On the Number of Quasi-Kernels in Digraphs

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Received July 18, 2002; Revised September 23, 2003

DOI 10.1002/jgt.10169

Abstract: A vertex set X of a digraph $D=(V, A)$ is a *kernel* if X is independent (i.e., all pairs of distinct vertices of X are non-adjacent) and for every $v \in V-X$ there exists $x \in X$ such that $vx \in A$. A vertex set X of a digraph $D=(V, A)$ is a *quasi-kernel* if X is independent and for every $v \in V-X$ there exist $w \in V-X$, $x \in X$ such that either $vx \in A$ or vw, $wx \in A$. In 1974, Chvátal and Lovász proved that every digraph has a

Contract grant sponsor: National University of Singapore.

quasi-kernel. In 1996, Jacob and Meyniel proved that if a digraph D has no kernel, then D contains at least three quasi-kernels. We characterize digraphs with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels. In particular, we prove that every strong digraph of order at least three, which is not a 4-cycle, has at least three quasi-kernels.

Keywords: *quasi-kernels; semi-kernels; kernels; digraphs*

1. INTRODUCTION, TERMINOLOGY, AND NOTATION

A vertex set X of a digraph $D = (V, A)$ is a *kernel* if X is independent (i.e., all pairs of distinct vertices of X are non-adjacent) and for every $v \in V - X$ there exists $x \in X$ such that $vx \in A$. A vertex set X of a digraph $D = (V, A)$ is a *quasi-kernel* if X is independent and for every $v \in V - X$ there exist $w \in V - X, x \in X$ such that either $vx \in A$ or $vw, wx \in A$. A digraph $T = (V, A)$ is a *tournament* if for every pair x, y of distinct vertices in V, either $xy \in A$ or $yx \in A$, but not both. A vertex of out-degree zero is called a *sink*.

While not every digraph has a kernel (e.g., a directed cycle $\vec{C}_n$ has a kernel if and only if n is even), Chvátal and Lovász [3] (see also Chapter 12 in [2]) proved that every digraph has a quasi-kernel. Jacob and Meyniel [6] proved that if a digraph D has no kernel, then D contains at least three quasi-kernels. While the assertion of Chvátal and Lovász generalizes the fact that every tournament has a 2-serf, that is a quasi-kernel of cardinality 1, the Jacob–Meyniel theorem extends the result of Moon [7] that every tournament with no sink has at least three 2-serfs.

While the Jacob–Meyniel theorem provides sufficient conditions for a digraph to have at least three quasi-kernels, in Section 2, we characterize digraphs with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels (see Theorem 2.4). In particular, we prove that every strong digraph, of order at least three, different from the 4-cycle $\vec{C}_4$ has at least three quasi-kernels. Note that, in our proofs, we naturally use the Chvátal–Lovász theorem, but not the more powerful Jacob–Meyniel theorem.

We use the standard terminology and notation on digraphs as given in [2]. We still provide most of the necessary definitions for the convenience of the reader.

For a digraph D, the vertex (arc) set is denoted by $V(D)(A(D))$. Let x, y be a pair of vertices in D. If $xy \in A(D)$, we say x *dominates* y, and y *is dominated by* x, and denote it by $x \rightarrow y$. A digraph D is *strong* if, for every ordered pair x, y of distinct vertices in D, there is a path from x to y. An *orientation* of a digraph D is an oriented graph obtained from D by deleting exactly one arc from each 2-cycle in D. A *biorientation* of D is a digraph, which is a subdigraph of D and superdigraph of an orientation of D. The *closed in-neighbourhood* (*closed* *out-neighbourhood*) of a set X of vertices of a digraph $D=(V,A)$ is defined as follows.

$$N_D^-[X]=X\cup\{y\in V:\ \exists x\in X, y\rightarrow x\}\quad (N_D^+[X]=X\cup\{y\in V:\ \exists x\in X, x\rightarrow y\}).$$

For disjoint subsets X and Y of $V(D)$, let $X\times Y=\{xy:\ x\in X, y\in Y\}$, $(X,Y)_D=(X\times Y)\cap A(D)$; $D[X]$ is the subdigraph of D induced by X. If the digraph under consideration is clear from the context, then we will omit the subscript D.

2. DIGRAPHS WITH EXACTLY ONE AND TWO QUASI-KERNELS

We start with the following.

Lemma 2.1. *Let x be a vertex in a digraph D. If x is a non-sink, then D has a quasi-kernel not including x.*

Proof. Let $y\in N^+[x]-\{x\}$ be arbitrary. If $N^-[y]=V(D)$, then y is the required quasi-kernel. If $N^-[y]\neq V(D)$, let Q' be a quasi-kernel in $D-N^-[y]$. If y dominates a vertex in Q', then Q' is a quasi-kernel in D, which does not contain x. If y does not dominate a vertex in Q', then $Q'\cup\{y\}$ is a quasi-kernel in D, which does not include x. ■

The following is an easy characterization of digraphs with merely one quasi-kernel.

Theorem 2.1. *A digraph D has only one quasi-kernel if and only if D has a sink and every non-sink of D dominates a sink of D. If a digraph D has only one quasi-kernel Q, then Q is a kernel and consists of the sinks of D.*

Proof. Assume that D has a sink and every non-sink of D dominates a sink of D. Let S be the set of sinks in D. To see that S is a unique quasi-kernel of D, it is enough to observe that every sink must be in a quasi-kernel.

Let D have only one quasi-kernel Q. To see that Q is the set of sinks in D, observe that Q contains all sinks in D and, by Lemma 2.1, Q does not have non-sinks. If x is a non-sink and x does not dominate a vertex in Q, then $Q\cup\{x\}$ is another quasi-kernel of D, a contradiction. Thus, we have proved that D has a sink and every non-sink of D dominates a sink of D. ■

In view of Theorem 2.1, the following assertion is a strengthening of the Jacob–Meyniel theorem for the case of digraphs with no sinks.

Theorem 2.2. *Let D be a digraph with no sink. Then D has precisely two quasi-kernels if and only if D has an induced 4-cycle or 2-cycle, C, such that no vertex of C dominates a vertex in $D-V(C)$ and every vertex in $D-V(C)$ dominates at least two adjacent vertices in C.*

To prove Theorem 2.2, we will extensively use the following.

Lemma 2.2. *Let a digraph D have exactly two quasi-kernels, R and Q. Then the following claims hold:*

(i) *If a vertex x in R dominates some vertex y such that $V(D) \neq N^-[y]$, then $Q - y$ is the only quasi-kernel in $D - N^-[y]$;*
(ii) *$\{R, Q\}$ is the set of quasi-kernels of every biorientation of D, in which both R and Q contain non-sinks.*

Proof. Let $R_1, R_2, \ldots, R_k$ be the quasi-kernels in $D - N^-[y]$. Then $R'_1, R'_2, \ldots, R'_k$ are quasi-kernels in D, where $R'_i = R_i$ if $(y, R_i) \neq \emptyset$ and $R'_i = R_i \cup \{y\}$, otherwise, $i = 1, 2, \ldots, k$. Since D has only two quasi-kernels, $k \leq 2$. Since, $x \in N^-[y]$ and $x \in R$, we conclude that $R - y$ is not a quasi-kernel in $D - N^-[y]$. By the Chvátal–Lovász theorem, every digraph has a quasi-kernel, so $Q - y$ is the unique quasi-kernel in $D - N^-[y]$.

Let D' be a biorientation of D, in which both R and Q contain non-sinks. Clearly, every quasi-kernel in D' is a quasi-kernel in D. However, by Theorem 2.1, neither R nor Q can be the only quasi-kernel in D'. Thus $\{R, Q\}$ is the set of quasi-kernels of D'. ■

Proof of Theorem 2.2. We first show that if D has precisely two quasi-kernels, then D has the above-described structure. We will prove this assertion by induction on $|V(D)|$. The assertion is clearly true when $|V(D)| \leq 2$, so we may assume that it is true for all digraphs, D^*, with $|V(D^*)| < |V(D)|$. Let Q_1 and Q_2 be the only two quasi-kernels in D. Note that by Lemma 2.1, Q_1 and Q_2 must be disjoint (if $x \in Q_1 \cap Q_2$ then use Lemma 2.1 for x). We now prove the following claims.

Claim 2.1. *If $(Q_i, Q_j) \neq \emptyset$ ($\{i, j\} = \{1, 2\}$), then for every $w \in Q_i$, $(w, Q_j) \neq \emptyset$.*

Proof. Let $xy \in (Q_i, Q_j)$ and let w be a vertex in Q_i which has no arc into Q_j. By Lemma 2.2(i), $Q_j - y$ is the unique kernel in $D - N^-[y]$ and, thus, by Theorem 2.1, we must have an arc from w to $Q_j - y$ since $w \in V(D) - N^-[y]$, a contradiction.

Claim 2.2. *Both (Q_1, Q_2) and (Q_2, Q_1) are non-empty.*

Proof. Clearly $Q_1 \cup Q_2$ is not an independent set, as then it would be a quasi-kernel. Hence, without loss of generality, we may assume that $(Q_1, Q_2) \neq \emptyset$. Suppose that $(Q_2, Q_1) = \emptyset$. Since Q_1 is a quasi-kernel, there exists a 2-path from any given $x \in Q_2$ to Q_1, say xzy ($z \notin Q_1 \cup Q_2$ and $y \in Q_1$).

We now show that every vertex in Q_2 must dominate z. Suppose that this is not the case, and let w be a vertex not dominating z. By Lemma 2.2, Q_1 is the only quasi-kernel in $D - N^-[z]$. However, by Theorem 2.1, this is a contradiction against the fact that w dominates no vertex in Q_1 ($w \in V(D) - N^-[z]$). Thus, $Q_2 \subseteq N^-[z]$.

Let D' be any orientation of D for which $(z, Q_2)_{D'} = \emptyset$, and let ab be an arc in $(Q_1, Q_2)_{D'}$. Since $z \in V(D') - N^-_{D'}[b]$, we have $V(D') \neq N^-_{D'}[b]$. By Lemma 2.2, $Q_2 - b$ is the only quasi-kernel in $D' - N^-_{D'}[b]$. By Theorem 2.1, $Q_2 - b$ is a kernel in $V(D') - N^-_{D'}[b]$. However, $Q_2 - b$ is not a kernel in $D' - N^-_{D'}[b]$ as z dominates no vertex in $Q_2 - b$, a contradiction.

Claim 2.3. *Let $\{a, b\}$ be a set of two distinct vertices from Q_1 and let $\{c, d\}$ be a set of two distinct vertices from Q_2. Then we cannot have both $a \to c$ and $d \to b$.*

Proof. Assume that $a \to c$ and $d \to b$. Suppose first that $c \not\to b$. By Lemma 2.2, $Q_1 - b$ is the only quasi-kernel in $V(D) - N^-[b]$. However, since the arc $ac \in D - N^-[b]$ we see that $Q_1 - b$ contains a non-sink in $V(D) - N^-[b]$ in contradiction with Theorem 2.1. Suppose now that $c \to b$, and let D' equal $D - bc$ (if $bc \notin D$, then $D' = D$). By Lemma 2.1, $Q_2 - c$ is the only quasi-kernel in $V(D') - N^-[c]$. However, since the arc $db \in D' - N^-_{D'}[c]$ we see that $Q_2 - c$ contains a non-sink in contradiction with Theorem 2.1.

Claim 2.4. *Either $D[Q_1 \cup Q_2]$ is a 2-cycle or $D[Q_1 \cup Q_2]$ contains an induced 4-cycle.*

Proof. If either Q_1 or Q_2 has only one vertex, then without loss of generality we may assume that $|Q_1| = 1$. If $|Q_2| = 1$ then by Claim 2.2, $D[Q_1 \cup Q_2]$ is a 2-cycle, so assume that $|Q_2| \geq 2$. Let $Q_1 = \{x\}$ and observe that by Claim 2.1 and 2.2 there exists a pair a, b of distinct vertices in Q_2 such that $ax, xb \in A(D)$. Let D' be any orientation of D with $ax, xb \in A(D')$. By Lemma 2.2, $Q_1 - x$ is the only quasi-kernel in the non-empty digraph $D' - N^-_{D'}[x]$, which contradicts the fact that $Q_1 = \{x\}$.

Therefore, we may now assume that both Q_1 and Q_2 have cardinality at least two. By Claim 2.2, there exists an arc x_2x_1 in $(Q_2, Q_1)_D$. Let $y_1 \in Q_1 - \{x_1\}$ be arbitrary, and observe that $(y_1, Q_2) \neq \emptyset$, by Claim 2.1 and 2.2. By Claim 2.3, $y_1x_2 \in (y_1, Q_2)$. Let $y_2 \in Q_2 - \{x_2\}$ be arbitrary. Analogously, we have $y_2y_1 \in A(D)$. Finally, Claims 2.1 and 2.3 imply that $x_1y_2 \in A(D)$. Therefore, $C = x_2x_1y_2y_1x_2$ is a 4-cycle. Observe that C is an induced 4-cycle, by Claim 2.3 and the fact that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are independent sets (they are subsets of quasi-kernels).

Claim 2.5. *If abcda is a 4-cycle such that $\{a, c\} \subseteq Q_1$ and $\{b, d\} \subseteq Q_2$, then there is no arc from $\{a, b, c, d\}$ to any vertex in $D - \{a, b, c, d\}$.*

Proof. Assume that the claim is false and that there exists a vertex $z \in V(D) - \{a, b, c, d\}$ such that there is an arc from $\{a, b, c, d\}$ to z. Without loss of generality, assume that $az \in A(D)$, and consider the following two cases.

Case 1. $z \to c$. Let D' be any orientation of D with $zc, az \in A(D')$. By Lemma 2.2, $Q_2 - z$ is the only quasi-kernel in $D' - N^-_{D'}[z]$. However, the existence of the arc $bc \in D'$ contradicts Theorem 2.1.

Case 2. $z \not\rightarrow c$. By Lemma 2.2(i), $Q_1 - c$ is the only quasi-kernel in $D - N_D^-[c]$. However, the existence of the arc $az \in D - N^-[c]$ contradicts Theorem 2.1.

Claim 2.6. *If abcda is a* 4*-cycle such that* $\{a, c\} \subseteq Q_1$ *and* $\{b, d\} \subseteq Q_2$*, then every vertex in* $D - \{a, b, c, d\}$ *dominates two adjacent vertices on abcda.*

Proof. Let $x \in V(D) - \{a, b, c, d\}$ be arbitrary. If x has no arc into $\{a, b, c, d\}$, then consider the digraph $D^* = D - N^-[x]$. Clearly, $Q_1 - N^-[x]$ and $Q_2 - N^-[x]$ are distinct quasi-kernels in D^*; D^* cannot have another quasi-kernel as D has only two quasi-kernels. Therefore there are exactly two quasi-kernels in D^*, and by our induction hypothesis, these quasi-kernels are precisely $\{a, c\}$ and $\{b, d\}$. Observe that, by Claim 2.5, x is adjacent to no vertex from the set $\{a, b, c, d\}$. However, this means that both $\{x, a, c\}$ and $\{x, b, d\}$ are quasi-kernels in D, contradicting the fact that Q_1 and Q_2 are disjoint. Therefore, x must have an arc into $\{a, b, c, d\}$. Observe that since x is arbitrary, this implies that $\{a, c\}$ and $\{b, d\}$ are quasi-kernels in D.

Without loss of generality, assume that $x \rightarrow a$ in D. Suppose also that $x \not\rightarrow b$ and $x \not\rightarrow d$, as otherwise we would be done. However, these assumptions imply that $\{x, b, d\}$ also is a quasi-kernel, along with $\{a, c\}$ and $\{b, d\}$, a contradiction.

Claim 2.7. *If* $C = D[Q_1 \cup Q_2]$ *is a* 2*-cycle, then no vertex of* C *dominates a vertex in* $D - V(C)$ *and every vertex in* $D - V(C)$ *dominates both vertices in* C*.*

Proof. Let $C = xyx$. Assume there exists an arc xz, $z \neq y$. Consider an orientation, D', of D such that $D' - N_{D'}^-[x]$ contains z and does not contain y. On one hand, D' has no quasi-kernels other than $\{x\}$ and $\{y\}$; on the other hand, either Q or $Q \cup \{x\}$ is a quasi-kernel in D', where Q is a quasi-kernel in $D' - N_{D'}^-[x]$. We have arrived at a contradiction. Therefore $(V(C), V(D) - V(C)) = \emptyset$. Furthermore, every vertex $v \in V(D) - V(C)$ must dominate both vertices on C since otherwise there would be a quasi-kernel containing v.

Claim 2.4–2.7. *prove the assertion on the structure of D.*

Now assume that D has the structure described in this theorem, and C is the cycle in D. If C is a 2-cycle, then it is easy to see that each of the two vertices on C is a quasi-kernel (and kernel) in D, and that there are no other quasi-kernels in D. So now assume that $C = abcda$ is an induced 4-cycle in D. Observe that $\{a, c\}$ and $\{b, d\}$ are quasi-kernels in D. Since $(\{a, b, c, d\}, V(D) - \{a, b, c, d\}) = \emptyset$, any quasi-kernel in D must contain a vertex, x, in C. Since the successor x^+ of x in C has to be able to reach the quasi-kernel with a path of length at most two, $(x^+)^+$ must also belong to the quasi-kernel. Since all other vertices are adjacent to one of these vertices, the only quasi-kernels are $\{a, c\}$ and $\{b, d\}$. ■

As corollaries we obtain the following two theorems.

Theorem 2.3. *A strong digraph D of order at least three has at least three quasi-kernels, unless D is* $\vec{C}_4$.

Proof. Immediate from the previous theorems, Theorems 2.1 and 2.2. ■

Theorem 2.4. *Let D be a digraph, S the set of sinks in D, R the set of vertices that have an arc into S, and $H = D - S - R$. Then D has precisely two quasi-kernels, if and only if one of the following holds:*

(a) *There is a 2-cycle C in H such that at most one of the vertices in C has an arc into R, no vertex of C dominates a vertex in $H - V(C)$, and every vertex in $H - V(C)$ dominates both vertices in C.*
(b) *There is an induced 4-cycle, C, in H such that no vertex of C dominates a vertex in $D - V(C)$ and every vertex in $H - V(C)$ dominates two adjacent vertices in C.*
(c) *The digraph H has at least two vertices. There is a vertex x in H such that no vertex of H is dominated by x, all the vertices of $H - x$ dominate x, i.e., $(V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\}$, and there is a kernel Q in $H - x$, consisting only of sinks in $H - x$. Moreover, there is no arc from Q to R.*
(d) *The digraph H has exactly one vertex and this vertex dominates a vertex in R.*

Proof. We first show that if D has precisely two quasi-kernels, then D has the above-described structure. Let D be a digraph with exactly two quasi-kernels. If D has no sinks, then by Theorem 2.2, D has the structure described in part (a) or (b) with $R \cup S = \emptyset$. Hence, we may assume that D contains some sinks, and let S, R, and H be as defined in the formulation of this theorem. Let us first prove that H has at most one sink.

Suppose that there are at least two sinks in H. Let x and y be two distinct sinks in H. Note that both x and y have arcs into R, since otherwise they would belong to S or R. Let Q_1 be a quasi-kernel in H, Q_2 a quasi-kernel in $H - x$, and Q_3 a quasi-kernel in $H - y$. Since $\{x, y\} \subseteq Q_1$, $\{x, y\} \cap Q_2 = \{y\}$, and $\{x, y\} \cap Q_3 = \{x\}$ we see that $Q_1 \cup S$, $Q_2 \cup S$ and $Q_3 \cup S$, are three different quasi-kernels in D, a contradiction. Hence, H has at most one sink.

Suppose that there is exactly one sink x in H. Since the case of H having exactly one vertex is trivial, we may assume that H contains at least two vertices. Let Q_1 be a quasi-kernel in H, and let Q_2 be a quasi-kernel in $H - x$. Note that $S \cup Q_1$ and $S \cup Q_2$ are different quasi-kernels in D (as $x \in Q_1$ and x has an arc into R). Therefore, Q_2 must be the unique quasi-kernel in $H - x$, and, by Theorem 2.1, Q_2 is a kernel in $H - x$ consisting only of sinks in $H - x$. Since x is the only sink in H, every vertex in Q_2 dominates x. Therefore, $\{x\}$ is a quasi-kernel in H. Since x must be the unique quasi-kernel in H and x is a sink, we must have $(V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\}$. Thus, $S \cup \{x\}$ and $S \cup Q_2$ are quasi-kernels in D. If there is a vertex $w \in Q_2$ which dominates a vertex in R, then let Q_3 be a quasi-kernel in $H - w - x$, and observe that $Q_3 \cup S$ is a third quasi-kernel, a contradiction. Therefore, D has the structure described in part (c).

Suppose now that H has no sink. (Since D has more than one quasi-kernel, H is non-empty.) By Theorem 2.1, there are at least two quasi-kernels, Q_1 and Q_2, in H. If Q is a quasi-kernel in H, then $S \cup Q$ is a quasi-kernel in D. Hence, Q_1 and Q_2 are the only quasi-kernels in H, and, thus, the structure of H is provided by Theorem 2.2. Let C be the 2-cycle or induced 4-cycle given in Theorem 2.2.

If C is a 2-cycle, xyx, then, by Theorem 2.2, to show that D has the structure described in part (a) it suffices to prove that at most one of the vertices x and y has an arc into R. Assume that both x and y have arcs into R. Let Q_3 be a quasi-kernel in $H-x-y$, if $V(H) \neq \{x, y\}$, and the empty set, otherwise. However, $S \cup x$, $S \cup y$, and $S \cup Q_3$ are three different quasi-kernels in D, a contradiction.

If C is an induced 4-cycle, $abcda$, then, by Theorem 2.2, to show that D has the structure described in part (b) it suffices to prove that no vertex in $V(C)$ dominates a vertex in R. Without loss of generality, assume that a dominates a vertex in R. By Lemma 2.1, there exists a quasi-kernel, Q, in $H-a$, which does not contain b, as b is not a sink in $H-a$. However, $Q \cup S$, $\{a, c\} \cup S$, and $\{b, d\} \cup S$ are three different quasi-kernels in D, a contradiction.

This proves that if D has exactly two quasi-kernels, then D has the structure described in the formulation of this theorem. If D has the structure provided in part (a), (b), (c), or (d), then it is not too difficult to check that there are exactly two quasi-kernels in D. ∎

3. DISJOINT QUASI-KERNELS

If a digraph D has a sink x, then every quasi-kernel in D must contain x. Hence, a digraph with sinks has no disjoint quasi-kernels. However, one may suspect that every digraph with no sink has a pair of disjoint quasi-kernels. By Lemma 2.1, this is true for digraphs with exactly two quasi-kernels: see the first paragraph in the proof of Theorem 2.2. One can show that this is also true for every digraph which possesses a quasi-kernel of cardinality at most two.

Unfortunately, in general, the above claim does not hold. Consider the following construction suggested to us by the referee. Let T be a tournament having the property that for every pair x, y of vertices there exists a vertex z such that $x \to z$ and $y \to z$. (The existence of such tournaments was first proved by Erdös [4], see also Section 1.2 in [1]. It was shown by Graham and Spencer [5] that some quadratic residue tournaments are such tournaments, see also Section 9.1 in [1].) Extend T to a digraph D by adding, for every vertex x in T, a new vertex x' together with the arc $x'x$.

Clearly, D has no sink and every quasi-kernel of D contains exactly one vertex in T. If Q_x and Q_y are a pair of quasi-kernels of D containing the vertices x and y, respectively, then they are not disjoint because they both have to contain z', where $x \to z$ and $y \to z$.

ACKNOWLEDGMENTS

We are grateful to the referee and David Cohen for their useful comments and suggestions. Parts of this work were done when G.G. was visiting the Department of Mathematics, National University of Singapore. The hospitality and financial support of the department are highly appreciated.

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