CMC constant angle hypersurfaces in semi-riemannian space forms

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Preliminaries

Given a vector field Z in a semi-riemannian manifold N, a hypersurface M have **constant angle** relative to Z if the $\langle Z/|Z|, \xi\rangle$ is a constant function on M. When N is a Riemannian manifold the above function is called the angle function and has a geometric interpretation. In a general semi-riemannian ambient, we are not defining an angle, just the concept of hypersurface of constan angle.

We will consider the case when Z is a closed and conformal vector field.

A vector field $Z \in \mathfrak{X}(N)$ is *closed conformal* if and only if there exist a differentiable function φ defined on N

$$\bar{\nabla}_Y Z = \varphi Y \tag{1}$$

for every $Y \in \mathfrak{X}(N)$. Examples:

- A parallel vector field
- A radial vector field in a semi-euclidean space
- Semi-riemannian space forms are plenty of closed conformal vector fields

Notation

Semi-riemannian space forms: $\mathbb{R}^{n+1}_s$, $\mathbb{S}^{n+1}_s(r)$ and $\mathbb{H}^{n+1}_s(r)$ by $\overline{\mathbb{M}}^{n+1}_s(c)$ where $c = 0$, $c = 1/r^2$ or $c = -1/r^2$ respectively.

$\mathbb{S}^{n+1}_s(r) := \{x \in \mathbb{R}^{n+2}_s \mid -x_1^2 - \ldots - x_s^2 + x_{s+1}^2 + \ldots + x_{n+2}^2 = r^2\}$

$\mathbb{H}^{n+1}_s(r) := \{x \in \mathbb{R}^{n+2}_{s+1} | -x_1^2 - \ldots - x_{s+1}^2 + x_{s+2}^2 + \ldots + x_{n+2}^2 = -r^2\}$

$\bar{\nabla}$ is the Levi-Civita de $\overline{\mathbb{M}}^{n+1}_s(c)$.

Z closed and conformal in $\overline{\mathbb{M}}^{n+1}_s(c)$.

ξ is a unit vector field everywhere normal to M.

$\epsilon_Z := \langle Z/|Z|, Z/|Z|\rangle$ We have the Gauss and Weingarten equations:

$$\bar{\nabla}_Y X = \nabla_Y X + \alpha(Y, X), \quad \bar{\nabla}_Y \xi = -A_\xi Y,$$

$X, Y \in \mathfrak{X}(M)$

α is the second fundamental form of M

A_ξ is the shape operator associated to ξ

α and A_ξ

$$\langle \alpha(Y, X), \xi\rangle = \langle A_\xi Y, X\rangle.$$

Let us define $Z^T, T \in \mathfrak{X}(M)$ by

$$Z^T = Z - \langle \xi, \xi\rangle\langle Z, \xi\rangle\xi \quad \text{and} \quad T = \frac{Z^T}{|Z^T|}.$$

A Simons type formula

Let M be a semi-Riemannian hypersurface isometrically immersed in $\overline{\mathbb{M}}^{n+1}_s(c)$ with CMC and let ξ a local unitary vector field orthogonal to M. If Z is a closed and conformal vector field on $\overline{\mathbb{M}}^{n+1}_s(c)$ with associated function φ, then

$$\triangle\langle Z, \xi\rangle + \langle \alpha, \alpha\rangle\langle Z, \xi\rangle + \varphi\langle H, \xi\rangle = 0, \tag{2}$$

where α is the second fundamental form of the immersion and H is the mean curvature vector of M.

The shape operator of a constant angle hypersurface

Let $M \subset \overline{\mathbb{M}}^{n+1}_s(c)$ be a constant angle hypersurface with respect to a closed and conformal vector field Z with associated function φ. Then $A_\xi(Z^\top) = kZ^\top$, *where* $k = -\frac{\epsilon_Z \varphi\langle Z,\xi\rangle}{|Z|^2}$.

- $Z^\top$ is a principal direction of M with principal curvature k.
- The integral curves of $T = Z^\top/|Z^\top|$ are geodesics of M.

Corollary

Let M be a semi-Riemannian hypersurface immersed in a semi-Euclidean space making a constant angle with respect to a parallel vector field Z. Then $Ric(Z^\top, Z^\top) = 0$ and the integral curves of $Z^\top$ are straight line segments in the ambient, i.e. M is ruled. Moreover, if Z is not tangent to M then M has constant mean curvature in the direction $Z^\top$ (i.e. $Z^\top \cdot H = 0$) if and only if $\langle \alpha, \alpha\rangle = 0$.

CMC + Constant Angle in a Semi-euclidean Space

Let M be a semi-Riemannian hypersurface immersed in a semi-Euclidean space with constant angle with respect to a parallel vector field Z. Let us assume that Z is not tangent to M. If M has constant mean curvature in the direction $Z^\top$ then any of the two following conditions imply that M is an open part of a hyperplane:

1. Either M is spacelike or A_ξ is diagonalizable.
2. The semi-Euclidean ambient is the Minkowski space, M is timelike and Z^T is timelike.

The case when Z is tangent

Let M be a CMC hypersurface isometrically immersed in $\overline{\mathbb{M}}^{n+1}_s(c)$. If Z is tangent to M then it has constant zero Gauss-Kronecker curvature and either M has zero mean curvature or $c = 0$, i.e. $\overline{\mathbb{M}}^{n+1}_s(c)$ is a semi-Euclidean space $\mathbb{R}^{n+1}_s$. Moreover, the Ricci curvature of M in the direction $Z^\top$ is given by $Ric(Z^\top, Z^\top) = (n-1)\langle Z^\top, Z^\top\rangle c$. In particular, if $\dim M = 2$ then either M is totally geodesic or $c = 0$.

SURFACES: Let M be a CMC semi-Riemannian surface in a three dimensional space form of non-vanishing curvature and with constant angle with respect to a closed and conformal Z. Then M is totally umbilical or totally geodesic when - $M \subset \mathbb{S}^3_1(r)$ is spacelike, - $M \subset \mathbb{H}^3_1(r)$ is timelike, - Z and ξ have the same causality.

Surfaces continued:
If the mean curvature of the surface is not $\pm 2/\sqrt{3}$ they are either totally umbilic or totally geodesic. In particular when the surface has zero mean curvature it is totally geodesic.

Surfaces are isoparametric

Let M be a CMC semi-Riemannian surface isometrically immersed in $\mathbb{R}^3_s$ with constant angle with respect to a closed and conformal vector field Z. Then either

1. M is an open portion of a plane.
2. M is an open portion of a quadric.
3. M is an open portion of a cylinder in direction Z over a curve with constant curvature in a plane orthogonal to Z.

This means that M is **isoparametric**.

These results are in [1] and were in collaboration with Dr. Didier Solis and Dr Matias Navarro from University of Yucatan México.

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