Electronic Journal of Probability

Electron. J. Probab. **18** (2013), no. 43, 1–17.
ISSN: 1083-6489 DOI: 10.1214/EJP.v18-2043

Regularity of affine processes on general state spaces

Martin Keller-Ressel* Walter Schachermayer† Josef Teichmann‡

Abstract

We consider a stochastically continuous, affine Markov process in the sense of Duffie, Filipovic and Schachermayer [9], with càdlàg paths, on a general state space D, i.e. an arbitrary Borel subset of $\mathbb{R}^d$. We show that such a process is always regular, meaning that its Fourier-Laplace transform is differentiable in time, with derivatives that are continuous in the transform variable. As a consequence, we show that generalized Riccati equations and Lévy-Khintchine parameters for the process can be derived, as in the case of $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ studied in Duffie et al. [9]. Moreover, we show that when the killing rate is zero, the affine process is a semi-martingale with absolutely continuous characteristics up to its time of explosion. Our results generalize the results of Keller-Ressel, Schachermayer and Teichmann [15] for the state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ and provide a new probabilistic approach to regularity.

Keywords: affine process; regularity; semimartingale; generalized Riccati equation.
AMS MSC 2010: 60J25.
Submitted to EJP on May 22, 2012, final version accepted on February 8, 2013.
Supersedes arXiv:1105.0632v2.

1 Introduction

A time-homogeneous, stochastically continuous Markov process X on the state space $D \subset \mathbb{R}^d$ is called affine, if its transition kernel $p_t(x, d\xi)$ has the following property: There exist functions Φ and ψ, taking values in $\mathbb{C}$ and $\mathbb{C}^d$ respectively, such that

$$\int_D e^{\langle \xi, u \rangle} p_t(x, d\xi) = \Phi(t, u) \exp(\langle x, \psi(t, u) \rangle)$$

for all $t \in \mathbb{R}_{\geqslant 0}$, $x \in D$ and u in the set $\mathcal{U} = \left\{ u \in \mathbb{C}^d : \sup_{x \in D} \operatorname{Re} \langle u, x \rangle < \infty \right\}$.

The class of stochastic processes resulting from this definition is a rich class that includes Brownian motion, Lévy processes, squared Bessel processes, continuous-state branching processes with and without immigration [14], Ornstein-Uhlenbeck-type processes [17, Ch. 17], Wishart processes [1] and several models from mathematical finance, such as the affine term structure models of interest rates [10] and the affine stochastic volatility models [13] for stock prices.

*TU Berlin, Germany. E-mail: mkeller@math.tu-berlin.de
†University of Vienna, Austria. E-mail: walter.schachermayer@univie.ac.at
‡ETH Zurich, Switzerland. E-mail: jteichma@math.ethz.ch

For a state space of the form $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ the class of affine processes has been originally defined and systematically studied in [9], under a regularity condition. In this context, regularity means that the time-derivatives

$$F(u) = \left.\frac{\partial \Phi(t,u)}{\partial t}\right|_{t=0+}, \qquad R(u) = \left.\frac{\partial \psi(t,u)}{\partial t}\right|_{t=0+}$$

exist for all $u \in \mathcal{U}$ and are continuous on subsets $\mathcal{U}_k$ of $\mathcal{U}$ that exhaust $\mathcal{U}$. Once regularity is established, the process X can be described completely in terms of the functions F and R. The problem of showing that regularity of a stochastically continuous affine process X always holds true was originally considered for processes on the state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ and was proven – giving a positive answer – in [15], building on results from [8].

Already in [9] it has been remarked that affine processes can be considered on other state spaces $D \neq \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, where also no reduction to the 'canonical' case by embedding or linear transformation is possible. One such example is given by the Wishart process (for $d \geq 2$), which is an affine process taking values in S_d^+, the cone of positive semidefinite $d \times d$-matrices. Recently, in [6] a full characterization of all affine processes with state space S_d^+ has been given and in [7] the even more general case when D is an 'irreducible symmetric cone' in the sense of [11] is considered, which includes the S_d^+ case.[1]

In both articles, regularity of the process remains a crucial ingredient, and the authors give direct proofs showing that regularity follows from the definition of the process, as in the case of $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$. Even though the affine processes on $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ and on symmetric cones are regular and have been completely classified, it is known that this does not amount to a full classification of all affine processes on a general state space D. A simple example is given by the process $X_t^{(x,x^2)} = (B_t + x, (B_t + x)^2)_{t \geq 0}$, where B is a standard Brownian motion. This process is an affine process that lives on the parabola $D = \left\{(y, y^2), y \in \mathbb{R}\right\} \subset \mathbb{R}^2$, and can be characterized by the functions

$$\Phi(t,u) = \frac{1}{\sqrt{1 - 2tu_2}} \exp\left(\frac{u_1^2 t}{2(1 - 2tu_2)}\right), \quad \psi(t,u) = (u_1, u_2)/(1 - 2tu_2).$$

It can even be extended into an affine process on the parabola's epigraph $\left\{(y,z) : z \geq y^2, y \in \mathbb{R}\right\}$ (see [9, Sec. 12.2]), but not into a process on the state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, or on any symmetric cone. For more general results in this direction we refer to [18], where a classification of affine diffusion processes on polyhedral cones and state spaces which are level sets of quadratic functions ('quadratic state spaces') is provided. The authors of [18] start from a slightly different definition of an affine process through a stochastic differential equation, which also immediately implies the regularity of the process.

The contribution of this article is to show regularity of an affine process on a general state space $D \subset \mathbb{R}^d$ under the only assumptions that D is a non-empty Borel set whose affine span is $\mathbb{R}^d$ and that the affine process has càdlàg paths. All existing regularity proofs, with the notable exception of [4] – which has been prepared in parallel to this article – use some particular properties of the state space: In the case of S_d^+ and the symmetric cones the fact that the set $\mathcal{U}$ has open interior, and in the case $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ a degeneracy argument that reduces the problem to $\mathbb{R}^m_{\geqslant 0}$, which is again a symmetric cone. The existence of an non-empty interior of $\mathcal{U}$ leads to a purely analytical proof based in

[1] A symmetric cone is a self-dual convex cone D, such that for any two points $x, y \in D$ a linear automorphism f of D exists, which maps x into y. It is called irreducible if it cannot be written as a non-trivial direct sum of two other symmetric cones.

broad terms on the theory of differentiable transformation semigroups of Montgomery and Zippin ([16]); see [15]. For general state spaces D it is not true that $\mathcal{U}$ has non-empty interior, and the analytic technique ceases to work. An empty interior of $\mathcal{U}$ also causes problems when applying the techniques from [9] and [6] to show the Feller property of the process. Therefore, we present in this paper a substantially different – probabilistic – technique that is independent of the nature of the state space under consideration. It should be considered as an alternative to the approach in [4], where another probabilistic regularity proof for affine processes on general state spaces is given. Our proof is largely self-contained, while the approach in [4] uses the theory of *full* and *complete* function classes put forward in [3]. On the other hand, in [4] also the automatic right-continuity of the augmented natural filtration and the existence of a càdlàg modification of the affine process is shown, which results in slightly weaker assumptions than in this article.

2 Definitions and Preliminaries

Let D be a non-empty Borel subset of the real Euclidian space $\mathbb{R}^d$, equipped with the Borel σ-algebra $\mathcal{D}$, and assume that the affine hull of D is the full space $\mathbb{R}^d$. To D we add a point δ that serves as a 'cemetery state', define

$$\widehat{D} = D \cup \{\delta\}, \qquad \widehat{\mathcal{D}} = \sigma(\mathcal{D}, \{\delta\}),$$

and equip $\widehat{D}$ with the Alexandrov topology, in which any open set with a compact complement in D is declared an open neighborhood of δ.[2] Any continuous function f defined on D is tacitly extended to $\widehat{D}$ by setting $f(\delta) = 0$.
Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered space, on which a family $(\mathbb{P}^x)_{x\in\widehat{D}}$ of probability measures is defined, and assume that $\mathcal{F}$ is $\mathbb{P}^x$-complete for all $x \in \widehat{D}$ and that $\mathbb{F}$ is right-continuous. Finally let X be a càdlàg process taking values in $\widehat{D}$, whose transition kernel

$$p_t(x, A) = \mathbb{P}^x(X_t \in A), \qquad (t \geq 0, x \in \widehat{D}, A \in \widehat{\mathcal{D}}) \tag{2.1}$$

is a normal time-homogeneous Markov kernel, for which δ is absorbing. That is, $p_t(x, .)$ satisfies the following:

(a) $x \mapsto p_t(x, A)$ is $\widehat{\mathcal{D}}$-measurable for each $(t, A) \in \mathbb{R}_{\geqslant 0} \times \widehat{\mathcal{D}}$.

(b) $p_0(x, \{x\}) = 1$ for all $x \in \widehat{D}$,

(c) $p_t(\delta, \{\delta\}) = 1$ for all $t \geq 0$

(d) $p_t(x, \widehat{D}) = 1$ for all $(t, x) \in \mathbb{R}_{\geqslant 0} \times \widehat{D}$, and

(e) the Chapman-Kolmogorov equation

$$p_{t+s}(x, d\xi) = \int p_t(y, d\xi)\, p_s(x, dy)$$

holds for each $t, s \geq 0$ and $(x, d\xi) \in \widehat{D} \times \widehat{\mathcal{D}}$.

We equip $\mathbb{R}^d$ with the canonical inner product $\langle , \rangle$, and associate to D the set $\mathcal{U} \subseteq \mathbb{C}^d$ defined by

$$\mathcal{U} = \left\{ u \in \mathbb{C}^d : \sup_{x\in D} \operatorname{Re} \langle u, x\rangle < \infty \right\}. \tag{2.2}$$

[2] Note that the topology of $\widehat{\mathcal{D}}$ enters our assumptions in a subtle way: We require later that X is càdlàg on $\widehat{\mathcal{D}}$, which is a property for which the topology matters.

Note that the set $\mathcal{U}$ is the set of complex vectors u such that the exponential function $x \mapsto e^{\langle u,x\rangle}$ is bounded on D. It is easy to see that $\mathcal{U}$ is a convex cone and always contains the set of purely imaginary vectors $i\mathbb{R}^d$. We will also need the sets

$$\mathcal{U}_k = \left\{ u \in \mathbb{C}^d : \sup_{x\in D} \operatorname{Re} \langle u, x\rangle \leq k \right\}, \qquad k \in \mathbb{N}, \tag{2.3}$$

for which we note that $\mathcal{U} = \bigcup_{k\in\mathbb{N}} \mathcal{U}_k$.

Definition 2.1 (Affine Process)**.** *The process X is called* affine *with state space D, if its transition kernel $p_t(x, d\xi)$ satisfies the following:*

(i) it is stochastically continuous, i.e. $\lim_{s\to t} p_s(x, .) = p_t(x, .)$ weakly for all $t \geq 0, x \in D$, and

(ii) its Fourier-Laplace transform depends on the initial state in the following way: there exist $\Phi : \mathbb{R}_{\geqslant 0} \times \mathcal{U} \to \mathbb{C}$ and a continuous $\psi : \mathbb{R}_{\geqslant 0} \times \mathcal{U} \to \mathbb{C}^d$, such that

$$\int_D e^{\langle \xi,u\rangle} p_t(x, d\xi) = \Phi(t, u) \exp(\langle x, \psi(t, u)\rangle) \tag{2.4}$$

for all $t \in \mathbb{R}_{\geqslant 0}$, $x \in D$ and $u \in \mathcal{U}$.

Remark 2.2. *Note that this definition does not specify $\psi(t, u)$ in a unique way. However there is a natural unique choice for ψ that will be discussed in Prop. 2.4 below. Also note that as long as $\Phi(t, u)$ is non-zero, there exists $\phi(t, u)$ such that $\Phi(t, u) = e^{\phi(t,u)}$ and* (2.4) *becomes*

$$\int_D e^{\langle \xi,u\rangle} p_t(x, d\xi) = \exp(\phi(t, u) + \langle x, \psi(t, u)\rangle). \tag{2.5}$$

This is the essentially the definition that was used in [9]; with this notation the Fourier-Laplace transform is the exponential of an affine function *of x. This is usually interpreted as the reason for the name 'affine process', even though affine functions also appear in other aspects of affine processes, e.g. in the coefficients of the infinitesimal generator, or in the differentiated semi-martingale characteristics. We use* (2.4) *instead of* (2.5)*, as it leads to a slightly more general definition that avoids the a-priori assumption that the left hand side of* (2.4) *is non-zero. Interestingly, in the paper [14] also the 'big-Φ' notation is used to define a 'continuous-state branching process with immigration', which corresponds to an affine process on $\mathbb{R}_{\geqslant 0}$ in our terminology.*

Remark 2.3. *It has recently been shown in [5] (see also [4]), that any affine process on a general state space D has a càdlàg modification under every $\mathbb{P}^x, x \in D$. Moreover, when X is an affine process relative to an arbitrary filtration $\mathbb{F}_0$, then the $\mathbb{P}^x$-augmentation $\mathbb{F}^x$ of $\mathbb{F}_0$ is right-continuous, for any $x \in D$. This implies that the assumptions that we make on the path properties of X are in fact automatically satisfied after a suitable modification of the process.*

Before we explore the first consequences of Definition 2.1, we introduce some additional notation. For any $u \in \mathcal{U}$ define

$$\sigma(u) := \inf \{t \geq 0 : \Phi(t, u) = 0\}, \tag{2.6}$$

and

$$\mathcal{Q}_k := \{(t, u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}_k : t < \sigma(u)\}, \tag{2.7}$$

for $k \in \mathbb{N}$. We set $\mathcal{Q} := \cup_k \mathcal{Q}_k$. Finally on $\mathcal{Q}$ let ϕ be a function such that

$$\Phi(t,u) = e^{\phi(t,u)} \qquad ((t,u) \in \mathcal{Q}).$$

The uniqueness of ϕ will be discussed. The functions ϕ and ψ have the following properties:

Proposition 2.4. *Let X be an affine process on D. Then*

(i) It holds that $\sigma(u) > 0$ for any $u \in \mathcal{U}$.

(ii) The functions ϕ and ψ are uniquely defined on $\mathcal{Q}$ by requiring that they are jointly continuous on $\mathcal{Q}_k$ for $k \in \mathbb{N}$ and satisfy $\phi(0,0) = \psi(0,0) = 0$.

(iii) The function ψ maps $\mathcal{Q}$ into $\mathcal{U}$.

(iv) The functions ϕ and ψ satisfy the semi-flow *property. For any $u \in \mathcal{U}$ and $t, s \geq 0$ with $(t+s,u) \in \mathcal{Q}$ and $(s,\psi(t,u)) \in \mathcal{Q}$ it holds that*

$$\begin{aligned} \phi(t+s,u) &= \phi(t,u) + \phi(s,\psi(t,u)), \quad & \phi(0,u) = 0 \\ \psi(t+s,u) &= \psi(s,\psi(t,u)), & \psi(0,u) = u \end{aligned} \tag{2.8}$$

Proof. Choose some $x \in D$, and for $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}$ define the function

$$f(t,u) = \Phi(t,u) e^{\langle \psi(t,u), x \rangle} = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi). \tag{2.9}$$

Fix $k \in \mathbb{N}$ and let $(t_n, u_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{\geqslant 0} \times \mathcal{U}_k$ converging to $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}_k$. For any $\epsilon > 0$ we can find a function $\rho : D \to [0,1]$ with compact support, such that $\int_D (1 - \rho(\xi)) p_t(x, d\xi) < \epsilon$. Moreover, there exists a $N_0 \in \mathbb{N}$ such that

$$\left| e^{\langle u_n, \xi \rangle} - e^{\langle u, \xi \rangle} \right| < \epsilon, \quad \forall\, n \geq N_0, \xi \in \operatorname{supp} \rho.$$

By stochastic continuity of $p_t(x, d\xi)$ we can find $N_1 \geq N_0$ such that

$$\int_D (1 - \rho(\xi)) p_{t_n}(x, d\xi) < \epsilon, \quad \forall\, n \geq N_1,$$

and also

$$\left| \int_D e^{\langle u, \xi \rangle} p_{t_n}(x, d\xi) - \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) \right| < \epsilon, \quad \forall\, n \geq N_1,$$

For $n \geq N_1$, we now have

$$\begin{aligned} |f(t_n,u_n) - f(t,u)| &= \left| \int_D e^{\langle u_n, \xi \rangle} p_{t_n}(x, d\xi) - \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) \right| \leq \\ &\leq \left| \int_D e^{\langle u_n, \xi \rangle} \rho(\xi) p_{t_n}(x, d\xi) - \int_D e^{\langle u, \xi \rangle} \rho(\xi) p_{t_n}(x, d\xi) \right| + \\ &+ \left| \int_D e^{\langle u_n, \xi \rangle} (1 - \rho(\xi)) p_{t_n}(x, d\xi) - \int_D e^{\langle u, \xi \rangle} (1 - \rho(\xi)) p_{t_n}(x, d\xi) \right| + \\ &+ \left| \int_D e^{\langle u, \xi \rangle} p_{t_n}(x, d\xi) - \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) \right| \leq \\ &\leq \epsilon + k\epsilon + \epsilon = \epsilon(2 + k). \end{aligned}$$

Since ϵ was arbitrary this shows the continuity of $f(t,u)$ on $\mathbb{R}_{\geqslant 0} \times \mathcal{U}_k$. Hence we conclude that $(t,u) \mapsto f(t,u)$ is continuous on $\mathbb{R}_{\geqslant 0} \times \mathcal{U}_k$ for each $k \in \mathbb{N}$. Moreover $f(t,u) = 0$ if and only if $\Phi(t,u)=0$ and $f(0,u)=e^{\langle u,x\rangle}\neq 0$ for all $u\in\mathcal{U}$. We conclude from the continuity of f that $\sigma(u)=\inf\{t\geq 0: f(t,u)=0\}>0$ for all $u\in\mathcal{U}$ and (i) follows.

To obtain (ii), note that for each $x\in D$, we have just shown that the function $(t,u)\mapsto\int_D e^{\langle u,\xi\rangle}p_t(x,\xi)$ maps $\mathcal{Q}_k$ *continuously* into $\mathbb{C}\setminus\{0\}$ for each $k\in\mathbb{N}$. We claim that the mapping has a unique continuous logarithm[3], i.e. for each $x\in D$ there exists a unique function $g(x;.,.):\mathcal{Q}\to\mathbb{C}$ being continuous on $\mathcal{Q}_k$ for $k\in\mathbb{N}$, such that $g(x;0,0)=0$ and $\int_D e^{\langle u,\xi\rangle}p_t(x,\xi)=e^{g(x;t,u)}$. For each $n\in\mathbb{N}$ define the set

$$K_n=\{(t,u):u\in\mathcal{U}_n,\|u\|\leq n,t\in[0,\sigma(u)-1/n]\}\,.$$

Clearly, the K_n are compact subsets of $\mathcal{Q}_n\subset\mathcal{Q}$ and exhaust $\mathcal{Q}$ as $n\to\infty$. We show that every K_n is contractible to 0. Let $\gamma=(t(r),u(r))_{r\in[0,1]}$ be a continuous curve in K_n. For each $\alpha\in[0,1]$ define $\gamma_\alpha=(\alpha t(r),u(r))_{r\in[0,1]}$. Then γ_α depends continuously on α, stays in K_n for each α and satisfies $\gamma_1=\gamma$ and $\gamma_0=(0,u(r))_{r\in[0,1]}$. Thus any continuous curve in K_n is homotopically equivalent to a continuous curve in $\{0\}\times\mathcal{U}$. Moreover, all continuous curves in $\{0\}\times\mathcal{U}$ are contractible to 0, since $\mathcal{U}$ is a convex cone. We conclude that each K_n is contractible to 0 and in particular connected. Let $H_n:[0,1]\times K_n\to K_n$ be a corresponding contraction, and for some fixed $x\in D$ write $f_n(t,u)$ for the restriction of $(t,u)\mapsto\int_D e^{\langle u,\xi\rangle}p_t(x,\xi)$ to K_n. Since H_n and f_n are continuous and K_n is compact, we have that $\lim_{t\to s}\|f_n(H_n(t,.))-f_n(H_n(s,.))\|_\infty=0$. Hence $f_n\circ H_n$ is a continuous curve in $C_b(K_n)$ from f_n to the constant function 1. By [2, Thm. 1.3] there exists a continuous logarithm $g_n\in C_b(K_n)$ that satisfies $f_n(t,u)=e^{g_n(t,u)}$ for all $(t,u)\in K_n$. It follows that for arbitrary $m\leq n$ in $\mathbb{N}$ we have

$$g_m(t,u)=g_n(t,u)+2\pi i\,l(t,u)\qquad\text{for all}\quad(t,u)\in K_m,$$

where $l(t,u)$ is a continuous function from K_m to $\mathbb{Z}$ satisfying $l(0,0)=0$. But K_m is connected, hence also the image of K_m under l. We conclude that $l(t,u)=0$, and that $g_m(t,u)=g_n(t,u)$ for all $(t,u)\in K_m$. Taking $m=n$ this shows that g_n is uniquely defined on each subset K_n of $\mathcal{Q}$. Taking $m<n$ it shows that g_n extends g_m. Since the $(K_n)_{n\in\mathbb{N}}$ exhaust $\mathcal{Q}$, it follows that there exists indeed, for each $x\in D$, a unique function $g(x;.):\mathcal{Q}\to\mathbb{C}$ such that $g(x;0,0)=0$ and $\int_D e^{\langle u,\xi\rangle}p_t(x,\xi)=e^{g(x;t,u)}$. Since the affine span of D is $\mathbb{R}^d$ we may assume without loss of generality that $0\in D$ and obtain that $\phi(t,u)=g(0;t,u)$ is the unique choice of $\phi(t,u)$ with $\phi(0,0)=0$. Moreover we know from (2.9) that

$$g(x;t,u)=\phi(t,u)+\langle\psi(t,u),x\rangle+2\pi i\,l(x;t,u),$$

for all $x\in D,(t,u)\in\mathcal{Q}_k$ and where l is an $\mathbb{Z}$-valued function that satisfies $l(x;0,0)=0$ for all $x\in D$. By definition 2.1 $\psi(t,u)$ is continuous on $\mathbb{R}_{\geqslant 0}\times\mathcal{U}$ and hence also $l(x;t,u)$ is, for each fixed $x\in D$, a continous function on $\mathcal{Q}_k$. Connected sets are mapped to connected sets by continuous functions and we conclude that in fact $l(x;t,u)\equiv 0$ on $D\times\mathcal{Q}_k$, which completes the proof of (ii).

Next note that the rightmost term of (2.9) is uniformly bounded for all $x\in D$. Thus also the middle term is, and we obtain that $\psi(t,u)\in\mathcal{U}$, as claimed in (iii). Applying the Chapman-Kolmogorov equation to (2.4) and writing $\Phi(t,u)=e^{\phi(t,u)}$ yields that

$$\begin{aligned}\exp\left(\phi(t+s,u)+\langle x,\psi(t+s,u)\rangle\right)&=\int_D e^{\langle\xi,u\rangle}p_{t+s}(x,d\xi)=\\&=\int_D p_s(x,dy)\int_D e^{\langle\xi,u\rangle}p_t(y,d\xi)=e^{\phi(t,u)}\int_D e^{\langle y,\psi(t,u)\rangle}p_s(x,dy)=\\&=\exp\left(\phi(t,u)+\phi(s,\psi(t,u))+\langle x,\psi(s,\psi(t,u))\rangle\right)\end{aligned}\quad(2.10)$$

[3] We adapt a proof from [2, Thm.2.5] to our setting.

for all $x \in D$ and for all $u \in \mathcal{U}$ such that $(t+s,u) \in \mathcal{Q}$ and $(s,\psi(t,u)) \in \mathcal{Q}$. Taking (continuous) logarithms on both sides (iv) follows. □

Remark 2.5. *From now on ϕ and ψ shall always refer to the unique choice of functions described in Proposition 2.4.*

3 Main Results

3.1 Presentation of the main result

We now introduce the important notion of *regularity*.

Definition 3.1. *An affine process X is called* regular *if the derivatives*

$$F(u) = \left.\frac{\partial \phi(t,u)}{\partial t}\right|_{t=0+}, \qquad R(u) = \left.\frac{\partial \psi(t,u)}{\partial t}\right|_{t=0+} \tag{3.1}$$

exist for all $u \in \mathcal{U}$ and are continuous on $\mathcal{U}_k$ for each $k \in \mathbb{N}$.

Remark 3.2. *Note that in comparison with the definition given in the introduction, we now define $F(u)$ as the derivative at $t=0$ of $t \mapsto \phi(t,u)$ instead of $t \mapsto \Phi(t,u)$. In light of Proposition 2.4 these definitions coincide, since $\phi(t,u)$ is always defined for t small enough and satisfies $\Phi(t,u) = e^{\phi(t,u)}$ with $\phi(0,u) = 0$.*

Our main result is the following.

Theorem 3.3. *Let X be a càdlàg affine process on $D \subset \mathbb{R}^d$. Then X is regular.*

Before this result is proved in the subsequent sections, we illustrate why regularity is a crucial property. The following result has originally been established in [9] for affine processes on the state-space $\mathbb{R}^n \times \mathbb{R}^m_{\geqslant 0}$.

Proposition 3.4. *Let X be a* regular *affine process with state space D. Then there exist $\mathbb{R}^d$-vectors $b, \beta^1, \dots, \beta^d$; $d \times d$-matrices $a, \alpha^1, \dots, \alpha^d$; real numbers $c, \gamma^1, \dots, \gamma^d$ and signed Borel measures $m, \mu^1, \dots, \mu^d$ on $\mathbb{R}^d \setminus \{0\}$, such that for all $u \in \mathcal{U}$ the functions $F(u)$ and $R(u)$ can be written as*

$$F(u) = \frac{1}{2}\langle u, au\rangle + \langle b, u\rangle - c + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{\langle \xi, u\rangle} - 1 - \langle h(\xi), u\rangle\right) m(d\xi)\,, \tag{3.2a}$$

$$R_i(u) = \frac{1}{2}\left\langle u, \alpha^i u\right\rangle + \left\langle \beta^i, u\right\rangle - \gamma^i + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{\langle \xi, u\rangle} - 1 - \langle h(\xi), u\rangle\right) \mu^i(d\xi)\,, \tag{3.2b}$$

with truncation function $h(x) = x\mathbf{1}_{\{\|x\| \leq 1\}}$, and such that for all $x \in D$ the quantities

$$A(x) = a + x_1\alpha^1 + \dots + x_d\alpha^d, \tag{3.3a}$$

$$B(x) = b + x_1\beta^1 + \dots + x_d\beta^d, \tag{3.3b}$$

$$C(x) = c + x_1\gamma^1 + \dots + x_d\gamma^d, \tag{3.3c}$$

$$\nu(x, d\xi) = m(d\xi) + x_1\mu^1(d\xi) + \dots + x_d\mu^d(d\xi) \tag{3.3d}$$

have the following properties: $A(x)$ is positive semidefinite, $C(x) \leq 0$ and $\int_{\mathbb{R}^d \setminus \{0\}} \left(\|\xi\|^2 \wedge 1\right) \nu(x, d\xi) < \infty$.
Moreover, for $u \in \mathcal{U}$ the functions ϕ and ψ satisfy the ordinary differential equations

$$\frac{\partial}{\partial t}\phi(t,u) = F(\psi(t,u)), \qquad \phi(0,u) = 0 \tag{3.4a}$$

$$\frac{\partial}{\partial t}\psi(t,u) = R(\psi(t,u)), \qquad \psi(0,u) = u. \tag{3.4b}$$

for all $t \in [0, \sigma(u))$.

Remark 3.5. *The differential equations* (3.4) *are called* generalized Riccati equations*, since they are classical Riccati differential equations, when* $m(d\xi) = \mu^i(d\xi) = 0$*. Moreover equations* (3.2) *and* (3.3) *imply that* $u \mapsto F(u) + \langle R(u), x\rangle$ *is a function of Lévy-Khintchine form for each* $x \in D$*. Note that the Proposition makes no claim on the uniqueness of the solutions of the generalized Riccati equations. In fact examples are known where uniqueness does not hold true (cf. [9, Ex. 9.3]).*

Proof. The equations (3.4) follow immediately by differentiating the semi-flow equations (2.8). The form of F, R follows by the following argument: By (3.1) and the affine property (2.4) it holds for all $x \in D$ and $u \in \mathcal{U}$ that

$$\begin{aligned} F(u) + \langle x, R(u)\rangle &= \lim_{t\downarrow 0} \frac{1}{t}\left\{e^{\phi(t,u)+\langle x,\psi(t,u)-u\rangle} - 1\right\} = \\ &= \lim_{t\to 0} \frac{1}{t}\left\{\int_D e^{\langle \xi - x, u\rangle} p_t(x, d\xi) - 1\right\} = \\ &= \lim_{t\to 0}\left\{\frac{1}{t}\int_D \left(e^{\langle \xi - x, u\rangle} - 1\right) p_t(x, d\xi) + \frac{p_t(x, D) - 1}{t}\right\} = \\ &= \lim_{t\to 0}\left\{\frac{1}{t}\int_{D-x} \left(e^{\langle \xi, u\rangle} - 1\right) \widetilde{p}_t(x, d\xi)\right\} + \lim_{t\to 0}\frac{p_t(x, D) - 1}{t}, \end{aligned} \tag{3.5}$$

where we write $\widetilde{p}_t(x, d\xi) := p_t(x, d\xi + x)$ for the shifted transition kernel. Inserting $u = 0$ into the above equation shows that $\lim_{t\downarrow 0}(p_t(x, D) - 1)/t$ converges to $F(0) + \langle x, R(0)\rangle$. Set $c = -F(0)$ and $\gamma = -R(0)$ and write $\widetilde{F}(u) = F(u) + c$ and $\widetilde{R}(u) = R(u) + \gamma$, such that

$$\exp\left(\widetilde{F}(u) + \left\langle x, \widetilde{R}(u)\right\rangle\right) = \lim_{t\downarrow 0} \exp\left\{\frac{1}{t}\int_{D-x}\left(e^{\langle \xi, u\rangle} - 1\right)\widetilde{p}_t(x, d\xi)\right\}. \tag{3.6}$$

For each $t \geq 0$ and $x \in D$, the exponential on the right hand side is the Fourier-Laplace transform of a compound Poisson distribution with jump measure $\widetilde{p}_t(x, d\xi)$ and jump intensity $\frac{1}{t}$ (cf. [17, Ch. 4]). The Fourier-Laplace transforms converge pointwise for $u \in \mathcal{U}$ – and in particular for all $u \in i\mathbb{R}^d$ – as $t \to 0$. By the assumption of regularity the pointwise limit is continuous at $u = 0$ as function on $i\mathbb{R}^d \subset \mathcal{U}_k$ for each $k \in \mathbb{N}$, which implies by Lévy's continuity theorem that the compound Poisson distributions converge weakly to a limiting probability distribution. Moreover, as the weak limit of compound Poisson distributions, the limiting distribution must be infinitely divisible. Let us denote the law of the limiting distribution, for given $x \in D$, by $K(x, dy)$. Since it is infinitely divisible, its characteristic exponent is of Lévy-Khintchine form, and we obtain the identity

$$\begin{aligned} \widetilde{F}(u) + \left\langle x, \widetilde{R}(u)\right\rangle &= \log \int_{\mathbb{R}^d} e^{\langle \xi, u\rangle} K(x, d\xi) = \\ &= -\frac{1}{2}\langle uA(x), u\rangle + \langle B(x), u\rangle - \int_{\mathbb{R}^d}\left(e^{\langle \xi, u\rangle} - 1 - \langle h(\xi), u\rangle\right)\nu(x, d\xi), \end{aligned} \tag{3.7}$$

where for each $x \in D$, $A(x)$ is a positive semi-definite $d \times d$-matrix, $B(x) \in \mathbb{R}^d$, and $\nu(x, d\xi)$ a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ and $\int \left(\|\xi\|^2 \wedge 1\right)\nu(x, d\xi) < \infty$. Note that in the step from (3.6) from (3.7) we have used that $\widetilde{F}(u)$ and $\widetilde{R}(u)$ are continuous on every $\mathcal{U}_k, k \in \mathbb{N}$, and hence that $\widetilde{F}(u) + \left\langle x, \widetilde{R}(u)\right\rangle$ is the unique continuous logarithm of $\exp\left(\widetilde{F}(u) + \left\langle x, \widetilde{R}(u)\right\rangle\right)$ on each $\mathcal{U}_k$ and for all $x \in D$. Since (3.7) holds for all $x \in D$, and D contains at least $d + 1$ affinely independent points, we conclude that $A(x)$, $B(x)$ and $\nu(x, d\xi)$ are of the form given in (3.3) and the decompositions in (3.2) follow. □

In general, the parameters $(a, \alpha^i, b, \beta^i, c, \gamma^i, m, \mu^i)_{i\in\{1,\dots,d\}}$ of F and R have to satisfy additional conditions, called *admissibility* conditions, that guarantee the existence of an affine Markov process X with state space D and prescribed F and R. It is clear that such conditions depend strongly on the geometry of the state space D, in particular of its boundary. Finding such (necessary and sufficient) conditions on the parameters for different types of state spaces has been the focus of several publications. For $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ the admissibility conditions have been derived in [9], for $D = S_d^+$, the cone of semi-definite matrices in [6], and for cones D that are symmetric and irreducible in the sense of [11] in [7]. Finally for affine diffusions ($m = \mu^i = 0$) on polyhedral cones and on quadratic state spaces the admissiblility conditions have been given in [18]. The purpose of this article is not to derive these admissibility conditions for conrete specifications of D, but merely to show that for any arbitrary state space D there are parameters in terms of which admissibility conditions can be formulated.

3.2 Auxiliary Results

For the sake of simpler notation we define

$$\varrho(t,u) = \psi(t,u) - u.$$

Note that we have $\varrho(0,u) = 0$ for all $u \in \mathcal{U}$. The following Lemma is a purely analytical result that will be needed later.

Lemma 3.6. *Let K be a compact subset of $\mathcal{U}_l$ for some $l \in \mathbb{N}$ and assume that*

$$\limsup_{t\to 0} \sup_{u\in K} \left(\frac{|\phi(t,u)|}{t} + \frac{\|\varrho(t,u)\|}{t} \right) = \infty \,. \tag{3.8}$$

Then there is $x \in D$, $\varepsilon > 0$, $\eta > 0$, $z \in \mathbb{C}$ with $|z| = 1$, a sequence $(t_k)_{k=1}^\infty$ of positive real numbers, a sequence $(M_k)_{k=1}^\infty$ of integers satisfying

$$\lim_{k\to\infty} t_k = 0, \qquad \lim_{k\to\infty} M_k = \infty, \qquad \lim_{k\to\infty} M_k t_k = 0, \tag{3.9}$$

and a sequence of complex vectors $(u_k)_{k=0}^\infty$ in K such that $u_k \to u_0$ and

$$|\phi(t_k,u_k) + \langle x, \varrho(t_k,u_k)\rangle| \geq \eta \, \|\varrho(t_k,u_k)\| \,. \tag{3.10}$$

Moreover, for all $\xi \in \mathbb{R}^d$ satisfying $\|x - \xi\| < \varepsilon$,

$$M_k(\phi(t_k,u_k) + \langle \xi, \varrho(t_k,u_k)\rangle) = z + e_{k,\xi}, \tag{3.11}$$

where the complex numbers $e_{k,\xi}$ describing the deviation from z satisfy $|e_{k,\xi}| < \frac{1}{2}$ and $\lim_{k\to\infty} \sup_{\{\xi : \|x-\xi\|<\varepsilon\}} |e_{k,\xi}| = 0$.

Remark 3.7. *The essence of the above Lemma is that the behavior of $\phi(t,u)$ and $\varrho(t,u)$ as t approaches 0 can be crystallized along the sequences t_k and M_k. Equation* (3.9) *then states that $t_k = o\left(\frac{1}{M_k}\right)$, and* (3.11) *asserts that the asymptotic equivalence*

$$|\phi(t_k,u_k) + \langle \xi, \varrho(t_k,u_k)\rangle| \sim \frac{1}{M_k},$$

holds uniformly for all ξ in an ε-ball around x.

Proof. We first show all assertions of the Lemma for a sequence $(\widetilde{M}_k)_{k\in\mathbb{N}}$ of positive but not necessarily integer numbers. In the last step of the proof we show that it is possible to switch from $(\widetilde{M}_k)_{k\in\mathbb{N}}$ to the integer sequence $(M_k)_{k\in\mathbb{N}}$.

By Assumption (3.8) we can find a sequence $(t_k)_{k=0}^{\infty} \downarrow 0$ and a sequence $(u_k)_{k=0}^{\infty}$ with $u_k \in K$, such that

$$\frac{|\phi(t_k, u_k)| + \|\varrho(t_k, u_k)\|}{t_k} \to \infty .$$

Passing to a subsequence, and using the compactness of K, we may assume that u_k converges to some point $u_0 \in K$. For more concise notation, we write from now on $\phi_k = \phi(t_k, u_k)$ and $\varrho_k = \varrho(t_k, u_k)$. Note that $\phi_k \to 0$ and $\varrho_k \to 0$, by joint continuity of ϕ and ϱ on $\mathcal{U}_l$, and the fact that $\phi(0, u) = 0$ and $\varrho(0, u) = 0$.
Let us now show (3.10). By assumption, D contains $d + 1$ affinely independent vectors $x_0, x_1, \ldots, x_d$. Assume for a contradiction that

$$\lim_{k\to\infty} \frac{|\phi_k + \langle \varrho_k, x_j\rangle|}{\|\varrho_k\|} \to 0 \tag{3.12}$$

for all x_j, $j \in \{0, \ldots, d\}$. Since the vectors x_j affinely span $\mathbb{R}^d$, the vectors $\{x_1 - x_0, \ldots, x_d - x_0\}$ are linearly independent, and we can find some numbers $\alpha_{j,k} \in \mathbb{C}$, such that

$$\varrho_k / \|\varrho_k\| = \sum_{j=1}^{d} \alpha_{j,k} (x_j - x_0) , \tag{3.13}$$

for all $k \in \mathbb{N}$. Moreover, since $\varrho_k / \|\varrho_k\|$ is bounded also the $|\alpha_{j,k}|$ are bounded by a constant. By direct calculation we obtain

$$\sum_{j=1}^{d} \alpha_{j,k} \left(\frac{\phi_k + \langle \varrho_k, x_j\rangle}{\|\varrho_k\|} - \frac{\phi_k + \langle \varrho_k, x_0\rangle}{\|\varrho_k\|} \right) = \frac{\langle \varrho_k / \|\varrho_k\|, \varrho_k\rangle}{\|\varrho_k\|} = 1, \tag{3.14}$$

for all $k \in \mathbb{N}$. On the other hand, (3.12) implies that the left hand side of (3.14) converges to 0 as $k \to \infty$, which is a contradiction. We conclude that there exists $x^* \in D$ for which

$$\frac{|\phi_k + \langle \varrho_k, x^*\rangle|}{\|\varrho_k\|} \geq \eta \tag{3.15}$$

for some $\eta > 0$ after possibly passing to subsequences, whence (3.10) follows.

To show (3.11), set $\widetilde{M}_k = |\phi_k + \langle x^*, \varrho_k\rangle|^{-1}$. Passing once more to a subsequence, and using the compactness of the complex unit circle, we can find some $\alpha \in [0, 2\pi)$ such that $\arg(\phi_k + \langle x^*, \varrho_k\rangle) \to \alpha$. Now

$$\phi_k + \langle \xi, \varrho_k\rangle = (\phi_k + \langle x^*, \varrho_k\rangle) + \langle \xi - x^*, \varrho_k\rangle = \frac{1}{\widetilde{M}_k}(e^{i\alpha} + e_k^{(1)}) + \langle \xi - x^*, \varrho_k\rangle$$

where $e_k^{(1)} \to 0$ as $k \to \infty$. Multiplying by $\widetilde{M}_k$ and setting $z = e^{i\alpha}$ we obtain

$$\widetilde{M}_k(\phi_k + \langle \xi, \varrho_k\rangle) = z + e_k^{(1)} + e_{k,\xi}^{(2)}$$

where we can estimate $|e_{k,\xi}^{(2)}| \leq \widetilde{M}_k \varepsilon \|\varrho_k\|$. Since $\widetilde{M}_k \|\varrho_k\| \leq \frac{1}{\eta}$ by (3.10) we can make $e_{k,\xi}^{(2)}$ arbitrarily small by choosing a small enough ε. Setting $e_{k,\xi} = e_k^{(1)} + e_{k,\xi}^{(2)}$ we obtain (3.11). Finally, for each $k \in \mathbb{N}$ let M_k be the nearest integer greater than $\widetilde{M}_k$. It is clear that after possibly removing a finite number of terms from all sequences, the assertion of the Lemma is not affected from switching from $\widetilde{M}_k$ to M_k. □

Lemma 3.8. *Let $X = (X_t)_{t\geq 0}$ be an affine process starting at X_0 and let $u \in \mathcal{U}$, $\Delta > 0$, $\varepsilon > 0$. Define*

$$L(n,\Delta,u) = \exp\left(\langle u, X_{n\Delta} - X_0\rangle - \sum_{j=1}^{n}\left(\phi(\Delta,u) + \left\langle \varrho(\Delta,u), X_{(j-1)\Delta}\right\rangle\right)\right) \tag{3.16}$$

and the stopping time $N_\Delta = \inf\{n \in \mathbb{N} : \|X_{n\Delta} - X_0\| > \varepsilon\}$ Then $n \mapsto L(n \wedge N_\Delta, \Delta, u)$ is a $(\mathcal{F}_{n\Delta})_{n\in\mathbb{N}}$-martingale under every measure $\mathbb{P}^x$, $x \in D$.

Proof. It is obvious that each $L(n,\Delta,u)$ is $\mathcal{F}_{n\Delta}$-measurable. The definition of the stopping time N_Δ guarantees the integrability of $L(n\wedge N_\Delta,\Delta,u)$. We show the martingale property by combining the affine property of X with the tower law for conditional expectations. Write

$$S_n = \sum_{j=1}^{n}\left(\phi(\Delta,u) + \left\langle \varrho(\Delta,u), X_{(j-1)\Delta}\right\rangle\right),$$

and note that S_n is $\mathcal{F}_{(n-1)\Delta}$-measurable. On $\{n \leq N_\Delta\}$ we have that

$$\begin{aligned}\mathbb{E}^x\left[L(n,\Delta,u)|\,\mathcal{F}_{(n-1)\Delta}\right] &= \mathbb{E}^x\left[\exp\left(\langle u, X_{n\Delta} - X_0\rangle\right)|\,\mathcal{F}_{(n-1)\Delta}\right]e^{-S_n} = \\ &= \exp\left(\phi(\Delta,u) + \left\langle \psi(\Delta,u), X_{(n-1)\Delta}\right\rangle - \langle u, X_0\rangle - S_n\right) = \\ &= \exp\left(\left\langle u, X_{(n-1)\Delta} - X_0\right\rangle - S_{n-1}\right) = L(n-1,\Delta,u),\end{aligned}$$

showing that $n \mapsto L(n\wedge N_\Delta,\Delta,u)$ is indeed a $(\mathcal{F}_{n\Delta})_{n\in\mathbb{N}}$-martingale under every $\mathbb{P}^x, x \in D$. □

We combine the two preceding Lemmas to show the following.

Proposition 3.9. *Let X be a càdlàg affine process. Then the associated functions $\phi(t,u)$ and $\varrho(t,u) = \psi(t,u) - u$ satisfy*

$$\limsup_{t\downarrow 0}\sup_{u\in K}\left(\frac{|\phi(t,u)|}{t} + \frac{\|\varrho(t,u)\|}{t}\right) < \infty \tag{3.17}$$

for each compact subset K of $\mathcal{U}_l$ and each $l \in \mathbb{N}$.

Proof. We argue by contradiction: Fix $l \in \mathbb{N}$ and assume that (3.17) fails to hold true. Then by Lemma 3.6 there exist $\varepsilon > 0$ and sequences $u_k \to u_0$ in K, $t_k \downarrow 0$ and $M_k \uparrow \infty$ such that $t_k M_k \to 0$ and equations (3.10), (3.11) hold. Define the $(\mathcal{F}_{nt_k})_{n\in\mathbb{N}}$-stopping times $N_k = \inf\{n \in \mathbb{N} : \|X_{nt_k} - X_0\| > \varepsilon\}$. Then setting $\Delta = t_k$ in Lemma 3.8 yields that

$$\begin{aligned}&n \mapsto L(n\wedge N_k, t_k, u_k) = \\ &\quad = \exp\left(\left\langle u_k, X_{(n\wedge N_k)t_k} - X_0\right\rangle - \sum_{j=1}^{n\wedge N_k}\left(\phi(t_k,u_k) + \left\langle \varrho(t_k,u_k), X_{(j-1)t_k}\right\rangle\right)\right)\end{aligned} \tag{3.18}$$

is a $(\mathcal{F}_{nt_k})_{n\in\mathbb{N}}$-martingale. It follows in particular that $\mathbb{E}\left[L(M_k\wedge N_k, t_k, u_k)\right] = 1$ for all $k \in \mathbb{N}$. By (3.11), we have the uniform bound

$$\begin{aligned}|L(M_k\wedge N_k, t_k, u_k)| &\leq C\exp\left(\left|\sum_{j=1}^{M_k\wedge N_k}\left(\phi(t_k,u_k) + \left\langle \varrho(t_k,u_k), X_{(j-1)t_k}\right\rangle\right)\right|\right) \leq \\ &\leq C\exp(3/2),\end{aligned} \tag{3.19}$$

where $C = \exp\left(-\mathrm{Re}\,\langle u, X_0\rangle\right)$. Let $\delta > 0$ and $x \in D$. Since X is càdlàg we can find a $T > 0$ such that $\mathbb{P}^x\left(\sup_{t\in[0,T]}\|X_t - X_0\| > \varepsilon\right) < \delta$. For k large enough $t_k M_k \leq T$ and hence $\mathbb{P}(M_k > N_k) < \delta$. We conclude that $\mathbb{P}^x\left(\lim_{k\to\infty}\frac{M_K\wedge N_k}{M_k}=1\right)\geq 1-\delta$, and since δ was arbitrary $\lim_{k\to\infty}\frac{M_k\wedge N_k}{M_k}=1$ holds $\mathbb{P}^x$-a.s. for any $x\in D$. Together with (3.11) and (3.19) we obtain by dominated convergence that

$$\lim_{k\to\infty}\mathbb{E}^x\left[L(M_k\wedge N_k,T_k,u_k)\right]=\mathbb{E}^x\left[\lim_{k\to\infty}L(M_k\wedge N_k,T_k,u_k)\right]=$$
$$=\mathbb{E}^x\left[\lim_{k\to\infty}\exp\left((M_k\wedge N_k)\left(\phi(t_k,u_k)+\langle\varrho(t_k,u_k),x\rangle\right)\right)\right]=e^{-z}. \quad (3.20)$$

where $|z|=1$. But $\mathbb{E}^x\left[L(M_k\wedge N_k,T_k,u_k)\right]=1$ by its martingale property, which is the desired contradiction. □

3.3 Affine processes are regular

In this section we prove the main result, Theorem 3.3.

Lemma 3.10. *Let a sequence $t_k(u)\downarrow 0$ be assigned to each $u\in\mathcal{U}$. Then each of these sequences has a subsequence $\mathbb{S}(u):=(s_k(u))_{k\in\mathbb{N}}$ such that the limits*

$$F_{\mathbb{S}}(u):=\lim_{s_k(u)\downarrow 0}\frac{\phi(s_k(u),u)}{s_k(u)},\qquad R_{\mathbb{S}}(u):=\lim_{s_k(u)\downarrow 0}\frac{\varrho(s_k(u),u)}{s_k(u)} \quad (3.21)$$

are well-defined and finite. Moreover the subsequences $\mathbb{S}(u)$ can be chosen such that the numbers $F_{\mathbb{S}}(u)$ and $R_{\mathbb{S}}(u)$ are bounded on each compact subset K of $\mathcal{U}_l$ for each $l\in\mathbb{N}$.

Proof. Let the sequences $t_k(u)\downarrow 0$ be given, but assume that the assertion of the Lemma does not hold true. Then either $t_k(u)$ for some $u\in\mathcal{U}$ has no subsequence for which the limits in (3.21) exist, or the limits $F(u)$ and $R(u)$ exist for each $u\in\mathcal{U}$, but at least one of them is not bounded in some compact $K\subset U_l$ for some $l\in\mathbb{N}$.
Consider the first case. By the Bolzano-Weierstrass theorem an $\mathbb{R}^d$-valued sequence that contains no convergent subsequence must be unbounded, and we conclude that

$$\limsup_{t_k(u)\downarrow 0}\left(\frac{|\phi(t_k(u),u)|}{t_k(u)}+\frac{\|\varrho(t_k(u),u)\|}{t_k(u)}\right)=\infty,$$

in contradiction to Proposition 3.9. Consider now the second assertion. Fix $l\in\mathbb{N}$. For each $u\in\mathcal{U}_l$ there is a sequence $s_k(u)$ such that (3.21) holds, but $F_{\mathbb{S}}(u)$ or $R_{\mathbb{S}}(u)$ is not bounded in $K\subset\mathcal{U}_l$, i.e. there exists a sequence $u_n\to u_0$ in K for which $|F(u_n)|+\|R(u_n)\|\to\infty$. Fix some $\eta>0$. Then for each $k\in\mathbb{N}$ there exists an $N_k\in\mathbb{N}$ such that

$$\left|\frac{\phi(s_{N_k}(u_k),u_k)}{s_{N_k}(u_k)}\right|\geq|F(u_k)|-\eta/2\qquad\text{and}\qquad\left\|\frac{\varrho(s_{N_k}(u_k),u_k)}{s_{N_k}(u_k)}\right\|\geq\|R(u_k)\|-\eta/2.$$

We conclude that

$$\limsup_{s_k\downarrow 0}\sup_{u\in K}\left(\frac{|\phi(s_k,u)|}{s_k}+\frac{\|\varrho(s_k,u)\|}{s_k}\right)\geq\limsup_{k\to\infty}|F(u_k)|+\|R(u_k)\|-\eta=\infty,$$

again in contradiction to Prop. 3.9. □

Having shown Lemma 3.10, only a small step remains to show regularity. Comparing with Definition 3.1 we see that two ingredients are missing: First we have to show that the limits $F(u)$ and $R(u)$ do not depend on the choice of subsequence, i.e. they are proper limits and hence the proper derivatives of ϕ and ψ at $t=0$, and second we have to show that F and R are continuous on $\mathcal{U}_l$ for each $l\in\mathbb{N}$.

Proof of Theorem 3.3. Our first step is to show that the derivatives $F(u)$ and $R(u)$ in (3.1) exist. By Lemma 3.10 we already know that they exist as limits along a sequence $\mathbb{S}(u)$ which depends on the point $u \in \mathcal{U}$ and has been chosen as a particular subsequence of a given sequence $(t_k(u))_{k\in\mathbb{N}}$. We show now that the limit is in fact independent of the choice of $\mathbb{S}(u)$ and even of the original sequence $(t_k(u))_{k\in\mathbb{N}}$, and hence that $F(u)$ and $R(u)$ are proper derivatives in the sense of (3.1). To this end, fix some $u \in \mathcal{U}$, and let $\widetilde{\mathbb{S}}(u)$ be an arbitrary other sequence $\widetilde{s}_k(u) \downarrow 0$, such that

$$\widetilde{F}_{\mathbb{S}}(u) := \lim_{\widetilde{s}_k(u)\downarrow 0} \frac{\phi(\widetilde{s}_k(u), u)}{\widetilde{s}_k(u)}, \qquad \widetilde{R}_{\mathbb{S}}(u) := \lim_{\widetilde{s}_k(u)\downarrow 0} \frac{\varrho(\widetilde{s}_k(u), u)}{\widetilde{s}_k(u)}. \tag{3.22}$$

We want to show that $F_{\mathbb{S}}(u) = \widetilde{F}_{\mathbb{S}}(u)$ and $R_{\mathbb{S}}(u) = \widetilde{R}_{\mathbb{S}}(u)$. Assume for a contradiction that this were not the case. Then we can find $x \in D$ and $r > 0$ such that the convex set $\{F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), \xi\rangle : \|\xi - x\| \le r\}$ and its counterpart involving $\widetilde{\mathbb{S}}$ are disjoint, i.e.

$$\Big\{F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), \xi\rangle : \|\xi - x\| \le r\Big\} \cap \Big\{\widetilde{F}_{\mathbb{S}}(u) + \Big\langle \widetilde{R}_{\mathbb{S}}(u), \xi\Big\rangle : \|\xi - x\| \le r\Big\} = \emptyset. \tag{3.23}$$

For the next part of the proof, we set $\tau = \inf\{t \ge 0 : \|X_t - X_0\| \ge r\}$, and introduce the following notation:

$$\begin{aligned} a^u_t &:= F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_t\rangle, & A^u_t &:= \int_0^t a^u_{s-}\,ds, \\ G^u_t &:= \exp(A^u_t), & Y^u_t &:= \exp(\langle u, X_t - X_0\rangle) \end{aligned}$$

with $\widetilde{a}^u_t$, $\widetilde{A}^u_t$ and $\widetilde{G}^u_t$ the corresponding counterparts for $\widetilde{F}_{\mathbb{S}}$ and $\widetilde{R}_{\mathbb{S}}$. We show that

$$L^u_{t\wedge\tau} = \frac{Y^u_{t\wedge\tau}}{G^u_{t\wedge\tau}} = \exp\left(\langle u, X_{t\wedge\tau} - X_0\rangle - \int_0^{t\wedge\tau} \left(F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_{s-}\rangle\right) ds\right) \tag{3.24}$$

is a martingale under every $\mathbb{P}^x, x \in D$. This reduces to showing that

$$\mathbb{E}^x\left[\exp\left(\langle u, X_{h\wedge\tau} - X_0\rangle - \int_0^{h\wedge\tau} \left(F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_{s-}\rangle\right) ds\right)\right] = 1\,,$$

since then by the Markov property of X

$$\begin{aligned} &\mathbb{E}^x\left[\exp\left(\left\langle u, X_{(t+h)\wedge\tau} - X_{t\wedge\tau}\right\rangle - \int_{t\wedge\tau}^{(t+h)\wedge\tau} \left(F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_{s-}\rangle\right) ds\right)\middle|\,\mathcal{F}_t\right] = \\ &\mathbb{E}^x\left[\exp\left(\left\langle u, X_{(t+h)\wedge\tau} - X_{t\wedge\tau}\right\rangle - \int_{t\wedge\tau}^{(t+h)\wedge\tau} \left(F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_{s-}\rangle\right) ds\right) 1_{\tau\ge t}\middle|\,\mathcal{F}_t\right] + 1_{\tau\le t} = \\ &\quad = \mathbb{E}^{X_t}\left[\exp\left(\langle u, X_{h\wedge\tau} - X_0\rangle - \int_0^{h\wedge\tau} \left(F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_{s-}\rangle\right) ds\right)\right] 1_{\tau\ge t} + 1_{\tau\le t} = 1 \end{aligned} \tag{3.25}$$

holds true. Now, use the sequence $\mathbb{S}(u) = (s_n(u))_{n\in\mathbb{N}} \downarrow 0$ to define a sequence of Riemannian sums approximating the above integral. Define $M_k = \lfloor h/s_k\rfloor$ and $N_k = \inf\{n \in \mathbb{N} : \|X_{ns_k} - X_0\| > r\}$. First we show that $s_k N_k \to \tau$ almost surely under every $\mathbb{P}^x$. Fix $\omega \in \Omega$ such that $t \to X_t(\omega)$ is a càdlàg function. Let $\widetilde{N}_k(\omega)$ be a sequence in $\mathbb{N}$ such that $s_k\widetilde{N}_k(\omega) \downarrow \tau(\omega)$. It follows from the right-continuity of $t \mapsto X_t(\omega)$ that for large enough k it holds that $\left\|X_{s_k\widetilde{N}_k} - X_0\right\| > r$ and hence that eventually $\widetilde{N}_k(\omega) \ge N_k(\omega)$. On the other hand $\|X_{s_k N_k} - X_0\| > r$ for all $k \in \mathbb{N}$, which implies that $N_k(\omega)s_k \geq \tau(\omega)$. Hence, for large enough $k \in \mathbb{N}$ it holds that

$$s_k \widetilde{N}_k(\omega) \geq s_k N_k(\omega) \geq \tau(\omega).$$

We also know that $s_k \widetilde{N}_k(\omega) \to \tau(\omega)$ as $k \to \infty$, such that we conclude that $s_k N_k \to \tau$ $\mathbb{P}^x$-almost surely, as claimed. By Riemann approximation and the fact that X is càdlàg it then holds that

$$\sum_{j=1}^{M_k \wedge N_k} \left(F_{\$}(u) + \left\langle R_{\$}(u), X_{(j-1)s_k}\right\rangle\right) s_k \to \int_0^{h\wedge\tau} \left(F_{\$}(u) + \langle R_{\$}(u), X_{s-}\rangle\right) ds$$

$\mathbb{P}^x$-almost-surely as $k \to \infty$ for all $x \in D$.

From Lemma 3.10 we know that $\phi(s_k, u) = F_{\$}(u)s_k + o(s_k)$ and $\phi(s_k, u) = R_{\$}(u)s_k + o(s_k)$. Moreover $(M_k \wedge N_k)o(s_k) \to 0$ since $M_k s_k \to 0$. Thus we have that

$$\begin{aligned}
&L(M_k \wedge N_k, s_k, u) = \\
&= \exp\left(\left\langle u, X_{(M_k\wedge N_k)s_k} - X_0\right\rangle - \sum_{j=1}^{M_k\wedge N_k} \left(\phi(t_k, u) + \left\langle \varrho(t_k, u), X_{(j-1)s_k}\right\rangle\right)\right) = \\
&= \exp\Bigg(\left\langle u, X_{(M_k\wedge N_k)s_k} - X_0\right\rangle - \\
&- \sum_{j=1}^{M_k\wedge N_k} \left(F_{\$}(u) + \left\langle R_{\$}(u), X_{(j-1)s_k}\right\rangle\right) s_k + (M_k\wedge N_k)o(s_k)\Bigg) \to \\
&\to \exp\left(\langle u, X_{h\wedge\tau} - X_0\rangle - \int_0^{h\wedge\tau} \left(F_{\$}(u) + \langle R_{\$}(u), X_{s-}\rangle\right) ds\right),
\end{aligned}$$

as $k \to \infty$ almost surely with respect to all $\mathbb{P}^x, x \in D$. But by Lemma 3.8 and optional stopping, $\mathbb{E}^x\left[L(M_k \wedge N_k, s_k, u)\right] = 1$, such that by dominated convergence (cf. (3.19)) we conclude that

$$\mathbb{E}\left[\exp\left(\langle u, X_{h\wedge\tau} - X_0\rangle - \int_0^{h\wedge\tau} \left(F_{\$}(u) + \langle R_{\$}(u), X_{s-}\rangle\right) ds\right)\right] = 1,$$

and hence that $t \mapsto L^u_{t\wedge\tau}$ is a martingale. Summing up we have established that $Y^u_{t\wedge\tau} = L^u_{t\wedge\tau} G^u_{t\wedge\tau}$, where $L^u_{t\wedge\tau}$ is a martingale and hence a semimartingale. Clearly, the process $G^u_{t\wedge\tau}$ is predictable and of finite variation and hence a semimartingale too. We conclude that also the product $Y^u_{t\wedge\tau} = \exp\left(\langle u, X^x_{t\wedge\tau} - x\rangle\right)$ is a semimartingale. It follows from [12, Thm. I.4.49] that $M^u_{t\wedge\tau} = Y^u_{t\wedge\tau} - \int_0^{t\wedge\tau} L^u_{s-} dG^u_s$ is a local martingale. We can rewrite M^u_t as

$$M^u_t = Y^u_{t\wedge\tau} - \int_0^t L^u_{s-} G^u_{s-} dA^u_s = Y^u_t - \int_0^t Y^u_{s-} dA^u_s = Y^u_t - \int_0^t Y^u_{s-} a^u_{s-} ds.$$

Hence $Y^u_{t\wedge\tau} = M^u_{t\wedge\tau} + \int_0^{t\wedge\tau} Y^u_{s-} a^u_{s-} ds$ is the decomposition of the semi-martingale $Y^u_{t\wedge\tau}$ into a local martingale and a finite variation part. But $\int_0^{t\wedge\tau} Y^u_{s-} a^u_{s-} ds$ is even predictable, such that Y^u is a special semi-martingale, and the decomposition is unique. The same derivation goes through with A^u replaced by $\widetilde{A}^u$ and by the uniqueness of the special semi-martingale decomposition we conclude that

$$\int_0^{t\wedge\tau} Y^u_{s-} a^u_{s-} ds = \int_0^{t\wedge\tau} Y^u_{s-} \widetilde{a}^u_{s-} ds,$$

up to a $\mathbb{P}^x$-nullset. Taking derivatives we see that $Y^u_{t-}a^u_{t-} = Y^u_{t-}\widetilde{a}^u_{t-}$ on $\{t \le \tau\}$. As long as $t \le \tau$ it holds that $Y^u_{t-} \neq 0$, and dividing by Y^u_{t-}, we see that $a^u_{t-} = \widetilde{a}^u_{t-}$, that is

$$F_{\mathbb{S}}(u) + \langle R_{\mathbb{S}}(u), X_{t-}\rangle = \widetilde{F}_{\mathbb{S}}(u) + \left\langle \widetilde{R}_{\mathbb{S}}(u), X_{t-}\right\rangle \quad \text{on } \{t \le \tau\},$$

$\mathbb{P}^x$-a.s, in contradiction to (3.23). We conclude that the limits $F_{\mathbb{S}}$ and $R_{\mathbb{S}}$ are independent from the sequence $\mathbb{S}$, and hence that $F(u)$ and $R(u)$ exist as proper derivatives in the sense of (3.1).

It remains to show that $F(u)$ and $R(u)$ are continuous on $\mathcal{U}_l$ for each $l \in \mathbb{N}$. Fix $l \in \mathbb{N}$ and suppose for a contradiction that there exists a sequence $u_k \to u_0$ in $\mathcal{U}_l$ such that $F(u_k) \to F^*$ and $R(u_k) \to R^*$, such that either $F(u_0) \neq F^*$ or $R(u_0) \neq R^*$. Since D affinely spans $\mathbb{R}^d$ this means that there is $x \in D$ with

$$F(u_0) + \langle R(u_0), x\rangle \neq F^* + \langle R^*, x\rangle\,.$$

Using the fact that $\mathbb{E}^x\left[L^{u_k}_{t\wedge\tau}\right] = 1$ for all $k \in \mathbb{N}$ we obtain

$$\begin{aligned}\frac{1}{t}\left(\exp(\langle \phi(t,u_0) + \psi(t,u_0), x\rangle) - 1\right) &= \lim_{k\to\infty}\frac{1}{t}\mathbb{E}^x\left[e^{\langle u_0, X_t - X_0\rangle} - L^{u_k}_{t\wedge\tau}\right] = \\ &= \lim_{k\to\infty}\frac{1}{t}\mathbb{E}^x\left[e^{\langle u_0, X_t - X_0\rangle}\left(1 - \exp(-\int_0^{t\wedge\tau}(F(u_k) + \langle R(u_k), X_{s-}\rangle)ds\right)\right] = \\ &= \mathbb{E}^x\left[\frac{1}{t}e^{\langle u_0, X_t - X_0\rangle}\left(1 - \exp(-\int_0^{t\wedge\tau}(F^* + \langle R^*, X_{s-}\rangle)ds\right)\right]. \qquad (3.26)\end{aligned}$$

for all $t \le \sigma(0)$ (see (2.6) for the definition of $\sigma(.)$) by dominated convergence. Writing $C = |F^*| + \|R^*\|\,\varepsilon$ and using the elementary inequality $|1 - e^z| \le |z|e^{|z|}$ we can bound

$$\left|\frac{1}{t}e^{\langle u_0, X_t - X_0\rangle}\left(1 - \exp(-\int_0^{t\wedge\tau}(F^* + \langle R^*, X_{s-}\rangle)ds\right)\right| \le Ce^{2l + Ct}$$

and therefore apply again dominated convergence to the right hand side of (3.26) as $t \to 0$. Taking the limit on both sides, we obtain

$$F(u_0) + \langle R(u_0), x\rangle = F^* + \langle R^*, x\rangle$$

leading to the desired contradiction. □

We conclude with a corollary that gives conditions for an affine process to be a D-valued semimartingale, up to its explosion time. Let $\tau_n = \inf\{t \ge 0 : \|X_t - X_0\| > n\}$ and define the explosion time τ_{exp} as the pointwise limit $\tau_{\text{exp}} = \lim_{n\to\infty}\tau_n$. Note that τ_{exp} is predictable.

Corollary 3.11. *Let X be a càdlàg affine process and suppose that the killing terms vanish, i.e. $c = 0$ and $\gamma = 0$. Then under every $\mathbb{P}^x, x \in D$ the process X is a D-valued semi-martingale on $[0, \tau_{\text{exp}})$ with absolutely continuous semimartingale characteristics*

$$\begin{aligned}A_t &= \int_0^t A(X_{s-})ds \\ B_t &= \int_0^t B(X_{s-})ds \\ K([0,t], d\xi) &= \int_0^t \nu(X_{s-}, d\xi)ds.\end{aligned}$$

where $A(.), B(.)$ and $\nu(., d\xi)$ are given by (3.3).

Proof. In the proof of Theorem 3.3 we have shown that $t \mapsto L^u_{t\wedge\tau}$, with L^u_t defined in (3.24) and $\tau = \inf\{t \geq 0 : \|X_t - X_0\| > r\}$, is a martingale under every $\mathbb{P}^x, x \in D$ and for every $u \in \mathcal{U}$. Since $r > 0$ was arbitrary, also $L^u_{t\wedge\tau_n}$ is a martingale for every $n \in \mathbb{N}$. By dominated convergence and using that $F(0) + \langle R(0), x\rangle = c + \langle \gamma, x\rangle = 0$ for all $x \in D$ we obtain

$$\mathbb{P}^x\left(X_{t\wedge\tau_{\mathrm{exp}}} \neq \delta\right) = \lim_{n\to\infty} \mathbb{P}\left(X_{t\wedge\tau_n} \neq \delta\right) = \mathbb{E}\left[L^0_{t\wedge\tau_n}\right] = 1.$$

Hence X_t and X_{t-} stay $\mathbb{P}^x$-almost surely in $D \subset \mathbb{R}^d$ for $t \in [0, \tau_{\mathrm{exp}})$. Moreover $t \mapsto L^u_t$ is a local martingale on $[0, \tau_{\mathrm{exp}})$ for all $u \in \mathcal{U}$. Thus [12, Cor. II.2.48b] can be applied to the local martingale L^u_t with $u \in i\mathbb{R}^d$ and the assertion follows. □

References

[1] M.-F. Bru, Wishart processes, J. Theoret. Probab. **4** (1991), no. 4, 725–751. MR-1132135

[2] A. Di Bucchianico, Banach algebras, logarithms, and polynomials of convolution type, J. Math. Anal. Appl. **156** (1991), no. 1, 253–273. MR-1102610

[3] E. Çinlar et al., Semimartingales and Markov processes, Z. Wahrsch. Verw. Gebiete **54** (1980), no. 2, 161–219. MR-0597337

[4] C. Cuchiero and J. Teichmann, Path properties and regularity of affine processes on general state spaces, arXiv:1107.1607, 2011.

[5] Christa Cuchiero, Affine and polynomial processes, Ph.D. thesis, ETH Zürich, 2011.

[6] C. Cuchiero et al., Affine processes on positive semidefinite matrices, Ann. Appl. Probab. **21** (2011), no. 2, 397–463. MR-2807963

[7] Christa Cuchiero, Martin Keller-Ressel, Eberhard Mayerhofer, and Josef Teichmann, Affine processes on symmetric cones, arXiv:1112.1233, 2011.

[8] D. A. Dawson and Z. Li, Skew convolution semigroups and affine Markov processes, Ann. Probab. **34** (2006), no. 3, 1103–1142. MR-2243880

[9] D. Duffie, D. Filipović and W. Schachermayer, Affine processes and applications in finance, Ann. Appl. Probab. **13** (2003), no. 3, 984–1053. MR-1994043

[10] Darrell Duffie and Rui Kan, A yield-factor model of interest rates, Mathematical Finance **6** (1996), 379 – 406.

[11] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, Oxford Univ. Press, New York, 1994. MR-1446489

[12] J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes*, Grundlehren der Mathematischen Wissenschaften, 288, Springer, Berlin, 1987. MR-0959133

[13] Jan Kallsen, A didactic note on affine stochastic volatility models, From Stochastic Calculus to Mathematical Finance (Y. Kabanov, R. Liptser, and J. Stoyanov, eds.), Springer, Berlin, 2006, pp. 343 – 368.

[14] K. Kawazu and S. Watanabe, Branching processes with immigration and related limit theorems, Teor. Verojatnost. i Primenen. **16** (1971), 34–51. MR-0290475

[15] M. Keller-Ressel, W. Schachermayer and J. Teichmann, Affine processes are regular, Probab. Theory Related Fields **151** (2011), no. 3-4, 591–611. MR-2851694

[16] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York, 1955. MR-0073104

[17] K. Sato, *Lévy processes and infinitely divisible distributions*, translated from the 1990 Japanese original, Cambridge Studies in Advanced Mathematics, 68, Cambridge Univ. Press, Cambridge, 1999. MR-1739520

[18] P. Spreij and E. Veerman, Affine diffusions with non-canonical state space, Stoch. Anal. Appl. **30** (2012), no. 4, 605–641. MR-2946041

Acknowledgments. The first and third author gratefully acknowledge the support by the ETH foundation. The second author gratefully acknowledges financial support from

the Austrian Science Fund (FWF) under grant P19456, from the European Research Council (ERC) under grant FA506041 and from the Vienna Science and Technology Fund (WWTF) under grant MA09-003. Furthermore this work was financially supported by the Christian Doppler Research Association (CDG).
The authors would like to thank Enno Veerman and Maurizio Barbato for comments on an earlier draft