Abstract. Let $D$ be a digraph with minimum indegree at least one. The following results are proved: a digraph $D$ has a semikernel if and only if its line digraph $L(D)$ does; the number of $(k, l)$-kernels in $L(D)$ is less than or equal to that in $D$; if the number of $(k, l)$-kernels in $D$ is less than or equal to the number of $(2, l)$-kernels in $L(D)$, and if $L(D)$ has a $(k, l)$-kernel, then $D$ has a $(k', l')$-kernel for $k' + l \leq k, l \leq l'$. As a consequence, it obtains previous results about kernels and quasikernels in the line digraph.

It is also proved that any digraph has a $(k, l)$-kernel with $l \geq 2k - 2, k \geq 1$, generalizing a previous result on the existence of quasikernels in digraphs.

Key words. kernel, $(k, l)$-kernels, line digraph, semikernels

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1. Introduction. For general concepts we refer the reader to [1].

DEFINITION 1.1. Let $D = (V(D), A(D))$ be a digraph. The line digraph $L(D)$ of $D$ is the digraph $L(D) = (V(L(D)), A(L(D)))$ with set of vertices the set of arcs of $D$, and for any $h, k \in A(D)$ there is $(h, k) \in A(L(D))$ if and only if the corresponding arcs $h, k$ induce a directed walk in $D$, i.e., the terminal endpoint of $h$ is the initial endpoint of $k$. In what follows we denote the arc $h = (u, v) \in A(D)$ and the vertex $h \in V(L(D))$ by the same symbol. If $H$ is a set of arcs in $D$, it is also a set of vertices of $L(D)$. When we want to emphasize our interest in $H \subseteq A(D)$ as a set of vertices of $L(D)$, we use the symbol $H_L$ instead of $H$.

DEFINITION 1.2. A set $K \subseteq V(D)$ is said to be a kernel if it is both independent (a vertex in $K$ has no successor in $K$) and absorbing (a vertex not in $K$ has a successor in $K$).

This concept was introduced by Von Neumann [11] and it has found many applications [1], [2]. Several authors have been investigating sufficient conditions for the existence of kernels in digraphs, namely, Von Neumann and Morgenstern [11], Richardson [13], Duchet and Meyniel [4], [5], and Galeana-Sánchez and Neumann-Lara [7]. In [9], Harminc considered the existence of kernels in the line digraph of a given digraph $D$ and he proved the following result.

THEOREM 1.1 (see [8]). The number of kernels of a digraph $D$ is equal to the number of kernels in its line digraph $L(D)$.

DEFINITION 1.3 (see [12]). A semikernel $S$ of $D$ is an independent set of vertices such that, for every $z \in V(D) \setminus S$ for which there exists an $Sz$-arc, there also exists a $zS$-arc.

The concept of semikernel is closely related to that of kernel, and is very useful to find kernels in digraphs, where every induced subdigraph of a digraph $D$ has a semikernel then $D$ has a kernel (see [12]). In [8] it was proved that the number of
semikernels of a digraph $D$ is less than or equal to the number of semikernels of $L(D)$. In this paper we prove that a digraph $D$ has a semikernel if and only if $L(D)$ does.

**Definition 1.4.** A quasikenrel $Q$ of a digraph $D$ is an independent set of vertices such that $V(D) = Q \cup \Gamma^{-}(Q) \cup \Gamma^{-}(\Gamma^{-}(Q))$ (where for any $A \subseteq X$, $\Gamma^{-}(A) = \{x \in X | x \text{ has a successor in } A\}$).

In [3], Chvátal and Lovász proved that any digraph has a quasikenrel; a generalization of this result was obtained by Duchet, Hamidoune, and Meyniel [6]. In [8] the following result was proved.

**Theorem 1.2** (see [8]). If $D$ is a digraph such that every vertex has indegree at least one, then the number of quasikenrels of $D$ is less than or equal to the number of quasikenrels of its line digraph $L(D)$.

**Definition 1.5.** Let $D$ be a digraph. By the directed distance $d_{D}(x, y)$ from the vertex $x$ to the vertex $y$ in $D$ we mean the length of a shortest directed path from $x$ to $y$ in $D$.

**Definition 1.6** (see [10]). Let $k$ and $l$ be natural numbers with $k \geq 2$, $l \geq 1$. A set $J \subseteq V(D)$ will be called a $(k, l)$-kernel of the digraph $D$ if

1. for each $x' \neq x$, $\{x, x'\} \subseteq J$ we have $d_{D}(x, x') \geq k$,
2. for each $y \in (V(D) \setminus J)$, there exists $x \in J$ such that $d_{D}(y, x) \leq l$.

Notice that, for $k = 2$, $l = 1$, we have that a $(k, l)$-kernel is a kernel and that for $k = 2$, $l = 2$, a $(k, l)$-kernel is a quasikenrel.

### 2. Semikenrels and $(k, l)$-kenrels in the line digraph.

**Definition 2.1** (see [9]). Let $D = (V(D), A(D))$ be a digraph. We denote by $\mathcal{P}(X)$ the set of all the subsets of the set $X$, and $f : \mathcal{P}(V(D)) \rightarrow \mathcal{P}(A(D))$ will denote the function defined as follows: for each $Z \subseteq V(D)$, $f(Z) = \{(u, x) \in A(D) | x \in Z\}$. Also, we denote by $\bar{f} : \mathcal{P}(A(D)) \rightarrow \mathcal{P}(V(D))$ the function defined as follows: for each $A \subseteq A(D)$, $\bar{f}(A) = \{x \in V(D) | (u, x) \in A\}$.

**Lemma 2.1** (see [9]). If $Z \subseteq V(D)$ is an independent set of $D$, then $f(Z)^{\bar{f}}$ is an independent set in $L(D)$.

**Theorem 2.1**. If $D$ is a digraph such that every vertex has indegree at least one, then $D$ has a semikenrel if and only if $L(D)$ has a semikenrel.

Proof. If $D$ has a semikenrel $S$, then from the proof of Theorem 2.1 [8], we know that $f(S)^{\bar{f}}$ is a semikenrel of $L(D)$.

Conversely, if $L(D)$ has a semikenrel $A$, then we will show that $f(A)$ is a semikenrel of $D$.

First we prove that $\bar{f}(A)$ is independent. By contradiction, if $\bar{f}(A)$ is not independent, then there are two vertices $x, y \in \bar{f}(A)$ such that $(x, y) \in A(D)$. Since $x \in \bar{f}(A)$, there exists a vertex $u \in V(D)$ such that $(u, x) \in A$. Since $((u, x), (x, y))$ is an $A(x, y)$-arc in $L(D)$ and $A$ is a semikenrel of $L(D)$, there must be an arc $(y, v) \in A(D)$ such that $(y, v) \in A$ and $((x, y), (y, v)) \in A(L(D))$. Since $y \in \bar{f}(A)$, there is a $t \in V(D)$ such that $(t, y) \in A$. Then we have $\{(t, y), (y, v)\} \subseteq A$, with $((t, y), (y, v)) \in A(L(D))$, which contradicts the indepndence of $A$. We conclude that $\bar{f}(A)$ is independent.

Now, let $y \in V(D)$ such that there is a $f(A)y$-arc; there exists $x \in f(A)$ with $(x, y) \in A(D)$. Since $x \in \bar{f}(A)$, there is an arc $(z, x) \in A$. Thus $((z, x), (x, y))$ is an $A(x, y)$-arc in $L(D)$. Since $A$ is a semikenrel of $L(D)$, there exists an $(x, y)A$-arc in $L(D)$. Let that arc be $((x, y), (y, u))$ so that $(y, u) \in A$ and then $u \in \bar{f}(A)$. We have proved that there is a $y\bar{f}(A)$-arc in $D$. Hence $\bar{f}(A)$ is a semikenrel of $D$.

**Theorem 2.2**. Let $D$ be a digraph such that each vertex has indegree at least one. Then the number of $(k, 1)$-kenrels in $L(D)$ is less than or equal to the number of $(k, 1)$-kenrels in $D$. 
Clearly, from Definition 2.1, \( x \) contradicts part (1) of Definition 1.6, so it follows from Definition 1.1 that the terminal endpoint of \( u \) is contained in \( K \).

3. Consider \( D \) no \((u, x)\) of Definition 1.6 as \( K \). Let \( a, a' \) be the set of all \( (u, x) \) contained in \( D \) of length \( n < k \) with \( \{u, x\}, a, a' \subseteq \bar{K} \); this contradicts part (1) of Definition 1.6, as \( \bar{K} \) is a \((k, 1)\)-kernel of \( L(D) \).

If \( a_n = (x_{n-1}, x_n) \notin \bar{K} \), then it follows from part (2) of Definition 1.6 that there exists \( (x_n, z) \in \bar{K} \) such that \( ((x_{n-1}, x_n), (x_n, z)) \in A(L(D)) \) (as \( \bar{K} \) is a \((k, 1)\)-kernel of \( L(D) \)). On the other hand, \( x' = x_n \in \bar{K} \), so there exists \( v \in V(D) \) with \( (v, x_n) \in \bar{K} \) and then \( ((v, x_n), (x_n, z)) \in A(L(D)) \) with \( \{v, x_n\}, (x_n, z) \subseteq \bar{K} \), contradicting part (1) of Definition 1.6 as \( \bar{K} \) is a \((k, 1)\)-kernel of \( L(D) \), \( k \geq 2 \).

(b) If \( y \in V(D) \setminus \bar{K} \), then there exists \( x \in \bar{K} \) such that \( (y, x) \in A(D) \).

Since \( y \in V(D) \), it follows from the hypothesis of Theorem 2.1 that there exists \( u \in V(D) \) with \( (u, y) \in A(D) \). Now \( y \in V(D) \setminus \bar{K} \) implies \( (u, y) \in V(L(D)) \setminus \bar{K} \); it follows from part (2) of Definition 1.6 that there exists \( (y, x) \in \bar{K} \) such that \( ((u, y), (y, x)) \in A(L(D)) \) (because \( \bar{K} \) is a \((k, 1)\)-kernel of \( L(D) \)). Since \( (y, x) \in \bar{K} \), we have \( x \in \bar{K} \) and (b) is proved.

Let \( K_1 \) be the set of all \((k, 1)\)-kernels of \( L(D) \) and \( K \) the set of all \((k, 1)\)-kernels of \( D \). We will prove that \( f' : K_1 \to K \), where \( f' \) is the restriction of \( f \) to \( K_1 \), is an injective function.

(c) If \( K_1, K_2 \subseteq K_1, K_1 \neq K_2 \), then \( f'(K_1) \neq f'(K_2) \).

Suppose, without loss of generality, that \( K_1 \setminus K_2 \neq \emptyset \) and take \( (u, v) \in K_1 \setminus K_2 \). Clearly, from Definition 2.1 \( v \in f'(K_1) \) and we will show that \( v \notin f'(K_2) \). By contradiction, assume \( v \in f'(K_2) \); hence there exists \((z, v) \in K_2 \). Since \( (u, v) \notin K_2 \), it follows from part (2) of Definition 1.6 that there exists \( (v, y) \) in \( K_2 \) such that \( ((z, v), (v, y)) \subseteq K_2 \), contradicting part (1) of Definition 1.6, because \( K_2 \) is a \((k, 1)\)-kernel of \( L(D) \). We conclude that \( v \notin f'(K_2) \), and so \( f'(K_1) \neq f'(K_2) \) and \( f' \) is injective.

Remark 2.1. The hypothesis that each vertex has indegree at least one cannot be omitted in Theorem 2.2 for \( k \geq 3 \). It suffices to consider \( D \) with \( V(D) = \{u_1, u_2, u_3, u_4, u_5, u_6\} \) and \( A(D) = \{(u_1, u_2), (u_2, u_3), (u_4, u_5), (u_5, u_6)\} \). Here \( D \) has no \((k, 1)\)-kernel but \( L(D) \) has one \((k, 1)\)-kernel for any \( k \geq 3 \).

Remark 2.2. The inequality announced in Theorem 2.2 can be strict for \( k \geq 3 \). Consider \( D \) with \( V(D) = \{u_1, u_2, u_3\} \) and \( A(D) = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\} \). Then \( D \) has a \((k, 1)\)-kernel and \( L(D) \) does not have any \((k, 1)\)-kernel for \( k \geq 3 \).

Theorem 2.3. Let \( D \) be a digraph such that every vertex has indegree at least one. Then the number of \((k, l)\)-kernels in \( D \) is less than or equal to the number of \((2, l)\)-kernels in \( L(D) \).

Proof. First we will prove that if \( K \) is a \((k, l)\)-kernel of \( D \), \( k \geq 2 \), then \( f(K) \) is a \((2, l)\)-kernel of \( L(D) \).

Let \( K \) be a \((k, l)\)-kernel of \( D \).

(a) If \( a \neq a' \), \( \{a, a'\} \subseteq f(K) \), then \( d_{L(D)}(a, a') \geq 2 \).

By contradiction, suppose that \( d_{L(D)}(a, a') \leq 1 \), as \( a \neq a' \), then \( d_{L(D)}(a, a') = 1 \); it follows from Definition 1.1 that the terminal endpoint of \( a \) is the initial endpoint
of \( a' \). Denoting \( a = (x, y), a' = (y, z) \), it follows from Definition 2.1 and the fact \\
\{a, a'\} \subseteq f(K) \) that \( \{y, z\} \subseteq K \), so \( (y, z) \in A(D) \) with \( \{y, z\} \subseteq K \), contradicting part \( (1) \) of Definition 1.6 as \( K \) is a \((k, l)\)-kernel of \( D \).

(b) If \( b \in V(L(D)) \setminus f(K) \), then there exists \( a \in f(K) \) such that \( d_{L(D)}(b, a) \leq l \).

Denoting \( b = (u, v) \) we have from Definition 2.1 and the fact \( b \notin f(K) \) that \( v \notin K \); now part \((2)\) of Definition 1.6 implies that there exists \( w \in K \) such that \( d_D(v, w) = n \leq l \). Let \( v = x_0, x_1, \ldots, x_n = w \) be a shortest directed path from \( v \) to \( w \) in \( D \) and denote \( a_i = (x_{i-1}, x_i) \in A(D) \). Then \( (b, a_1, a_2, \ldots, a_n) \) is a directed path in \( L(D) \) of length \( n \) from \( b \) to \( a_n \), and since \( w \in K \) we have \( a_n \in f(K) \), so taking \( a = a_n \), \((b)\) is proved.

Let \( \mathcal{K} \) be the set of all \((k, l)\)-kernels of \( D \), \( k \geq 2 \), and let \( \mathcal{K}_2 \) be the set of all \((2, l)\)-kernels of \( L(D) \). We will prove that \( f' : \mathcal{K} \to \mathcal{K}_2 \), where \( f' \) is the restriction of \( f \) to \( \mathcal{K} \), is an injective function.

(c) If \( K_1, K_2 \in \mathcal{K}, K_1 \neq K_2 \), then \( f'(K_1) \neq f'(K_2) \).

Suppose, without loss of generality, that \( K_1 \setminus K_2 \neq \emptyset \) and take \( v \in K_1 \setminus K_2 \). It follows from the hypothesis of Theorem 2.3 that there exists \((u, v) \in A(D) \); it follows from Definition 2.1 that \((u, v) \in f'(K_1) \setminus f'(K_2) \) and so \( f'(K_1) \neq f'(K_2) \). \( \square \)

**Remark 2.3.** The hypothesis that each vertex has indegree at least one cannot be omitted in Theorem 2.3 for \( l \geq 2 \). Consider that \( D \cong T_2 \) is the directed path of length two; \( L(D) \cong T_1 \) is the directed path of length one, \( D \) has two \((2, l)\)-kernels for any \( l \geq 2 \), and \( L(D) \) has just one \((2, l)\)-kernel for any \( l \geq 2 \).

**Remark 2.4.** The inequality announced in Theorem 2.3 can be strict for \( l \geq 2 \). Consider any \( k, k > 1 + l \) and \( T_{k-1} \) has no \((k, l)\)-kernel but that \( L(D) \cong T_{k-2} \) has a kernel, and hence a \((k, l)\)-kernel, for any \( l \geq 2 \).

**Remark 2.5.** As a direct consequence of Theorems 2.2 and 2.3 we obtain Theorem 1.1 in the case that each vertex has indegree at least one, as a kernel is a \((2, l)\)-kernel. In addition, Theorem 1.2 is a direct consequence of Theorem 2.3, as a quasikernel is a \((2, 2)\)-kernel.

**Corollary 2.1.** If \( D \) is a digraph such that each vertex has indegree at least one, then the number of \((2, l)\)-kernels in \( D \) is less than or equal to the number of \((2, l)\)-kernels in \( L(D) \).

The proof is a direct consequence of Theorem 2.3.

**Theorem 2.4.** Let \( D \) be a digraph such that each vertex has indegree at least one. If \( L(D) \) has a \((k, l)\)-kernel, then \( D \) has a \((k', l')\)-kernel, for \( k' + l \leq k \) and \( l \leq l' \).

**Proof.** Let \( D \) be a digraph as in the hypothesis, \( \tilde{K} \) a \((k, l)\)-kernel of \( L(D) \), \( k' + l \leq k \), and \( l \leq l' \). We will prove that \( \tilde{f}(\tilde{K}) \) is a \((k', l')\)-kernel of \( D \).

(a) If \( (x, y) \subseteq \tilde{f}(\tilde{K}) \), then \( d_D(x, y) \geq k' \).

By contradiction, suppose that \( d_D(x, y) = n < k' \), and let \( x = x_0, x_1, \ldots, x_n = y \) be a shortest directed path from \( x \) to \( y \) in \( D \). Since \( x \in \tilde{f}(\tilde{K}) \), there exists an arc \( a = (u, x) \in \tilde{K} \). Denoting \( a_i = (x_{i-1}, x_i) \in A(D) \), \( 1 \leq i \leq n \), we have from Definition 1.1 that \( (a, a_1, \ldots, a_n) \) is a directed path in \( L(D) \) of length \( n \). Now consider two possible cases.

If \( a_n \in \tilde{K} \), then \( d_{L(D)}(a, a_n) \leq n < k' < k \) with \( \{a, a_n\} \subseteq \tilde{K} \), contradicting part \((1)\) of Definition 1.6, as \( \tilde{K} \) is a \((k, l)\)-kernel of \( L(D) \).

If \( a_n \notin \tilde{K} \), then it follows from part \((2)\) of Definition 1.6 that there exists \( b \in \tilde{K} \) such that \( d_{L(D)}(a_n, b) \leq l \); let \( a_n = b_0, b_1, \ldots, b_m = b \) be a shortest directed path in \( L(D) \) from \( a_n \) to \( b \). On the other hand, since \( y = x_n \in \tilde{f}(\tilde{K}) \), there exists \( c = (v, y) \in \tilde{K} \). Now consider two possibilities.

If \( c \neq b \), then it follows from Definition 1.1 that \( (c, b_1, b_2, \ldots, b_m = b) \) is a directed path in \( L(D) \) from \( c \) to \( b \) in \( L(D) \) of length \( m \leq l < k \) with \( \{c, b\} \subseteq \tilde{K} \), contradicting part \((1)\) of Definition 1.6, as \( \tilde{K} \) is a \((k, l)\)-kernel of \( L(D) \).
If \( c = b \), then \((a,a_1,a_2,\ldots,a_n = b_0,b_1,\ldots,b_m = b)\) is a directed walk from \( a \) to \( b \) in \( L(D) \) of length \( n + m \); hence there exists in \( L(D) \) a directed path from \( a \) to \( b \) of length at most \( n + m \) and \( n + m < k' + l \leq k \). So \( d_{L(D)}(a,b) < k, a \neq n \) (because \( x \neq y, a = (u,x), b = c = (v,y) \), and \( \{a,b\} \subseteq K \). This contradicts part (1) of Definition 1.6, as \( K \) is a \((k,l)\)-kernel of \( L(D) \).

(b) If \( x \notin f(K) \), then there exists \( y \in f(K) \) such that \( d_D(x,y) \leq l' \).

Let \( x \in V(D) \setminus f(K) \). It follows from the hypothesis of Theorem 2.4 that there exists \( a = (u,x) \in A(D) \), and Definition 2.1 implies \( a \notin K \). Since \( a \notin K \) and \( K \) is a \((k,l)\)-kernel of \( L(D) \), it follows from part (2) of Definition 1.6 that there exists \( b \in K \) such that \( d_{L(D)}(a,b) \leq l \); let \( b = (v,y) \). Clearly \( y \in f(K) \) and \( d_D(x,y) \leq l \leq l' \).

**Theorem 2.5.** Let \( D \) be a digraph such that each vertex has indegree at least one. If \( L(D) \) has a \((k,l)\)-kernel \( \hat{A} \) with the properties that \( l < k \) and, for each arc \( a \in \hat{A} \), there is an arc \( b \neq a \) in \( \hat{A} \) such that the terminal endpoints of \( a \) and \( b \) are the same, then \( f(\hat{A}) \) is a \((k,l)\)-kernel of \( D \).

**Proof.** Let \( D \) be a digraph and \( \hat{A} \) a \((k,l)\)-kernel of \( L(D) \) as in the hypothesis of Theorem 2.5. We will prove that \( f(\hat{A}) \) is a \((k,l)\)-kernel of \( D \).

(a) If \( (x,y) \notin f(\hat{A}), x \neq y \), then \( d_D(x,y) \geq k \).

By contradiction, suppose that \( d_D(x,y) = n < k \) and let \( (x = x_0, x_1, \ldots, x_n = y) \) be a shortest directed path from \( x \) to \( y \) in \( D \). Since \( x \notin f(\hat{A}) \), there exists \( a = (u,x) \in \hat{A} \). Denote by \( a_i = (x_{i-1}, x_i), 1 \leq i \leq n \) and consider the following two possible cases.

If \( a_n = (x_{n-1}, y) \notin \hat{A} \), then \((a_1, a_2, \ldots, a_n)\) is a directed path of length \( n < k \) contained in \( L(D) \) from \( a \) to \( a_n \) with \( a \neq a_n \) and \( \{a, a_n\} \subseteq \hat{A} \). This contradicts part (1) of Definition 1.6, as \( \hat{A} \) is a \((k,l)\)-kernel of \( L(D) \).

If \( a_n = (x_{n-1}, y) \notin \hat{A} \), it follows from part (2) of Definition 1.6 that there exists \( b \in \hat{A} \) such that \( d_{L(D)}(a_n, b) \leq l < k \). On the other hand, since \( y \notin f(\hat{A}) \), there exists \( c = (v,y) \in \hat{A} \). Now consider two possibilities.

If \( b \neq c \), we consider a shortest directed path from \( a_n \) to \( b \), say \((a_n, b_0, b_1, \ldots, b_n = b)\), contained in \( L(D) \); then it follows from Definition 2.1 that \((c, b_1, b_2, \ldots, b_n = b)\) is also a directed path in \( L(D) \) of length \( n < k \) from \( c \) to \( b \) with \( c \neq b \) and \( \{c, b\} \subseteq \hat{A} \), contradicting part (1) of Definition 1.6, as \( \hat{A} \) is a \((k,l)\)-kernel of \( L(D) \).

If \( b = c \), we consider an arc \( d \in \hat{A}, d \neq b \) such that \( d \) and \( b \) have the same terminal endpoint (this is from the hypothesis of Theorem 2.5). It follows from Definition 2.1 that \((d, b_1, b_2, \ldots, b_n = b)\) is a directed path contained in \( L(D) \) from \( d \) to \( b \) of length \( n < k \) with \( d \neq b \), \( \{d, b\} \subseteq \hat{A} \), contradicting part (1) of Definition 1.6.

(b) If \( x \notin f(\hat{A}) \), then there exists \( y \in f(\hat{A}) \) such that \( d_D(x,y) \leq l \).

It follows from the hypothesis of Theorem 2.5 that there exists an arc \( a = (u,x) \in A(D) \); since \( x \notin f(\hat{A}) \), we have \( a \notin \hat{A} \). Now \( a \notin \hat{A} \) and \( \hat{A} \) is a \((k,l)\)-kernel of \( L(D) \), so there exists \( b \in \hat{A} \) such that \( d_{L(D)}(a,b) \leq l \). Let \((a = a_0, a_1, \ldots, a_n = b)\) be a shortest directed path in \( L(D) \) from \( a \) to \( b \), and \( a_i = (x_{i-1}, x_i) \) for \( 1 \leq i \leq n, b = (x_n, x_{n+1}) \); then \((x, x_1, \ldots, x_{n-1}, x_n)\) is a directed walk in \( D \) of length \( n \leq l \) from \( x \) to \( x_n \); clearly Definition 2.1 implies \( x_n \in f(\hat{A}) \). So, taking \( y = x_n \), (b) is thus proved.

**Corollary 2.2.** Let \( D \) be a digraph such that each vertex has indegree at least one and let \( 1 \leq l < k \). If each \((k,l)\)-kernel \( \hat{A} \) of \( L(D) \) satisfies that, for each arc \( a \in \hat{A} \), there is an arc \( b \in \hat{A} \) such that the terminal endpoints of \( a \) and \( b \) are the same, then the number of \((k,l)\)-kernels of \( L(D) \) is less than or equal to the number of \((k,l)\)-kernels of \( D \).

**Proof.** Let \( 1 \leq l < k \), \( K_1 \) be the set of all \((k,l)\)-kernels of \( L(D) \), let \( K \) be the set of all \((k,l)\)-kernels of \( D \), and let \( f' : K_1 \rightarrow K \) be the restriction of \( f \) to \( K_1 \). From Theorem 2.5 it suffices to prove that \( f' \) is an injective function.

(c) If \( K_1 \neq K_2 \) and \((K_1, K_2) \subseteq K_1 \), then \( f'(K_1) \neq f'(K_2) \).
Since \( \bar{K}_1 \neq \bar{K}_2 \), we can assume, without loss of generality, that \( \bar{K}_1 \setminus \bar{K}_2 \neq \emptyset \). Let \( a = (u, x) \in \bar{K}_1 \setminus \bar{K}_2 \). It follows from Definition 2.1 that \( x \in \bar{f}(\bar{K}_1) \), and we will show that \( x \notin \bar{f}(\bar{K}_2) \).

By contradiction, suppose that \( x \in \bar{f}(\bar{K}_2) \). Hence there exists \( b = (v, x) \in \bar{K}_2 \). Since \( a = (u, x) \notin \bar{K}_2 \) and \( \bar{K}_2 \) is a \((k, l)\)-kernel of \( L(D) \), there exists \( c \in \bar{K}_2 \) such that

\[
d_{L(D)}(a, c) \leq l < k.
\]

Let \( (a = a_0, a_1, \ldots, a_n = c) \) be a shortest directed path in \( L(D) \) from \( a \) to \( c \) and consider the following two possibilities:

If \( b \neq c \), then it follows from Definition 2.1 that \( (b, a_1, a_2, \ldots, a_n = c) \) is a directed path in \( L(D) \) from \( b \) to \( c \) of length \( n \leq l < k \) with \( \{b, c\} \subseteq \bar{K}_2 \), \( b \neq c \). This contradicts part (1) of Definition 1.6, as \( \bar{K}_2 \) is a \((k, l)\)-kernel of \( L(D) \).

If \( b = c \), we have from the hypothesis of Corollary 2.2 that there exists an arc \( d \in \bar{K}_2 \) such that \( d \neq b \) and that \( d \) and \( b \) have the same terminal endpoint \( x \). Then it follows from Definition 2.1 that \( (d, a_1, a_2, \ldots, a_n = c = b) \) is a directed path in \( L(D) \) of length \( n \leq k < l \) from \( d \) to \( b \) with \( d \neq b \), \( \{d, b\} \subseteq \bar{K}_2 \), contradicting part (1) of Definition 1.6, as \( \bar{K}_2 \) is a \((k, l)\)-kernel of \( L(D) \).

**Theorem 2.6.** Every digraph has a \((k, 2k - 2)\)-kernel.

**Proof.** We proceed by induction on \( |V(D)| \).

For \( D \) with \( |V(D)| = 1 \) it is obvious. Suppose that if \( D' \) is a digraph with \( |V(D')| < n \), then \( D' \) has a \((k, 2k - 2)\)-kernel, and let \( D \) be a digraph with \( |V(D)| = n \).

Let \( x_0 \in V(D) \) and \( D^* = D[V(D) \setminus \{x \in V(D) \setminus d_D(x, x_0) \leq k - 1\}] \). Clearly \( |V(D^*)| < n \), and hence \( D^* \) has a \((k, 2k - 2)\)-kernel, namely \( S^* \). Consider the following two possibilities:

If there exists a directed path in \( D \) of length less than or equal to \( k - 1 \), then \( S^* \) is a \((k, 2k - 2)\)-kernel of \( D \).

If there is no directed path in \( D \) from \( x_0 \) to some point of \( S^* \) of length less than or equal to \( k - 1 \), then \( S^* \cup \{x_0\} \) is a \((k, 2k - 2)\)-kernel of \( D \).

**Corollary 2.3.** Every digraph has a \((k, l)\)-kernel for \( l \geq 2k - 2 \).

The proof is a direct consequence of Theorem 2.6 and Definition 1.6, as a \((k, l)\)-kernel of a digraph \( D \) is also a \((k, l')\)-kernel for every \( l' \geq l \).

**Remark 2.6.** The hypothesis \( l \geq 2k - 2 \) cannot be omitted in Corollary 2.3. Consider \( C_{2k - 1} \) to be the directed cycle of length \( 2k - 1 \); for any \( l < 2k - 1 \), the digraph \( C_{2k - 1} \) has no \((k, l)\)-kernel.

**Corollary 2.4** (see [3]). Every digraph has a quasikernel.

The proof is a direct consequence of Theorem 2.6 by taking \( k = 2 \), as a quasikernel is a \((2, 2)\)-kernel.

**REFERENCES**