}) f(\rho_t) \mathbb{E}^{\mathrm{me}, t - \rho_t}(G)\big].$$

By time reversal at t, we see that

$$\mathbb{E}_0\big(F\big(X^{\leftarrow}\big)f(\rho_t)G\big(X^{\rightarrow}\big)\big)=\mathbb{E}_0\big[\hat{\mathbb{E}}^{\mathrm{me},\rho_t}(F)f(\rho_t)\mathbb{E}^{\mathrm{me},t-\rho_t}(G)\big]. \qquad \square$$

We now establish a Denisov-type decomposition for bridges of Lévy processes.

Proof of Theorem 3. Since 0 is regular for $(-\infty,0)$ under $\mathbb{P}_0$, using local absolute continuity between $\mathbb{P}^t_{0,0}$ and $\mathbb{P}_0$ we see that $\underline{X}_t<0$ and $\rho_t>0$ almost surely under $\mathbb{P}^t_{0,0}$. Time reversal and regularity of 0 for $(0,\infty)$ proves that $\rho_t<t$ almost surely under $\mathbb{P}^t_{0,0}$.

From the absolute continuity relationship between the meander and the Lévy process conditioned to stay positive, we see that $\mathbb{P}^{\uparrow,t}_{0,x}$ is a regular conditional probability of X given $X_t=x$ under $\mathbb{P}^{\mathrm{me},t}$. Hence, Theorem 2 allows the conclusion

$$\mathbb{E}_0\big(F_1\big(X^{\leftarrow}\big)f(\rho_t)g(\underline{X}_t,X_t)F_2\big(X^{\rightarrow}\big)\big)=\mathbb{E}_0\big[\hat{\mathbb{E}}^{\uparrow,\rho_t}_{0,-\underline{X}_t}(F_1)f(\rho_t)g(\underline{X}_t,X_t)\hat{\mathbb{E}}^{\uparrow,\rho_t}_{0,X_t-\underline{X}_t}(F_2)\big].$$

Hence, we see that for every continuous and bounded f, g_1, g_2, F_1, F_2:

$$\begin{aligned}&\int \mathbb{E}^t_{0,x}\big[F_1\big(X^{\leftarrow}\big)f(\rho_t)g_1(\underline{X}_t)F_2\big(X^{\rightarrow}\big)\big]g_2(x)p_t(0,x)\,\mathrm{d}x\\&\quad=\int \mathbb{E}^t_{0,x}\big[\hat{\mathbb{E}}^{\uparrow,\rho_t}_{0,\underline{X}_t}(F_1)f(\rho_t)g_1(\underline{X}_t)\mathbb{E}^{\uparrow,\rho_t}_{0,\underline{X}_t}(F_2)\big]g_2(x)p_t(0,x)\,\mathrm{d}x.\end{aligned}$$

Since both integrands are continuous because of weak continuity of the bridge laws (of the Lévy process, its dual, and their conditioning to remain positive), we see that

$$\mathbb{E}^t_{0,0}\big(F_1\big(X^{\leftarrow}\big)f(\rho_t)g_1(\underline{X}_t)F_2\big(X^{\rightarrow}\big)\big)=\mathbb{E}^t_{0,0}\big[\hat{\mathbb{E}}^{\uparrow,\rho_t}_{0,\underline{X}_t}(F_1)f(\rho_t)g_1(\underline{X}_t)\mathbb{E}^{\uparrow,\rho_t}_{0,\underline{X}_t}(F_2)\big]. \qquad \square$$

Theorems 2 and 3 imply the following corollary.

Corollary 2. *The joint law of* $(\rho_t,\underline{X}_t,X_t)$ *under* $\mathbb{P}_0$ *admits the expression*

$$\mathbb{P}_0(\rho_t\in\mathrm{d}s,-\underline{X}_t\in\mathrm{d}y,X_t-\underline{X}_t\in\mathrm{d}z)=\mathbb{P}_0(\rho_t\in\mathrm{d}s)\hat{\mathbb{P}}^{\mathrm{me},s}(X_s\in\mathrm{d}y)\mathbb{P}^{\mathrm{me},t-s}(X_{t-s}\in\mathrm{d}z).$$

The joint law of $(\rho_t,\underline{X}_t)$ *under* $\mathbb{P}^t_{0,0}$ *admits the expression*

$$\mathbb{P}^t_{0,0}(\rho_t\in\mathrm{d}s,-\underline{X}_t\in\mathrm{d}y)=\frac{\mathbb{P}_0(\rho_t\in\mathrm{d}s)}{p_t(0,0)}\hat{\mathbb{P}}^{\mathrm{me},s}(X_s\in\mathrm{d}y)\mathbb{P}^{\mathrm{me},t-s}(X_{t-s}\in\mathrm{d}y).$$

7. An extension of Vervaat's theorem

In this section, we prove Theorem 4.

Proof of Theorem 4. Let λ_t be Lebesgue measure on $(0,t)$; we will work under the law $\mathbb{P}^{\uparrow,t}_{0,0}\otimes\lambda_t$ and we keep the notation X for the canonical process (which is now defined on the product space

$\Omega \times (0, t))$ and U will be the projection in the second coordinate of this space. Then a regular version of the law of $X_{r\wedge U}$, $r \geq 0$, and $X_{(U+r)\wedge t}$, $r \geq 0$, given $U = u$ and $X_U = y$ is $\mathbb{P}^{\uparrow,t}_{0,y} \otimes \mathbb{P}^{\uparrow,t}_{y,0}$ and the law of (U, X_U) admits the following density:

$$(u, y) \mapsto \frac{p^{\uparrow}_s(0, y) p^{\uparrow}_{t-s}(y, 0)}{t \cdot p^{\uparrow}_t(0, 0)} \, du \, \lambda^{\uparrow}(dx).$$

On the other hand, the Vervaat transformation of X is the concatenation of $X^{\rightarrow}$ followed by the time-reversal of $X^{\leftarrow}$ at ρ_t; under $\mathbb{P}^t_{0,0}$, the joint law of $(X^{\rightarrow}, X^{\leftarrow})$ given $\rho_t = t - s$ and $\underline{X}_t = y$ is $\mathbb{P}^{\uparrow,s}_{0,y} \otimes \mathbb{P}^{\uparrow,t-s}_{y,0}$.

We finish the proof by identifying the law of $(t - \rho_t, \underline{X}_t)$ under $\mathbb{P}^t_{0,0}$ with that of (U, X_U) under $\mathbb{P}^{\uparrow,t}_{0,0} \times \lambda_1$. Indeed, by Corollary 2, a version of the density with respect to Lebesgue measure of $(t - \rho_t, -\underline{X}_t)$ at (s, y) is

$$\frac{\mathbb{P}_0(\rho_t \in t - ds) p^{\uparrow}_{t-s}(0, y) p^{\uparrow}_s(y, 0) h(y) \hat{h}(y)}{p_t(0, 0) \hat{\beta}_{t-s} \beta_s}.$$

However, by the Chapman–Kolmogorov equations we can obtain the marginal density of ρ_t under $\mathbb{P}^t_{0,0}$ at u:

$$\frac{\mathbb{P}_0(\rho_t \in t - ds) p^{\uparrow}_t(0, 0)}{p_t(0, 0) \hat{\beta}_{t-s} \beta_s}.$$

Since ρ_t has an uniform law under $\mathbb{P}^t_{0,0}$ as proved in Knight [17], then, actually the above expression is almost surely equal to $1/t$ so that a joint density of $(\rho_t, \underline{X}_t)$ under $\mathbb{P}^t_{0,0}$ is

$$(u, y) \mapsto \frac{p^{\uparrow}_s(0, y) p^{\uparrow}_{t-s}(y, 0)}{t \cdot p^{\uparrow}_t(0, 0)} \, du \, \lambda^{\uparrow}(dy).$$

□

Note added in proof

It has been pointed out to the author that the proof of what we state as Theorem 5 (taken from reference [9]) has an error. Since we use this theorem to construct our bridges, the reader should note that Theorem 5 has a simple proof when the Markov process in the statement has a Feller dual. This is the case both for Lévy processes and for Lévy processes conditioned to stay positive (thanks to Lemma 1 for the latter), and this ensures the validity of the results in this paper.

Acknowledgements

The author would like to thank Loïc Chaumont, Pat Fitzsimmons, and Jim Pitman for stimulating conversations, information, references which were helpful in developing this research, and for spotting some obscurities and errors in the paper. Further thanks are due to Jim Pitman who suggested to look for further extensions of Vervaat's theorem and performed the work of an engaging postdoctoral supervisor. Research partially conducted at the Department of Statistics of the University of California at Berkeley. Supported by a postdoctoral fellowship from UC MexUS – CoNaCyt, NSF Grant DMS-08-06118 and PAPIIT Grant IN100411.

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Received March 2012 and revised September 2012