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ON KERNELS AND SEMIKERNELS OF DIGRAPHS

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A kernel N of a digraph D is an independent set of vertices of D such that for every $w \in V(D)-N$ there exists an arc from w to N. If every induced subdigraph of D has a kernel, D is said to be an R-digraph. Minimal non-R-digraphs are called R^--digraphs. In this paper some structural results concerning R^--digraphs and sufficient conditions for a digraph to be an R-digraph are presented. In particular, it is proved that every vertex (resp. arc) in an R^--digraph is contained in an odd directed cycle not containing special pseudodiagonals. It is also proved that any digraph in which every odd directed cycle has two pseudodiagonals with consecutive terminal endpoints is an R-digraph. Previous results of other authors (Richardson, Meyniel, Duchet, and others) are generalized.

1. Introduction

For general concepts we refer the reader to [1].

Let D be a digraph; $V(D)$ and $F(D)$ (or FD) will denote the set of vertices and arcs of D respectively. Often we shall write u_1u_2 instead of (u_1, u_2). Let S_1, S_2 be subsets of $V(D)$. The arc u_1u_2 of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$. A directed S_1S_2-path is any u_1u_2-directed path with $u_1 \in S_1$ and $u_2 \in S_2$, $D[S_1]$ will denote the subdigraph of D induced by S_1 and $D[S_1, S_2]$ the subdigraph of D whose vertex-set is $S_1 \cup S_2$ and whose arcs are the S_1S_2-arcs of D. The length of a path P is denoted by $l(P)$.

Definition. A set $I \subset V(D)$ is *independent* if $F(D[I]) = \emptyset$.

A *kernel* N of D is an independent set of vertices such that for each $z \in V(D)-N$ there exists a zN-arc in D.

The concept of kernel was introduced by Von Neumann and Morgenstern [6] in the context of Game Theory. They also proved that any finite acyclic digraph has a (unique) kernel. The problem of the existence of a kernel in a given digraph has been studied by several authors, in particular by Richardson [7–9], Neumann-Lara [5] and recently by Duchet and Meyniel [2, 3]. A well-known result of

Richardson states that any digraph which does not contain directed cycles of odd length has a kernel. A short proof of this result was obtained in [5] (see also [1, p. 311]) by introducing the concepts of semikernel and R-digraphs.

Definition. A *semikernel S* of D is an independent set of vertices such that for every $z \in V(D)-S$ for which there exists a Sz-arc, there also exists a zS-arc; D is an *R-digraph* iff every non-empty induced subdigraph of D has a non-empty semikernel. We need the following results included in [5].

Lemma A. *Let S be a semikernel of D, $B=\{v \in V(D)-S \mid \exists\ vS\text{-arcs in } D\}$, and S' a semikernel (resp. kernel) of $D[B]$. Then $S \cup S'$ is a semikernel (resp. kernel) of D.*

Theorem 1.1. *D is an R-digraph if and only if every induced subdigraph of D has a kernel.*

Thus an R-digraph is just a kernel-perfect graph in the terminology of Duchet and Meyniel [3]. We say that D is an R^--digraph if D does not have a kernel but every proper induced subdigraph of D does have at least one (R^--digraphs are called kernel-perfect-critical graphs by Duchet and Meyniel [3]).

In the present work we study some general sufficient conditions for a digraph to be an R-digraph and some structural properties of R^--digraphs. To this end we introduce in Section 2 the concepts of strong semikernel of a digraph D modulo a subset of $V(D)$, and K-normality.

Finally we give some more notation.

We write $F^-(S_1)$ (resp. $F^+(S_1)$) instead of $FD[V(D), S_1]$ (resp. $FD[S_1, V(D)]$) and F_u^-, F_u^+ for $F^-(\{u\})$, $F^+(\{u\})$ resp. If D_0 is a subdigraph (resp. induced subdigraph) of D, we write $D_0 \subset D$ (resp. $D_0 \subset^* D$). An arc $uv \in F(D)$ is called a pseudodiagonal of $D_0 \subset D$ whenever $u, v \in V(D_0)$ and $uv \notin F(D_0)$. If $C = (u_0, u_1, \ldots, u_n, u_0)$ is a directed cycle, we put

$$C^0_{u_0} = \{u_i \mid i \equiv 0 \pmod 2;\ i \neq 0\}, \qquad C^1_{u_0} = \{u_i \mid i \equiv 1 \pmod 2\}.$$

For instance if $C=(u_0, u_1, u_2, u_0)$, $C^0_{u_0}=\{u_2\}$, $C^1_{u_0}=\{u_1\}$.

For a path $P=(u_0, \ldots, u_n)$ we put

$$P^0 = \{u_i \mid i \equiv 0 \pmod 2\}, \qquad P^1 = \{u_i \mid i \equiv 1 \pmod 2\}.$$

2. Semikernels modulo *R* and *K*-normal directed paths

In this section we introduce the concepts of semikernel and strong semikernel of a digraph modulo a set of vertices and state Theorem 2.1 which is the main tool used in this paper. Theorems 2.2 and 2.3 are useful variations of Theorem 2.1.

Definition 2.1. Let D be a digraph; $I, R \subset V(D)$ and consider the following conditions:

(i) $I \cap R^c$ is an independent set.

(i′) D does not contain $(I \cap R^c)I$-arcs.

(ii) If $uv \in F(D)$, $u \in I \cap R^c$ and $v \in I^c \cap R^c$, then there exists $w \in I$ such that $vw \in F(D)$.

If conditions (i) and (ii) are satisfied, I will be called a *semikernel of D modulo R*.

If condition (i′), which is stronger than (i), and (ii) are satisfied, I will be called a *strong semikernel of D modulo R*.

Definition 2.2. Suppose that $K \subset V(D)$. A directed path $T = (w_0, w_1, \ldots, w_n)$ in D will be called *K-normal* whenever T satisfies:

(i) $V(T) \cap K = \{w_j \mid 1 \leqslant j \leqslant n,\ j \text{ odd}\}$. or $V(T) \cap K = \{w_j \mid 0 \leqslant j \leqslant n,\ j \text{ even}\}$.

(ii) If $s < j < n$, $w_j \in K^c$, $w_s \in K$, then $w_j w_s \notin F(D)$.

Remark 2.1. Notice that any K-normal directed path passes by K and K^c alternately.

Theorem 2.1. *If I_0, I, $R \subset V(D)$ are such that $I_0 \subset I$, $I_0 \cap R = \emptyset$ and satisfy*

(a) *I is a strong semikernel of D modulo R,*

(b) *every I-normal, I_0R-directed path passes by $U = \Gamma^-(I_0) \cap R^c$,*

then $S = \{w \in I \mid$ there exists an I-normal, I_0w-directed path not passing by $U\}$ is a semikernel of D which satisfies $I_0 \subset S \subset I \cap R^c$.

Proof. By (a), $U \subset I^c \cap R^c$ and I_0 is an independent set. Therefore $I_0 \subset S$. By (b), $S \subset R^c$ and so $S \subset I \cap R^c$. Using (a) again, we conclude that D contains no SI-arcs. Therefore, S is an independent set. Suppose that S is not a semikernel of D. Then there exists $s \in S$ and $w \in V(D) - S$ such that $sw \in F(D)$ and D contains no wS-arc. Let $(w_0, w_1, \ldots, w_m)$, $w_m = s$, $w_0 \in I_0$, be an I-normal, I_0s-directed path not passing by U. Since $w \in I^c$, the directed path $(w_0, w_1, \ldots, w_m, w)$ is also I-normal. Then $w \notin R$, since otherwise (b) would be contradicted. Therefore $w \in I^c \cap R^c$. By (a), there exists $wz \in F(D)$ with $z \in I$. The directed path $(w_0, w_1, \ldots, w_m, w, z)$ is I-normal and does not pass by U. Therefore $z \in S$ and wz is a wS-arc in D, which contradicts the assumption that D contains no wS-arc. We conclude that S is a semikernel of D, (see Fig. 1).

Theorems 2.2 and 2.3 are useful variations of Theorem 2.1.

Theorem 2.2. *Suppose that I_0, I, $R \subset V(D)$ satisfy the following conditions:*

(i) *I is a strong semikernel of D modulo R.*

(ii) *D contains no semikernel S such that $I_0 \subset S \subset I \cap R^c$.*

Then, there exists a direct I-normal, I_0R-directed path $T = (t_0, \ldots, t_n)$, not passing

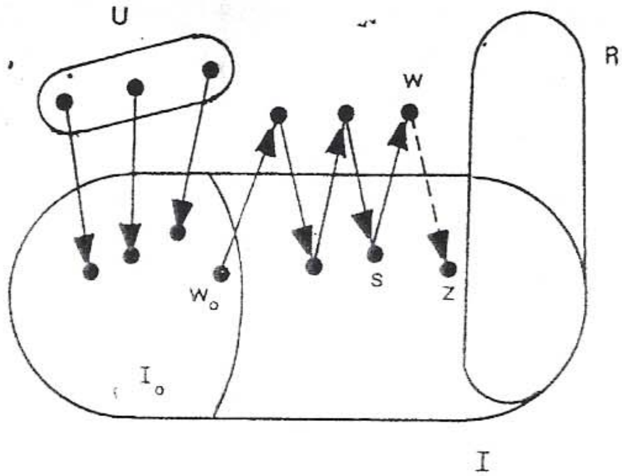

Fig. 1.

by $\Gamma^-(I_0)\cap R^c$ which satisfies the following properties:

(1) *T has no $(V(T)-t_n)T^0$-pseudodiagonals.*

(2) *$l(T)$ is even iff $t_n\in I$.*

Proof. By Theorem 2.1, there exists an I-normal, I_0R-directed path T not passing by $\Gamma^-(I_0)\cap R^c$. Choose T so that $l(T)$ takes the minimum possible value. Clearly T satisfies

(1a) $t_{2i}t_j\notin F(D)$ for all $0\leq 2i<j\leq n$, $j\neq 2i+1$,

(2a) $t_{2i+1}t_{2j}\notin F(D)$ for all $0<2i+1<2j\leq n$, $j\neq i+1$,

and by using the I-normality of T we conclude that T satisfies (1) and (2).

A special case of Theorem 2.2 is the following result.

Theorem 2.3. *Let I_0, I, $R\subset V(D)$ be such that $\emptyset\neq I_0\subset I$, $I_0\cap R=\emptyset$. Suppose that conditions* (i), (ii) *and* (iii) *are satisfied:*

(i) *I is a strong semikernel of D modulo R.*

(ii) *D has no kernel.*

(iii) *$D-(I_0\cup\Gamma^-(I_0))$ is an R-digraph.*

Then there exists a direct I-normal, I_0R-directed path T not passing by $\Gamma^-(I_0)\cap R^c$, which satisfies conditions (1) *and* (2) *of Theorem* 2.2.

Proof. If Theorem 2.3 were false, D would contain a semikernel S such that $\emptyset\neq I_0\subset S\subset I\cap R^c$. By (iii) and Lemma A, D would contain a kernel contradicting (ii).

3. Structural results on kernel theory

In this section we apply the results of Section 2.

Theorem 3.1. *Let D be a digraph, $u \in V(D)$ and N_u a kernel of $D-u$. Suppose that conditions (i) and (ii) are satisfied:*

(i) *$D-v$ is an R-digraph.*

(ii) *D has no kernel.*

Then there exists an N_u-normal, vu-directed path T without $(V(T)-u)T^i$-pseudodiagonals (where i is the residue of $l(T)+1$ modulo 2).

Proof. Take $I = N_u$, $R = \{u\}$ and define I_0 as follows: If $v \in N_u$, $I_0 = \{v\}$; otherwise $I_0 = \Gamma^+(v) \cap N_u$. Since the conditions of Theorem 2.3 are fulfilled, Theorem 3.1 follows.

Corollary 3.1. *Let $f = uv$ be an arc of D. Suppose that D has no kernel and satisfies:*

(i) *$D-u$ has no kernel.*

(ii) *$D-v$ is an R-digraph.*

Then there exists a directed cycle C, of odd length passing by f and having no $V(C)C_u^0$-pseudodiagonals. (In particular C_u^0 is an independent set.)

Theorem 3.2. *If $\emptyset \neq A \subset F_u^+$ and $I_0 = \{z \in V(D) \mid uz \in A\}$ satisfy:*

(i) *$D-A$ has a kernel but $D-A'$ has no kernel for $A' \subsetneq A$,*

(ii) *$D-(I_0 \cup \Gamma^-(I_0))$ is an R-digraph,*

then there exist $f \in A$ and a directed cycle C of odd length passing by f, not intersecting $\Gamma^-(I_0)-\{u\}$ and without $V(C)(C_u^1 \cup \{u\})$-pseudodiagonals.

Proof. By (i) D has no kernel. Let I be a kernel of $D-A$ and $R = \{u\}$. Clearly I is a strong semikernel of D modulo R and $I_0 \cup \{u\} \subset I$. By Theorem 2.3 there exists a direct I-normal, t_0u-directed path T with $t_0 \in I_0$ not passing by $\Gamma^-(I_0)-u$. Adding the arc ut_0 to T we get a directed cycle C with the required properties.

Corollary 3.2. *Let $f = uv \in F(D)$. If D does not have a kernel and $D-f$ is an R-digraph, then there exists a directed cycle C of odd length containing f and without $V(C)(C_u^1 \cup \{u\})$-pseudodiagonals.*

Corollary 3.3. *Let $u \in V(D)$. If D does not have a kernel and $D-u$ is an R-digraph, then there exists a directed cycle C of odd length containing u and without $V(C)(C_u^1 \cup \{u\})$-pseudodiagonals.*

Proof. Since $\{u\}$ is a semikernel of $D-F_u^+$ and $D-u$ is an R-diagraph, by Lemma A, $D-F_u^+$ has a kernel N_u containing u. Choose N_u so that $|F(D[N_u]) \cap F_u^+|$ takes the minimum possible value, take $A = F_u^+ \cap FD[N_u]$ and apply Theorem 3.2 to conclude the proof.

Theorem 3.3. *If $\emptyset \neq A \subset F_u^-$ has the following properties:*

(i) $D-A$ *has a kernel*,
(ii) $D-A'$ *has no kernel for* $A' \subsetneq A$,
(iii) $D-(\Gamma^-(u)\cup\{u\})$ *is an* R*-digraph*,

then there exist $f=wu\in A$ *and a directed cycle* C *of odd length passing by* f*, not intersecting* $\Gamma^-(u)-\{u, w\}$ *and without* $V(C)(C_w^1\cup\{w\})$*-pseudodiagonals.*

Proof. Let I be a kernel of $D-A$ and take $I_0=\{u\}$ and $R=\{z\in V(D) \mid zu\in A\}$. By (ii), $R\cup\{u\}\subset I$. By Theorem 2.3, there exists a direct I-normal, uw-directed path T not passing by $\Gamma^-(u)\cap R^c$, and such that $w\in R$. Adding wu to T we obtain a cycle which satisfies the required properties.

Corollary 3.4. *Let* u *be a vertex of* D*. If* D *has no kernel and* $D-u$ *is an* R*-digraph, then there exists* $f=vu\in F(D)$ *and a directed cycle* C *of odd length passing by* f *and having no* $V(C)(C_v^1\cup\{v\})$*-pseudodiagonals.*

Proof. Let N_u be a kernel of $D-u$ such that $|\Gamma^-(u)\cap N_u|$ takes the minimum possible value. Take $A=F_u^-\cap F(D[N_u])$ and apply Theorem 3.3.

4. R^--digraphs structure

The results of this section are corollaries of those of Section 3.

Theorem 4.1. *Let* D *be an* R^-*-digraph and* $u, v\in V(D)$*. Then there exists a* vu*-directed path* $T=(w_0, w_1, \ldots, w_n)$, $w_0=v$, $w_n=u$, *having no* $V(T)T^i$*-pseudodiagonals (where* i *is the residue of* $n+1$ *modulo 2).*

Proof. It follows directly from Theorem 3.1.

Corollary 4.1 (Duchet [2]). R^-*-digraphs are strongly connected.*

Theorem 4.2. *Let* D *be an* R^-*-digraph, and* $f=uv\in F(D)$*. Then there exists a directed cycle* C *of odd length containing* f *and having no* $V(C)C_u^0$*-pseudodiagonals. (In particular* C_u^0 *is an independent set.)*

Proof. It follows directly from Corollary 3.1.

Corollary 4.2. *Let* D *be an* R^-*-digraph and* $u\in V(D)$*. Then there exists a directed cycle* C *of odd length passing by* u *which contains neither* $V(C)C_u^0$*-pseudodiagonals nor* uC*-pseudodiagonals.*

Theorem 4.3. *Let* D *be an* R^-*-digraph,* $u\in V(D)$*. Then there exists a directed cycle* C *of odd length passing by* u *and having no* $V(C)(C_u^1\cup\{u\})$*-pseudodiagonals.*

Proof. It follows directly from Corollary 3.3

Theorem 4.4. *Let D be an R^--digraph and $u \in V(D)$. Then for some $f = vu \in F(D)$ there exists a directed cycle of odd length passing by f and having no $V(C)(C_v^1 \cup \{v\})$-pseudodiagonals.*

Proof. It follows directly from Corollary 3.4.

Theorem 4.5. *Let D be an R^--digraph which is not a directed cycle of odd length. $u \in V(D)$. Then there exist $f' \in F_u^-$ and $f'' \in F_u^-$ such that each of them belongs to at least two directed cycles of odd length.*

Proof. By Theorem 4.4 there exists a directed cycle C of odd length passing by some $f' = vu \in F_u^-$ and containing no $V(C)(C_v^1 \cup \{v\})$-pseudodiagonals. Let C' be a cycle of odd length passing by f' and without $V(C')C_v'^0$-pseudodiagonals (Theorem 4.2), $C' \neq C$, for otherwise C would be an induced subdigraph of D. In a similar way and applying Theorems 4.2 and 4.3 we prove the existence of f''.

Corollary 4.3. *Suppose that D is an R^--digraph which is not a directed cycle of odd length and $u \in V(D)$. Then u belongs to at least $\Delta_D(u) + 1$ directed cycles of odd length ($\Delta_D(u) = \max\{|\Gamma^-(u)|, |\Gamma^+(u)|\}$).*

Proof. It follows directly from Theorems 4.2 and 4.5.

5. R-digraphs

In this section we study some sufficient conditions for a digraph to be an R-digraph. Lemma 5.1 gives a general scheme for results and proofs included in this section.

5.1. *General results*

Lemma 5.1. *If $P_0 \subset P(D) = \{D_0 \subset^* D \mid \exists H$: H is an R^--digraph and $D_0 \subset^* H \subset^* D\}$, then D is an R-digraph iff every induced subdigraph D_0 of D not containing induced subdigraphs in P_0 is an R-digraph.*

Proof. If D were not an R-digraph, it would contain an induced R^--subdigraph H. Since H contains no induced subdigraph belonging to P_0, H is an R-digraph. This yields a contradiction. The converse is obvious.

Theorem 5.1. *Let D be a digraph and $T \subset V(D)$ such that $D - T$ is an R-digraph. Furthermore suppose that for every $u \in T$ either* (a) *or* (b) *is satisfied.*

(a) *Every directed cycle C passing by u has at least one $V(C)C_u^0$-pseudodiagonal.*

(b) *Every directed cycle C of odd length passing by u has at least one $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal. Then D is an R-digraph.*

Proof. If D is not an R-digraph, D contains an induced R^--subdigraph H. Since $D-T$ is an R-digraph, $V(H) \cap T \neq \emptyset$. Take any $u \in V(H) \cap T$. By Theorems 4.2 and 4.3 neither (a) nor (b) are satisfied in H and consequently in D. The hypothesis is thus contradicted.

Theorem 5.2. *Let D be a digraph and $A \subset F(D)$. Suppose that every $f = uv \in A$ satisfies:* (i) *Each directed cycle C of odd length passing by f has some $V(C)C_u^0$-pseudodiagonal*

Then D is an R-digraph if and only if every induced subdigraph H of D such that $F(H) \cap A = \emptyset$ is an R-digraph.

Proof. If D is not an R-digraph, D contains an induced R^--subdigraph H. It follows by hypothesis that $F(H) \cap A \neq \emptyset$. Take $f \in F(H) \cap A$ and apply Theorem 4.2. Condition (i) is thus contradicted. The converse is obvious.

Let C be a directed cycle of odd length and $p(C) = \{w \in V(C) \mid \exists V(C)w$-pseudodiagonal of $C\}$. By definition

$$C^{(0)} = \bigcup_{v \in p(C)} C_v^1, \qquad C^{(1)} = p(C) \cup \bigcup_{u \in p(C)} C_u^0.$$

Corollary 5.1 (to Theorem 5.1). *Let D be a digraph and $T \subset V(D)$. Suppose that $D-T$ is an R-digraph. If every directed cycle C of odd length such that $V(C) \cap T \neq \emptyset$ satisfies $C = C^{(1)}$, then D is an R-digraph.*

Proof. If C is a directed cycle in D of odd length such that $C = C^{(1)}$ and $u \in V(C)$, then C has at least one $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal.

Corollary 5.2 (to Theorem 5.2). *Let D be a digraph and $A \subset F(D)$. Suppose that every directed cycle C of odd length such that $F(C) \cap A \neq \emptyset$ satisfies $C = C^{(0)}$. Then D is an R-digraph if and only if every induced subdigraph H of D such that $F(H) \cap A = \emptyset$ is an R-digraph.*

Proof. If C is a directed cycle in D of odd length such that $C = C^{(0)}$ and $f = uv \in F(C)$, C has at least one $V(C)C_u^0$-pseudodiagonal.

Remark 5.1. Let $C = (u_0, u_1, \ldots, u_{2n}, u_0)$ be a directed cycle in D of odd length,

$p(C)=\{u_{i_1},\ldots,u_{i_k}\}$; $0\leq i_1<i_2<\cdots<i_k\leq 2n$. Then

(i) $V(C)=C^{(1)}$ if and only if

(i.1) there exists j, $1\leq j\leq k$ such that $i_{j+1}=i_j+1$, or

(i.2) there exist j, l, $1\leq j<l\leq k$, such that both, the $u_{i_j}u_{i_{j+1}}$-directed path and the $u_{i_l}u_{i_{l-1}}$-directed path contained in C, have odd length (addition is taken mod k).

(ii) $V(C)=C^{(0)}$ if and only if (i.2).

5.2. *Applications*

Proposition 1. *Let D be a digraph and $T\subset V(D)$. Suppose that $D-T$ is an R-digraph and that for every directed cycle $C=(u_0,u_1,\ldots,u_{2n},u_0)$ in D of odd length such that $V(C)\cap T\neq\emptyset$ there exists i such that u_i, $u_{i+1}\in p(C)$. Then D is an R-digraph.*

Proof. Notice that $C=C^{(1)}$ and apply Corollary 5.1.

This proposition implies the following result obtained by Duchet [2].

(Duchet) *If every directed cycle $C=(u_0,u_1,\ldots,u_{2n},u_0)$ in D of odd length has two diagonals of the form (u_k,u_{k+2}), (u_{k+1},u_{k+3}), then D has a kernel.*

The following conjecture due to Meyniel [2], was disproved by Galeana-Sánchez [4].

Conjecture (Meyniel [2]). If every directed cycle of odd length in D has at least two pseudodiagonals, D is an R-digraph.

Proposition 2. *D is an R-digraph if and only if $D-V_{\text{o.a.}}(D)$ is an R-digraph, where $V_{\text{o.a.}}(D)$ denotes the set of vertices of D which do not belong to a directed cycle of odd length.*

Proof. Apply Theorem 5.1 and Lemma 5.1.

Proposition 3. *If every directed cycle C in D, of odd length, such that for some $uv\in F(C)$, $vu\notin F(D)$, has a pseudodiagonal f_c such that for each directed cycle γ of odd length containing f_c, $\gamma=\gamma^{(0)}$, then D is an R-digraph.*

Proof. Let H be an R^--subdigraph of D and C any directed cycle of odd length in H. If C were not symmetric, H would contain an $f_c=uv$ which contradicts Theorem 4.2. Then every directed cycle of odd length is symmetric and $C^{(1)}=C$. By Corollary 5.1; H is an R-digraph which also yields a contradiction.

This generalizes the following result obtained by Romanowicz, Zbigniew [10].

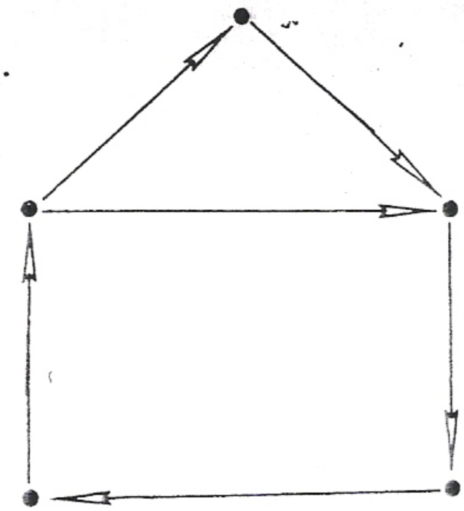

Fig. 2.

(Romanowicz, Zbigniew) *If every directed cycle C in D of odd length containing an asymmetric arc, contains an arc uv such that $vu \in F(D)$ and vu is contained in no directed cycle of odd length, then D has a kernel.*

Proposition 4. *Denote by $F_{\text{o.a.}}(D)$ the set of arcs of D contained in no directed cycle of odd length. Then D is an R-digraph if and only if every induced subdigraph H of D such that $F(H) \cap F_{\text{o.a.}}(D) = \emptyset$ is an R-digraph. In particular, D is an R-digraph whenever $D - F_{\text{o.a.}}(D)$ is an R-digraph. (The converse of this proposition is false: Consider the digraph of Fig. 2.)*

Proposition 5. *Let D be a digraph without induced directed cycles of odd length and $T \subset V(D)$. Suppose that every $u \in T$ belongs to at most $\Delta_D(u)$ directed cycles of odd length. Then D is an R-digraph if and only if $D - T$ is an R-digraph.*

Proof. Apply Lemma 5.1 and Corollary 4.3.

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