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Orientations of graphs in kernel theory

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Abstract

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In this paper we investigate structural properties of a certain class of graphs ($\mathfrak{M}$-free graphs) which are relevant in the study of kernel theory. $\mathfrak{M}$-free graphs satisfy the strong perfect graph conjecture of Berge. We investigate orientations of $\mathfrak{M}$-free graphs and other classes of graphs which produce kernel-perfect digraphs.

0. Introduction

We consider finite (except in Theorem 1.1), loopless graphs, without multiple edges. Undefined terms are in Berge [1].

If β is a class of graphs, a graph G is said to be a β-free graph whenever G has no induced subgraph isomorphic to a member of β. If β is a singular set, say $\beta = \{H\}$, we will write H-free graphs instead of $\{H\}$-free graphs. In what follows, we will denote by $\mathfrak{M}$ the set $\{M_1, M_2, M_3\}$ where M_1, M_2, M_3 are the graphs of Fig. 1.

The chromatic number χ of a graph G is the minimum number of colors necessary to color the vertices of G such that no two adjacent vertices are colored alike. The clique number w of a graph G is the maximum number of vertices in a complete subgraph G. A graph G is said to be perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Berge's strong perfect-graph conjecture states that G is perfect iff G does not contain C_{2n+1} and $\bar{C}_{2n+1}$, $n \geq 2$ as an induced subgraph.

If G is a graph; an orientation $\vec{G}$ of G is a digraph obtained from G by orientation of each edge of G in at least one of the two possible directions.

A kernel of a digraph $\vec{G} = (X, U)$ is a subset $K \subseteq X$ such that: K is independent ($K \cap \Gamma_{\vec{G}}^{-}(K) = \emptyset$) and K is absorbing ($K \cup \Gamma_{\vec{G}}^{-}(K) = X$). A semikernel S of $\vec{G}$ is an independent set of vertices such that for every $z \in (V(\vec{G}) - S)$ for

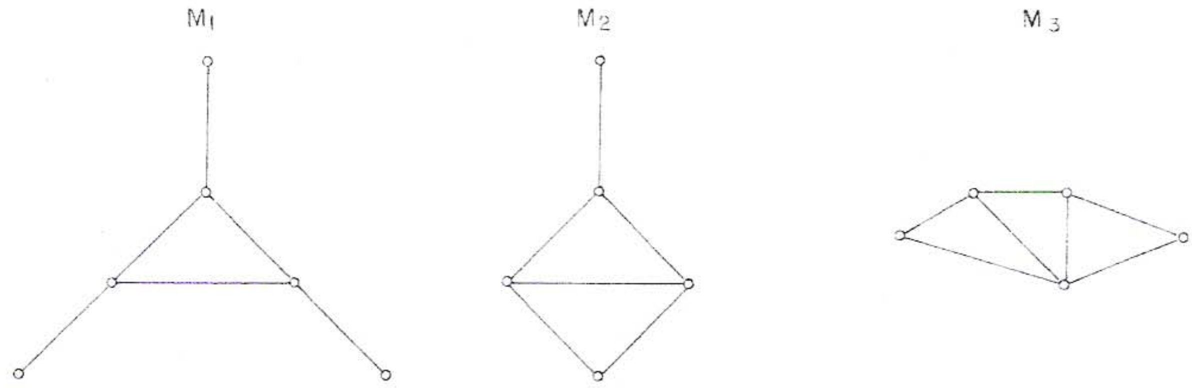

Fig. 1.

which there exists an Sz-arc, there also exists a zS-arc; $\vec{G}$ is an R-digraph iff every non-empty induced subdigraph of $\vec{G}$ has a non-empty semikernel. It was proved in [15] that $\vec{G}$ is an R-digraph if and only if every induced subdigraph of $\vec{G}$ has a kernel (i.e., D is a kernel-perfect digraph). We say that $\vec{G}$ is an R^{-}-digraph if $\vec{G}$ does not have a kernel but every proper induced subdigraph of $\vec{G}$ does have at least one (R^{-}-digraphs are also called kernel-perfect critical digraphs (see for instance [4])). Thus every complete subdigraph $\mathscr{C}$ of an R-digraph must have an absorbing vertex (i.e., a successor of all other vertices of $\mathscr{C}$).

A digraph $\vec{G}$ is called a normal orientation of G if every complete subgraph of G possesses an absorbing vertex. R-digraphs and R^{-}-digraphs have been investigated by several authors, namely Von Neumann and Morgenstern [16], Richardson [17–18] Duchet and Meyniel [4–7] and Galeana-Sánchez and Neumann-Lara [9–13, 15].

We end this section with some definitions and previous results:

If $\mathscr{C} = (u = u_0, u_1, \ldots, u_n, u_0)$ is a directed cycle, we put

$$\mathscr{C}_u^0 = \{u_i \mid i \equiv 0 \bmod 2; i \neq 0\}, \qquad \mathscr{C}_u^1 = \{u_i \mid i \equiv 1 \bmod 2\}.$$

An arc (z, w) is a pseudodiagonal of $\mathscr{C}$ if $z, w \in V(\mathscr{C})$ and $(z, w) \in (A(D) - A(\mathscr{C}))$.

We will need the following well-known result due to Richardson (see [1, p. 311]).

Theorem 0.1 (*Richardson's Theorem*). *Any digraph which does not contain directed cycles of odd length has a kernel.*

In this paper we will show that $\mathfrak{M}$-free graphs satisfy the following conjecture.

Berge–Duchet Conjecture (Berge and Duchet [2]). A graph G is perfect if and only if any normal orientation of G is kernel-perfect.

1. $\mathfrak{M}$-free graphs

In this section we study some structural properties of $\mathfrak{M}$-free graphs, in particular we show that $\mathfrak{M}$-free finite graphs satisfy the strong perfect graph conjecture of Berge.

Definition 1.1. For each $m \in \mathbb{N}$ let $U = \{u_0, \ldots, u_{m-1}\}$, $W = \{w_0, \ldots, w_{m-1}\}$ two disjoint sets of cardinality m. We will denote by S_m the digraph defined as follows:

$$V(S_m) = U \cup W, \qquad S_m[U] \cong K_m, \qquad S_m[W] \cong \bar{K}_m$$

and $w_i \operatorname{adj}_{S_m} u_j$ if and only if $j \leq i$. Similarly, if $U = \{u_0, u_1, \ldots\}$ and $W = \{w_0, w_1, \ldots\}$ are two disjoint countable sets (of type ω), we will denote by S_ω the digraph defined as follows.

$$V(S_\omega) = U \cup W$$

and

$$x \operatorname{adj}_{S_\omega} y \iff \begin{cases} x = u_i,\ y = u_j & \text{for } i \neq j, \quad \text{or} \\ x = w_i,\ y = u_j & \text{for } j \leq i. \end{cases}$$

Theorem 1.1. *Let G be a (possibly infinite) $\mathfrak{M}$-free graph, Q a maximal clique of G with $|V(Q)| \geq 3$ and I a maximal independent set of G. If $Q \cap I = \emptyset$, then G contains an induced subgraph isomorphic to S_ω with $\{u_0, u_1, \ldots\} \subseteq Q$ and $\{w_0, w_1, \ldots\} \subseteq I$.*

Proof. First of all we shall prove that G contains an induced subgraph isomorphic to S_3, with $\{u_0, u_1, u_2\} \subseteq Q$ and $\{w_0, w_1, w_2\} \subseteq I$. Since $Q \cap I - \emptyset$ and I is a maximal independent set; there exists $u_0 \in Q$ and $w_0 \in I$ such that $u_0 w_0 \in E(G)$. Since Q is a maximal clique, there exists $u_1 \in Q$ such that $w_0 u_1 \notin E(G)$; $u_1 \notin I$ and I is a maximal independent set. Hence there exists $w_1 \in I$ such that $u_1 w_1 \in E(G)$.

Now we analyze some possible cases.

Case 1: $u_0 w_1 \in E(G)$.

In this case, there exists $u_2 \in Q$ such that, $u_2 w_1 \notin A(G)$. Now we need to analyze two subcases.

1(i) $u_2 w_0 \in E(G)$. In this case

$$G[\{u_0, u_1, u_2, w_0, w_1\}] \cong M_3,$$

which is impossible.

1(ii) $u_2 w_0 \notin E(G)$. Since I is maximal independent set, it follows that there exists $w_2 \in I$ such that $w_2 u_2 \in E(G)$.

If $w_2 u_1 \notin E(G)$ and $w_2 u_0 \notin E(G)$,

$$G[\{u_0, u_1, u_2, w_1, w_2\}] \cong M_2.$$

If $w_2 u_0 \in E(G)$ and $w_2 u_1 \notin E(G)$, it follows that

$$G[\{u_0, u_1, u_2, w_1, w_2\}] \cong M_3.$$

If $w_2 u_0 \notin E(G)$ and $w_2 u_1 \in E(G)$, necessarily

$$G[\{u_0, u_1, u_2, w_2, w_0\}] \cong M_2$$

or

$$G[\{u_0, u_1, u_2, w_1, w_2\}] \cong M_3.$$

Therefore $w_2u_0 \in E(G)$ and $w_2u_1 \in E(G)$ and hence

$$G[\{u_0, u_1, u_2, w_0, w_1, w_2\}] \cong S_3.$$

Case 2: $u_0w_1 \notin E(G)$.

Since $|V(Q)| \geq 3$, there exists $u_2 \in (Q - \{u_0, u_1\})$. Now we analyze some cases.

2(i) In case $u_2w_0 \in E(G)$ and $u_2w_1 \in E(G)$, we have

$$G[\{u_0, u_1, u_2, w_0, w_1\} \cong M_3.$$

2(ii) In case $u_2w_0 \in E(G)$ and $u_2w_1 \notin E(G)$,

$$G[\{u_0, u_1, u_2, w_0, w_1\}] \cong M_2.$$

2(iii) In case $u_2w_0 \notin E(G)$ and $u_2w_1 \in E(G)$,

$$G[\{u_0, u_1, u_2, w_0, w_1\}] \cong M_2.$$

2(iv) In case $u_2w_0 \notin E(G)$ and $u_2w_1 \notin E(G)$, since I is a maximal stable set, there exists $w_2 \in I$ such that $w_2u_2 \in E(G)$.

If $w_2u_1 \notin E(G)$ and $w_2u_0 \notin E(G)$,

$$G[\{u_0, u_1, u_2, w_0, w_1, w_2\}] \cong M_1.$$

If $w_2u_0 \in E(G)$ and $w_2u_1 \notin E(G)$,

$$G[\{u_0, u_1, u_2, w_2, w_1\}] \cong M_2.$$

If $w_2u_0 \notin E(G)$ and $w_2u_1 \in E(G)$,

$$G[\{u_0, u_1, u_2, w_0, w_2\}] \cong M_2.$$

Finally, suppose $w_2u_1 \in E(G)$ and $w_2u_0 \in E(G)$ and take $u_0' = u_1$, $u_1' = u_2$, $w_0' = w_1$ and $w_1' = w_2$. Arguing as in Case 1, we conclude that G contains S_3 as an induced subgraph with $\{u_0, u_1, u_2\} \subseteq Q$ and $\{w_0, w_1, w_2\} \subseteq I$. Now we proceed by induction on r to prove that G contains S_r as an induced subgraph with $\{u_0, u_1, \ldots, u_{r-1}\} \subseteq Q$ and $\{w_0, w_1, \ldots, w_{r-1}\} \subseteq I$ for each $r \geq 3$. We know that G contains S_3. Assume that we have proved that G contains S_r as an induced subgraph with $\{u_0, u_1, \ldots, u_{r-1}\} \subseteq Q$, $\{w_0, w_1, \ldots, w_{r-1}\} \subseteq I$, and where $u_iw_j \in E(S_r)$ if and only if $i \leq j$.

We will prove that G contains an extension S_{r+1} of S_r as an induced subgraph, with $\{u_0, u_1, \ldots, u_r\} \subseteq Q$ and $\{w_0, w_1, \ldots, w_r\} \subseteq I$. Since Q is a maximal clique there exists $u_r \in (Q - \{u_0, u_1, \ldots, u_{r-1}\})$ such that $u_rw_{r-1} \notin E(G)$.

Observation 1. *$u_rw_i \notin E(G)$ for each $i \in \{0, 1, \ldots, r-2\}$.*

Proof. Suppose the contrary and let

$$k = \max\{i \in \{0, 1, \ldots, r-2\} \mid w_iu_r \in E(G)\}$$

then

$$u_rw_k \in E(G), \qquad u_rw_{k+1} \notin E(G) \quad \text{and}$$

$$G[\{w_k, u_k, u_{k+1}, w_{k+1}, u_r\}] \cong M_3. \qquad \square$$

Since I is a maximal independent set and $u_r \notin I$, there exists $w_r \in (I - \{w_0, w_1, \ldots, w_{r-1}\})$ such that $u_r w_r \in E(G)$.

Observation 2. *$w_r u_i \in E(G)$ for each $i \in \{0, 1, \ldots, r-1\}$.*

Proof. Suppose the contrary and let

$$t = \max\{i \in \{0, 1, \ldots, r-1\} / w_r u_i \notin E(G)\}$$

we analyze two possible cases:

Case 1: $t > 0$.

If $u_{t-1} w_r \in E(G)$, then

$$G[\{u_{t-1}, u_t, u_{t+1}, w_t, w_r\}] \cong M_3.$$

If $u_{t-1} w_r \notin E(G)$, then

$$G[\{u_{t-1}, u_t, u_{t+1}, w_t, w_r\}] \cong M_2.$$

Case 2: $t = 0$.

In this case

$$G[\{u_0, u_1, u_2, w_0, w_r\}] \cong M_2. \quad \square$$

It follows from Observations 1 and 2 and from Definition 1.1. that

$$G[\{u_0, u_1,, \ldots, u_r\} \cup \{w_0, w_1, \ldots, w_r\}] \cong S_{r+1}$$

with $\{u_0, u_1, \ldots, u_r\} \subseteq Q$ and $\{w_0, w_1, \ldots, w_r\} \subseteq I$. Theorem 1.1 follows. $\square$

As a direct consequence of Theorem 1.1 we obtain the following corollary.

Corollary 1.1. *Let G be a finite $\mathfrak{M}$-free graph. If Q is a maximal clique of G with $|V(Q)| \geq 3$ and I is a maximal stable set of G then $Q \cap I \neq \emptyset$.*

Theorem 1.2. *Let G be a finite graph. The following statements are equivalent:*

(i) *G is an $\mathfrak{M}$-free graph.*

(ii) *For each induced subgraph H of G, if I_H is a maximal independent set of H and Q_H is a maximal clique of H with $|V(Q_H)| \geq 3$ then we have $Q_H \cap I_H \neq \emptyset$.*

Proof. This result is a direct consequence of Corollary 1.1 and the fact that each M_i, $i \in \{1, 2, 3\}$ of Fig. 1 has a maximal independent set I_i and a maximal clique Q_i such that $I_i \cap Q_i = \emptyset$. $\square$

Notice that the class of $\mathfrak{M}$-free graphs neither contains nor is contained in the class of P_4-free graphs.

The following result is proved by Corneil et al. [3].

Theorem 1.3. *Let G be a graph with a point set X. The following statements are equivalent*:

(1) *G is a P_4-free graph.*

(2) *For each induced subgraph H of G, if I_H is a maximal independent set of H and Q_H is a maximal clique of H then $I_H \cap Q_H \neq \emptyset$.*

Theorem 1.4. *If G is an $\mathfrak{M}$-free graph then the following statements are equivalent*:

(i) *G does not contain C_{2n+1}, $\bar{C}_{2n+1}$, $n \geq 2$ as an induced subgraph.*

(ii) *For each non-bipartite induced subgraph H of G, if I_H is a maximal independent set of H and Q_H a maximum clique of H then we have $Q_H \cap I_H \neq \emptyset$.*

Proof. (i)$\Rightarrow$(ii). Let G be an $\mathfrak{M}$-free graph which does not contain C_{2n+1} as an induced subgraph for each $n \geq 2$, H an induced subgraph of G, I_H a maximal stable set and Q_H a maximum clique of H. It follows from (i) that H contains a triangle and since G is $\mathfrak{M}$-free it follows from Theorem 1.2 that $Q_H \cap I_H \neq \emptyset$.

(ii)$\Rightarrow$(i). If G contains C_{2n+1} as an induced subgraph for some $n \geq 2$, then C_{2n+1} is an induced subgraph which contains a maximal stable set I and a maximum clique Q such that $Q \cap I = \emptyset$. If G contains $\bar{C}_{2n+1}$ as an induced subgraph for some $n \geq 2$ then G contains C_5 as an induced subgraph or G contains M_3 as an induced subgraph. □

Corollary 1.2. *$\mathfrak{M}$-free graphs satisfy the Strong Perfect Graph Conjecture.*

Proof. It follows directly from property (ii) of Theorem 1.4. □

Remark 1.1. Let $\mathfrak{M}^* = \mathfrak{M} \cup \{C_{2n+1} \mid n \geq 2\}$. Then $\mathfrak{M}^*$-free graphs are perfect as a consequence of Corollary 1.2. The class of $\mathfrak{M}^*$-free graphs does not contain nor is contained in any of the following subclasses of perfect graphs: triangulated graphs, cotriangulated graphs, comparability graphs, co-comparability graphs.

For $m \geq 2$ we will denote by N_m the graph defined as follows:

$$V(N_m) = \{z_1, z_2, \ldots, z_m\} \cup \{x_1, \ldots, x_m\}, \qquad N_m[\{z_1, \ldots, z_m\}] \cong K_m,$$
$$N_m[\{x_1, \ldots, x_m\}] \text{ is an independent set} \quad \text{and} \quad x_i z_j \in E(N_m) \quad \text{iff } i = j.$$

Theorem 1.5. *Let G be a K_4-e free graph with $\omega(G) \geq 2$. The following statements are equivalent*:

(i) *G does not contain $N_{\omega(G)}$ as an induced subgraph.*

(ii) *For each maximal independent set I of G and each maximum clique Q of G we have $Q \cap I \neq \emptyset$.*

Proof. For $\omega(G) = 2$ Theorem 1.5 follows from Theorem 1.3. Suppose that $\omega(G) \geq 3$.

(i)$\Rightarrow$(ii). Let G be a K_4-e free graph, with $\omega(G) \geqslant 3$ which does not contain $N_{\omega(G)}$ as an induced subgraph and suppose that there exists a maximal independent set I of G, and a maximum clique Q of G such that $Q \cap I = \emptyset$.

Clearly each pont of Q is adjacent to at least one point of I and each point in I is adjacent to at most one point of Q. It follows that G contains $N_{\omega(G)}$ as an induced subgraph.

(ii)$\Rightarrow$(i) Let G be a graph which contains $N_{\omega(G)}$ as an induced subgraph with

$$V(N_{\omega(G)}) = \{z_1, \ldots, z_{\omega(G)}\} \cup \{x_1, \ldots, x_{\omega(G)}\},$$

$$N_{\omega(G)}[\{z_1, \ldots, z_{\omega(G)}\}] \cong K_{\omega(G)},$$

$N_{\omega(G)}[\{x_1, \ldots, x_{\omega(G)}\}]$ is an independent set and $z_i x_j \in E(N_{\omega(G)})$ iff $i = j$.

Clearly $\{x_1, \ldots, x_{\omega(G)}\}$ is contained in a maximal independent set I of G and $N_{\omega(G)}[\{z_1, \ldots, z_{\omega(G)}\}]$ is a maximum clique of G which does not intersect I. □

2. Orientations of certain classes of graphs

In this section we obtain some results relating normal orientations of $\mathfrak{M}$-free graphs and R-digraphs. In particular we prove that the class of $\mathfrak{M}$-free graphs satisfy the Berge–Duchet Conjecture, also we consider orientations of K_4-e free graphs which result in kernel-perfect digraphs.

Theorem 2.1. *Let G be an $\mathfrak{M}$-free graph. If there exists a normal orientation of G which is an R^{-}-digraph, then G is a triangle free graph.*

Proof. Let G be an $\mathfrak{M}$-free graph which has a triangle and suppose that there exists a normal orientation $\vec{G}$ of G which is an R^{-}-digraph. Consider Q a maximum clique of G and u a source of Q in $\vec{G}$ and N_u a kernel of $(\vec{G} - \{u\})$. If $\{u\} \cup N_u$ is an independent set it follows that $\{u\} \cup N_u$ is a kernel of $\vec{G}$, so we can assume that N_u is a maximal independent set of G and it follows from Theorem 1.1 that $Q \cap N_u \neq \emptyset$, $(Q - \{u\}) \cap N_u \neq \emptyset$ and so N_u is a kernel of $\vec{G}$. □

Theorem 2.2. *Let G be an $\mathfrak{M}$-free graph and let $\vec{G}$ be a normal orientation of G. If every triangle free induced subdigraph $\vec{G}_0$ of $\vec{G}$ is an R-digraph then $\vec{G}$ is an R-digraph.*

Proof. Suppose that $\vec{G}$ is not an R-digraph and let $\vec{H}$ be an induced subdigraph of $\vec{G}$ which is an R^{-}-digraph. By hypothesis the underlying graph H of $\vec{H}$ is $\mathfrak{M}$-free and has a triangle which, by Theorem 2.1, is impossible. □

Remark 2.1. Maffray [8] has proved that a graph such that every normal orientation is kernel-perfect neither contains C_{2n+1} nor $\bar{C}_{2n+1}$ for $n \geqslant 2$ as an induced subgraph.

The next Theorem asserts that $\mathfrak{M}$-free graphs satisfy the Berge-Duchet Conjecture.

Theorem 2.3. *Let G be an $\mathfrak{M}$-free graph. G is a perfect graph iff every normal orientation of G is kernel-perfect.*

Proof. Let G be a perfect $\mathfrak{M}$-free graph, $\vec{G}$ a normal orientation of G and $\vec{G}_0$ a triangle-free induced subdigraph of $\vec{G}$. Since G is a perfect graph, $\vec{G}_0$ has a bipartite underlying graph so $\vec{G}_0$ is an R-digraph by Theorem 0.1 and by Theorem 2.2 we have that $\vec{G}$ is an R-digraph.

If every normal orientation of G is kernel perfect, then by Maffray's [8] result does not contain C_{2n+1}, $\bar{C}_{2n+1}$ for $n \geq 2$. Then G is perfect by Corollary 1.2. □

Theorem 2.4. *Let G be a K_4-e free graph with $\omega(G) \geq 3$. If there exists a normal orientation of G which is an R^{-}-digraph then G contains $N_{\omega(G)}$ as an induced subgraph.*

Proof. Let G be a K_4-e free graph with $\omega(G) \geq 3$ such that G does not contain $N_{\omega(G)}$ as an induced subgraph. Suppose that there exists a normal orientation $\vec{G}$ of G which is an R^{-}-digraph. Let Q a maximum clique of G and u a source of Q in $\vec{G}$ and N_u a kernel of $\vec{G} - \{u\}$. If $\{u\} \cup N_u$ is an independent set it follows that $\{u\} \cup N_u$ is a kernel of $\vec{G}$, so we can assume that N_u is a maximal independent set of G and it follows from Theorem 1.5 that $Q \cap N_u \neq \emptyset$, $(Q - \{u\}) \cap N_u \neq \emptyset$ and so N_u is a kernel of $\vec{G}$. □

Theorem 2.5. *Let G be an $\mathfrak{M}$-free graph which does not contain an induced subgraph isomorphic to H_1 to H_2* (Fig. 2). *If $\vec{G}$ is a normal orientation of G such that $\vec{G}$ does not contain induced odd directed cycles then $\vec{G}$ is an R-digraph.*

Proof. We argue by induction on the number of vertices of G, $|V(G)|$. The proposition is obvious for graphs with at most three vertices. Assume that we have proved Theorem 2.5 for graphs with at most p vertices and let $\vec{G}$ and G be as in the hypothesis and G with $p+1$ vertices.

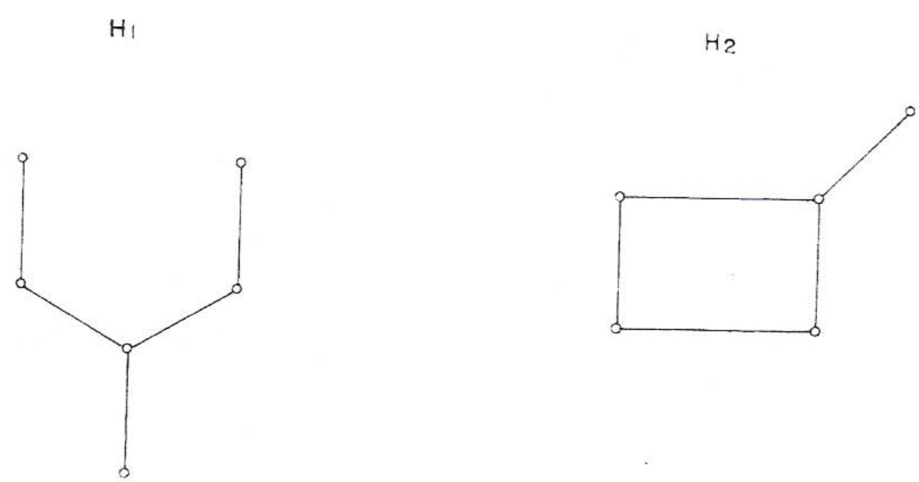

Fig. 2.

By the inductive hypothesis we know that for each $u \in V(\vec{G})$ we have $\vec{G} - u$ is an R-digraph; so it suffices to prove that $\vec{G}$ has a kernel. Now we analyze two possible cases:

Case 1: *For each* $z \in V(G)$, $\delta^+_{\vec{G}}(z) \leq 1$.

In this case $\vec{G}$ has no odd directed cycles and Theorem 0.1 implies that $\vec{G}$ is an R-digraph.

Case 2: *These exists* $u \in \vec{G}$ *such that* $\delta^+_{\vec{G}}(u) > 1$.

Again we consider two possible cases:

Case 2(a): *G is a triangle free graph.* Let $u \in V(G)$ such that $\delta^+_{\vec{G}}(u) > 1$, and let N_u be a kernel of $\vec{G} - u$. If $\{u\} \cup N_u$ is an independent set or if there exists some uN_u-arc in $\vec{G}$ we have that $\{u\} \cup N_u$ is a kernel of $\vec{G}$ or N_u, is a kernel of $\vec{G}$; so we can assume that there exists $n \in N_u$ such that $(n, u) \in A(\vec{G})$ and there is no uN_u-arc in $\vec{G}$. By the choice of u we have

$$u_1, u_2 \in V(\vec{G}), \quad uu_1 \in A(\vec{G}), \quad uu_2 \in A(\vec{G}); \quad n_1, n_2.N_u,$$
$$u_1n_1 \in A(\vec{G}) \quad u_2n_2 \in A(\vec{G}).$$

When $n_1 = n_2$ we have

$$G[\{u, n, u_1, u_2, n_1\}] \cong H_1.$$

If $n_1 \neq n_2$ and $u_2n_1 \in A(G)$,

$$G[\{u, n, u_1, u_2, n_1\}] \cong H_1$$

and if $n_1 \neq n_2$ and $n_2u_1 \in A(G)$,

$$G[\{n, u, u_1, u_2, n_2\}] \cong H_1.$$

In the other case we have

$$G[\{n, u, u_1, u_2, n_1, n_2\}] \cong H_2.$$

It follows that N_u is a kernel of $\vec{G}$.

Case 2(b): *G has a triangle.* Since for each $u \in G$ $\vec{G} - \{u\}$ is an R-digraph, if $\vec{G}$ has no kernel then $\vec{G}$ is an R^--digraph by Theorem 2.1 is impossible. So $\vec{G}$ has a kernel. □

Corollary 2.1. *Let G be an* $\mathfrak{M}$*-free graph which does not contain an induced subgraph isomorphic to* H_1 *or* H_2. *If* $\vec{G}$ *is an orientation of G which is an* R^-*-digraph then* $\vec{G} \cong \vec{C}_{2n+1}$, $n \geq 1$.

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