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SEMIKERNELS AND (k, l)-KERNELS IN DIGRAPHS*

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Abstract. Let D be a digraph with minimum indegree at least one. The following results are proved: a digraph D has a semikernel if and only if its line digraph $L(D)$ does; the number of $(k, 1)$-kernels in $L(D)$ is less than or equal to that in D; if the number of (k, l)-kernels in D is less than or equal to the number of $(2, l)$-kernels in $L(D)$, and if $L(D)$ has a (k, l)-kernel, then D has a (k', l')-kernel for $k' + l \leq k$, $l \leq l'$. As a consequence, it obtains previous results about kernels and quasikernels in the line digraph.

It is also proved that any digraph has a (k, l)-kernel with $l \geq 2k - 2$, $k \geq 1$, generalizing a previous result on the existence of quasikernels in digraphs.

Key words. kernel, (k, l)-kernels, line digraph, semikernels

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1. Introduction.

For general concepts we refer the reader to [1].

DEFINITION 1.1. *Let $D = (V(D), A(D))$ be a digraph. The line digraph $L(D)$ of D is the digraph $L(D) = (V(L(D)), A(L(D)))$ with set of vertices the set of arcs of D, and for any $h, k \in A(D)$ there is $(h, k) \in A(L(D))$ if and only if the corresponding arcs h, k induce a directed walk in D, i.e., the terminal endpoint of h is the initial endpoint of k. In what follows we denote the arc $h = (u, v) \in A(D)$ and the vertex $h \in V(L(D))$ by the same symbol. If H is a set of arcs in D, it is also a set of vertices of $L(D)$. When we want to emphasize our interest in $H \subseteq A(D)$ as a set of vertices of $L(D)$, we use the symbol H_L instead of H.*

DEFINITION 1.2. *A set $K \subseteq V(D)$ is said to be a kernel if it is both independent (a vertex in K has no successor in K) and absorbing (a vertex not in K has a successor in K).*

This concept was introduced by Von Neumann [11] and it has found many applications [1], [2]. Several authors have been investigating sufficient conditions for the existence of kernels in digraphs, namely, Von Neumann and Morgenstern [11] , Richardson [13], Duchet and Meyniel [4], [5], and Galeana-Sánchez and Neumann-Lara [7]. In [9], Harminc considered the existence of kernels in the line digraph of a given digraph D and he proved the following result.

THEOREM 1.1 (see [8]). *The number of kernels of a digraph D is equal to the number of kernels in its line digraph $L(D)$.*

DEFINITION 1.3 (see [12]). *A semikernel S of D is an independent set of vertices such that, for every $z \in (V(D) \setminus S)$ for which there exists an Sz-arc, there also exists a zS-arc.*

The concept of semikernel is nearly related to that of kernel, and is very useful to find kernels in digraphs, where every induced subdigraph of a digraph D has a semikernel then D has a kernel (see [12]). In [8] it was proved that the number of

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semikernels of a digraph D is less than or equal to the number of semikernels of $L(D)$. In this paper we prove that a digraph D has a semikernel if and only if $L(D)$ does.

DEFINITION 1.4. *A quasikernel Q of a digraph D is an independent set of vertices such that $V(D) = Q \cup \Gamma^-(Q) \cup \Gamma^-(\Gamma^-(Q))$ (where for any $A \subseteq X$, $\Gamma^-(A) = \{x \in X | x$ has a successor in $A\}$).*

In [3], Chvátal and Lovász proved that any digraph has a quasikernel; a generalization of this result was obtained by Duchet, Hamidoune, and Meyniel [6]. In [8] the following result was proved.

THEOREM 1.2 (see [8]). *If D is a digraph such that every vertex has indegree at least one, then the number of quasikernels of D is less than or equal to the number of quasikernels of its line digraph $L(D)$.*

DEFINITION 1.5. *Let D be a digraph. By the directed distance $d_D(x,y)$ from the vertex x to the vertex y in D we mean the length of a shortest directed path from x to y in D.*

DEFINITION 1.6 (see [10]). *Let k and l be natural numbers with $k \geq 2$, $l \geq 1$. A set $J \subseteq V(D)$ will be called a (k,l)-kernel of the digraph D if*

(1) *for each $x' \neq x$, $\{x, x'\} \subseteq J$ we have $d_D(x,x') \geq k$,*

(2) *for each $y \in (V(D) \setminus J)$, there exists $x \in J$ such that $d_D(y,x) \leq l$.*

Notice that, for $k = 2$, $l = 1$, we have that a (k,l)-kernel is a kernel and that for $k = 2$, $l = 2$, a (k,l)-kernel is a quasikernel.

2. Semikernels and (k,l)-kernels in the line digraph.

DEFINITION 2.1 (see [9]). *Let $D = (V(D), A(D))$ be a digraph. We denote by $\mathcal{P}(X)$ the set of all the subsets of the set X, and $f : \mathcal{P}(V(D)) \to \mathcal{P}(A(D))$ will denote the function defined as follows: for each $Z \subseteq V(D)$, $f(Z) = \{(u,x) \in A(D) | x \in Z\}$. Also, we denote by $\bar{f} : \mathcal{P}(A(D)) \to \mathcal{P}(V(D))$ the function defined as follows: for each $A \subseteq A(D)$, $\bar{f}(A) = \{x \in V(D) | (u,x) \in A\}$.*

LEMMA 2.1 (see [9]). *If $Z \subseteq V(D)$ is an independent set of D, then $f(Z)_L$ is an independent set in $L(D)$.*

THEOREM 2.1. *If D is a digraph such that every vertex has indegree at least one, then D has a semikernel if and only if $L(D)$ has a semikernel.*

Proof. If D has a semikernel S, then from the proof of Theorem 2.1 [8], we know that $f(S)_L$ is a semikernel of $L(D)$.

Conversely, if $L(D)$ has a semikernel A, then we will show that $\bar{f}(A)$ is a semikernel of D.

First we prove that $\bar{f}(A)$ is independent. By contradiction, if $\bar{f}(A)$ is not independent, then there are two vertices $x, y \in \bar{f}(A)$ such that $(x,y) \in A(D)$. Since $x \in \bar{f}(A)$, there exists a vertex $u \in V(D)$ such that $(u,x) \in A$. Since $((u,x),(x,y))$ is an $A(x,y)$-arc in $L(D)$ and A is a semikernel of $L(D)$, there must be an arc $(y,v) \in A(D)$ such that $(y,v) \in A$ and $((x,y),(y,v)) \in A(L(D))$. Since $y \in \bar{f}(A)$, there is a $t \in V(D)$ such that $(t,y) \in A$. Then we have $\{(t,y),(y,v)\} \subseteq A$, with $((t,y),(y,v)) \in A(L(D))$, which contradicts the independence of A. We conclude that $\bar{f}(A)$ is independent.

Now, let $y \in V(D)$ such that there is a $\bar{f}(A)y$-arc; there exists $x \in \bar{f}(A)$ with $(x,y) \in A(D)$. Since $x \in \bar{f}(A)$, there is an arc $(z,x) \in A$. Thus $((z,x),(x,y))$ is an $A(x,y)$-arc in $L(D)$. Since A is a semikernel of $L(D)$, there exists an $(x,y)A$-arc in $L(D)$. Let that arc be $((x,y),(y,u))$ so that $(y,u) \in A$ and then $u \in \bar{f}(A)$. We have proved that there is a $y\bar{f}(A)$-arc in D. Hence $\bar{f}(A)$ is a semikernel of D. ∎

THEOREM 2.2. *Let D be a digraph such that each vertex has indegree at least one. Then the number of $(k,1)$-kernels in $L(D)$ is less than or equal to the number of $(k,1)$-kernels in D.*

Proof. First we will prove that if $\bar{K}$ is a $(k,1)$-kernel of $L(D)$, then $\bar{f}(\bar{K})$ is a $(k,1)$-kernel of D.

Let $\bar{K}$ be a $(k,1)$-kernel of $L(D)$.

(a) If $x \neq x'$, $\{x, x'\} \subseteq \bar{f}(\bar{K})$, then $d_D(x, x') \geq k$.

By contradiction, suppose that $d_D(x, x') = n < k$. Take $\alpha = (x = x_0, x_1, \ldots, x_n = x')$, a shortest directed path from x to x' contained in D. Since $x \in \bar{f}(\bar{K})$, there exists $u \in V(D)$ such that $(u, x) \in \bar{K}$. Denote by $a_i = (x_{i-1}, x_i) \in A(D)$, $1 \leq i \leq n$, and consider the following two possibilities:

If $a_n = (x_{n-1}, x_n) \in \bar{K}$, consider that $((u, x), a_1, a_2, \ldots, a_n)$ is a directed path from (u, x) to a_n contained in $L(D)$ of length $n < k$ with $\{(u, x), a_n\} \subseteq \bar{K}$; this contradicts part (1) of Definition 1.6, as $\bar{K}$ is a $(k,1)$-kernel of $L(D)$.

If $a_n = (x_{n-1}, x_n) \notin \bar{K}$, then it follows from part (2) of Definition 1.6 that there exists $(x_n, z) \in \bar{K}$ such that $((x_{n-1}, x_n), (x_n, z)) \in A(L(D))$ (as $\bar{K}$ is a $(k,1)$-kernel of $L(D)$). On the other hand, $x' = x_n \in \bar{f}(\bar{K})$, so there exists $v \in V(D)$ with $(v, x_n) \in \bar{K}$ and then $((v, x_n), (x_n, z)) \in A(L(D))$ with $\{(v, x_n), (x_n, z)\} \subseteq \bar{K}$, contradicting part (1) of Definition 1.6 as $\bar{K}$ is a $(k,1)$-kernel of $L(D)$, $k \geq 2$.

(b) If $y \in V(D) \setminus \bar{f}(\bar{K})$, then there exists $x \in \bar{f}(\bar{K})$ such that $(y, x) \in A(D)$.

Since $y \in V(D)$, it follows from the hypothesis of Theorem 2.1 that there exists $u \in V(D)$ with $(u, y) \in A(D)$. Now $y \in V(D) \setminus \bar{f}(\bar{K})$ implies $(u, y) \in V(L(D)) \setminus \bar{K}$; it follows from part (2) of Definition 1.6 that there exists $(y, x) \in \bar{K}$ such that $((u, y), (y, x)) \in A(L(D))$ (because $\bar{K}$ is a $(k,1)$-kernel of $L(D)$). Since $(y, x) \in \bar{K}$, we have $x \in \bar{f}(\bar{K})$ and (b) is proved.

Let $\mathcal{K}_1$ be the set of all $(k,1)$-kernels of $L(D)$ and $\mathcal{K}$ the set of all $(k,1)$-kernels of D. We will prove that $\bar{f}' : \mathcal{K}_1 \to \mathcal{K}$, where $\bar{f}'$ is the restriction of $\bar{f}$ to $\mathcal{K}_1$, is an injective function.

(c) If $\bar{K}_1, \bar{K}_2 \in \mathcal{K}_1$, $\bar{K}_1 \neq \bar{K}_2$, then $\bar{f}'(\bar{K}_1) \neq \bar{f}'(\bar{K}_2)$.

Suppose, without loss of generality, that $\bar{K}_1 \setminus \bar{K}_2 \neq \emptyset$ and take $(u, v) \in \bar{K}_1 \setminus \bar{K}_2$. Clearly, from Definition 2.1 $v \in \bar{f}'(\bar{K}_1)$ and we will show that $v \notin \bar{f}'(\bar{K}_2)$. By contradiction, assume $v \in \bar{f}'(\bar{K}_2)$; hence there exists $(z, v) \in \bar{K}_2$. Since $(u, v) \notin \bar{K}_2$, it follows from part (2) of Definition 1.6 that there exists $(v, y) \in \bar{K}_2$. Hence $((z, v), (v, y)) \in A(L(D))$ with $\{(z, v), (v, y)\} \subseteq \bar{K}_2$, contradicting part (1) of Definition 1.6, because $\bar{K}_2$ is a $(k,1)$-kernel of $L(D)$. We conclude that $v \notin \bar{f}'(\bar{K}_2)$, and so $\bar{f}'(\bar{K}_1) \neq \bar{f}'(\bar{K}_2)$ and $\bar{f}'$ is injective. ∎

Remark 2.1. The hypothesis that each vertex has indegree at least one cannot be omitted in Theorem 2.2 for $k \geq 3$. It suffices to consider D with $V(D) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $A(D) = \{(u_1, u_2), (u_2, u_3), (u_4, u_5), (u_5, u_6)\}$. Here D has no $(k,1)$-kernel but $L(D)$ has one $(k,1)$-kernel for any $k \geq 3$.

Remark 2.2. The inequality announced in Theorem 2.2 can be strict for $k \geq 3$. Consider D with $V(D) = \{u_1, u_2, u_3\}$ and $A(D) = \{(u_1, u_2), (u_2, u_3), (u_3, u_1), (u_1, u_3)\}$. Then D has a $(k,1)$-kernel and $L(D)$ does not have any $(k,1)$-kernel for $k \geq 3$.

THEOREM 2.3. *Let D be a digraph such that every vertex has indegree at least one. Then the number of (k,l)-kernels in D is less than or equal to the number of $(2,l)$-kernels in $L(D)$.*

Proof. First we will prove that if K is a (k,l)-kernel of D, $k \geq 2$, then $f(K)$ is a $(2,l)$-kernel of $L(D)$.

Let K be a (k,l)-kernel of D.

(a) If $a \neq a'$, $\{a, a'\} \subseteq f(K)$, then $d_{L(D)}(a, a') \geq 2$.

By contradiction, suppose that $d_{L(D)}(a, a') \leq 1$, as $a \neq a'$, then $d_{L(D)}(a, a') = 1$; it follows from Definition 1.1 that the terminal endpoint of a is the initial endpoint of a'. Denoting $a = (x, y), a' = (y, z)$, it follows from Definition 2.1 and the fact $\{a, a'\} \subseteq f(K)$ that $\{y, z\} \subseteq K$, so $(y, z) \in A(D)$ with $\{y, z\} \subseteq K$, contradicting part (1) of Definition 1.6 as K is a (k, l)-kernel of D.

(b) If $b \in V(L(D)) \setminus f(K)$, then there exists $a \in f(K)$ such that $d_{L(D)}(b, a) \leq l$.

Denoting $b = (u, v)$ we have from Definition 2.1 and the fact $b \notin f(K)$ that $v \notin K$; now part (2) of Definition 1.6 implies that there exists $w \in K$ such that $d_D(v, w) = n \leq l$. Let $(v = x_0, x_1, \ldots, x_n = w)$ be a shortest directed path from v to w in D and denote $a_i = (x_{i-1}, x_i) \in A(D)$. Then $(b, a_1, a_2, \ldots, a_n)$ is a directed path in $L(D)$ of length n from b to a_n, and since $w \in K$ we have $a_n \in f(K)$, so taking $a = a_n$, (b) is proved.

Let $\mathcal{K}$ be the set of all (k, l)-kernels of D, $k \geq 2$, and let $\mathcal{K}_2$ be the set of all $(2, l)$-kernels of $L(D)$. We will prove that $f' : \mathcal{K} \to \mathcal{K}_2$, where f' is the restriction of f to $\mathcal{K}$, is an injective function.

(c) If $K_1, K_2 \in \mathcal{K}$, $K_1 \neq K_2$, then $f'(K_1) \neq f'(K_2)$.

Suppose, without loss of generality, that $K_1 \setminus K_2 \neq \emptyset$ and take $v \in K_1 \setminus K_2$. It follows from the hypothesis of Theorem 2.3 that there exists $(u, v) \in A(D)$; it follows from Definition 2.1 that $(u, v) \in f'(K_1) \setminus f'(K_2)$ and so $f'(K_1) \neq f'(K_2)$. □

Remark 2.3. The hypothesis that each vertex has indegree at least one cannot be omitted in Theorem 2.3 for $l \geq 2$. Consider that $D \cong T_2$ is the directed path of length two; $L(D) \cong T_1$ is the directed path of length one, D has two $(2, l)$-kernels for any $l \geq 2$, and $L(D)$ has just one $(2, l)$-kernel for any $l \geq 2$.

Remark 2.4. The inequality announced in Theorem 2.3 can be strict for $l \geq 2$. Consider any k, $k > l + 1$ and T_{k-1} has no (k, l)-kernel but that $L(D) \cong T_{k-2}$ has a kernel, and hence a (k, l)-kernel, for any $l \geq 2$.

Remark 2.5. As a direct consequence of Theorems 2.2 and 2.3 we obtain Theorem 1.1 in the case that each vertex has indegree at least one, as a kernel is a $(2, 1)$-kernel. In addition, Theorem 1.2 is a direct consequence of Theorem 2.3, as a quasikernel is a $(2, 2)$-kernel.

COROLLARY 2.1. *If D is a digraph such that each vertex has indegree at least one, then the number of $(2, l)$-kernels in D is less than or equal to the number of $(2, l)$-kernels in $L(D)$.*

The proof is a direct consequence of Theorem 2.3.

THEOREM 2.4. *Let D be a digraph such that every vertex has indegree at least one. If $L(D)$ has a (k, l)-kernel, then D has a (k', l')-kernel, for $k' + l \leq k$ and $l \leq l'$.*

Proof. Let D be a digraph as in the hypothesis, $\bar{K}$ a (k, l)-kernel of $L(D)$, $k' + l \leq k$, and $l \leq l'$. We will prove that $\bar{f}(\bar{K})$ is a (k', l')-kernel of D.

(a) If $\{x, y\} \subseteq \bar{f}(\bar{K})$, then $d_D(x, y) \geq k'$.

By contradiction, suppose that $d_D(x, y) = n < k'$, and let $(x = x_0, x_1, \ldots, x_n = y)$ be a shortest directed path from x to y in D. Since $x \in \bar{f}(\bar{K})$, there exists an arc $a = (u, x) \in \bar{K}$. Denoting $a_i = (x_{i-1}, x_i) \in A(D)$, $1 \leq i \leq n$, we have from Definition 1.1 that $(a, a_1, \ldots, a_n)$ is a directed path in $L(D)$ of length n. Now consider two possible cases.

If $a_n \in \bar{K}$, then $d_{L(D)}(a, a_n) \leq n < k' < k$ with $\{a, a_n\} \subseteq \bar{K}$, contradicting part (1) of Definition 1.6, as $\bar{K}$ is a (k, l)-kernel of $L(D)$.

If $a_n \notin \bar{K}$, then it follows from part (2) of Definition 1.6 that there exists $b \in \bar{K}$ such that $d_{L(D)}(a_n, b) \leq l$; let $(a_n = b_0, b_1, \ldots, b_m = b)$ be a shortest directed path in $L(D)$ from a_n to b. On the other hand, since $y = x_n \in \bar{f}(\bar{K})$, there exists $c = (v, y) \in \bar{K}$. Now consider two possibilities.

If $c \neq b$, then it follows from Definition 1.1 that $(c, b_1, b_2, \ldots, b_m = b)$ is a directed path in $L(D)$ from c to b in $L(D)$ of length $m \leq l < k$ with $\{c, b\} \subseteq \bar{K}$, contradicting part (1) of Definition 1.6, as $\bar{K}$ is a (k, l)-kernel of $L(D)$.

If $c = b$, then $(a, a_1, a_2, \ldots, a_n = b_0, b_1, \ldots, b_m = b)$ is a directed walk from a to b in $L(D)$ of length $n + m$; hence there exists in $L(D)$ a directed path from a to b of length at most $n + m$ and $n + m < k' + l \leq k$. So $d_{L(D)}(a, b) < k$, $a \neq n$ (because $x \neq y$, $a = (u, x)$,$b = c = (v, y)$), and $\{a, b\} \subseteq \bar{K}$. This contradicts part (1) of Definition 1.6, as $\bar{K}$ is a (k, l)-kernel of $L(D)$.

(b) If $x \notin \bar{f}(\bar{K})$, then there exists $y \in \bar{f}(\bar{K})$ such that $d_D(x, y) \leq l'$.

Let $x \in V(D) \setminus \bar{f}(\bar{K})$. It follows from the hypothesis of Theorem 2.4 that there exists $a = (u, x) \in A(D)$, and Definition 2.1 implies $a \notin \bar{K}$. Since $a \notin \bar{K}$ and $\bar{K}$ is a (k, l)-kernel of $L(D)$, it follows from part (2) of Definition 1.6 that there exists $b \in \bar{K}$ such that $d_{L(D)}(a, b) \leq l$; let $b = (v, y)$. Clearly $y \in \bar{f}(\bar{K})$ and $d_D(x, y) \leq l \leq l'$. $\square$

Theorem 2.5. *Let D be a digraph such that each vertex has indegree at least one. If $L(D)$ has a (k, l)-kernel $\bar{A}$ with the properties that $l < k$ and, for each arc $a \in \bar{A}$, there is an arc $b \neq a$ in $\bar{A}$ such that the terminal endpoints of a and b are the same, then $\bar{f}(\bar{A})$ is a (k, l)-kernel of D.*

Proof. Let D be a digraph and $\bar{A}$ a (k, l)-kernel of $L(D)$ as in the hypothesis of Theorem 2.5. We will prove that $\bar{f}(\bar{A})$ is a (k, l)-kernel of D.

(a) If $\{x, y\} \subseteq \bar{f}(\bar{A})$, $x \neq y$, then $d_D(x, y) \geq k$.

By contradiction, suppose that $d_D(x, y) = n < k$ and let $(x = x_0, x_1, \ldots, x_n = y)$ be a shortest directed path from x to y in D. Since $x \in \bar{f}(\bar{A})$, there exists $a = (u, x) \in \bar{A}$. Denote by $a_i = (x_{i-1}, x_i), 1 \leq i \leq n$ and consider the following two possible cases.

If $a_n = (x_{n-1}, y) \in \bar{A}$, then $(a, a_1, a_2, \ldots, a_n)$ is a directed path of length $n < k$ contained in $L(D)$ from a to a_n with $a \neq a_n$ and $\{a, a_n\} \subseteq \bar{A}$. This contradicts part (1) of Definition 1.6, as $\bar{A}$ is a (k, l)-kernel of $L(D)$.

If $a_n = (x_{n-1}, y) \notin \bar{A}$, it follows from part (2) of Definition 1.6 that there exists $b \in \bar{A}$ such that $d_{L(D)}(a_n, b) \leq l < k$. On the other hand, since $y \in \bar{f}(\bar{A})$, there exists $c = (v, y) \in \bar{A}$. Now consider two possibilities.

If $b \neq c$, we consider a shortest directed path from a_n to b, say $(a_n = b_0, b_1, \ldots, b_n = b)$, contained in $L(D)$; then it follows from Definition 2.1 that $(c, b_1, b_2, \ldots, b_n = b)$ is also a directed path in $L(D)$ of length $n < k$ from c to b with $c \neq b$ and $\{c, b\} \subseteq \bar{A}$, contradicting part (1) of Definition 1.6, as $\bar{A}$ is a (k, l)-kernel of $L(D)$.

If $b = c$, we consider an arc $d \in \bar{A}, d \neq b$ such that d and b have the same terminal endpoint (this is from the hypothesis of Theorem 2.5). It follows from Definition 2.1 that $(d, b_1, b_2, \ldots, b_n = b)$ is a directed path contained in $L(D)$ from d to b of length $n < k$ with $d \neq b$, $\{d, b\} \subseteq \bar{A}$, contradicting part (1) of Definition 1.6.

(b) If $x \notin \bar{f}(\bar{A})$, then there exists $y \in \bar{f}(\bar{A})$ such that $d_D(x, y) \leq l$.

It follows from the hypothesis of Theorem 2.5 that there exists an arc $a = (u, x) \in A(D)$; since $x \notin \bar{f}(\bar{A})$, we have $a \notin \bar{A}$. Now $a \notin \bar{A}$ and $\bar{A}$ is a (k, l)-kernel of $L(D)$, so there exists $b \in \bar{A}$ such that $d_{L(D)}(a, b) \leq l$. Let $(a = a_0, a_1, \ldots, a_n = b)$ be a shortest directed path in $L(D)$ from a to b, and $a_i = (x_{i-1}, x_i)$ for $1 \leq i \leq n$, $b = (x_{n-1}, x_n)$; then $(x, x_1, \ldots, x_{n-1}, x_n)$ is a directed walk in D of length $n \leq l$ from x to x_n; clearly Definition 2.1 implies $x_n \in \bar{f}(\bar{A})$. So, taking $y = x_n$, (b) is thus proved. $\square$

Corollary 2.2. *Let D be a digraph such that each vertex has indegree at least one and let $1 \leq l < k$. If each (k, l)-kernel $\bar{A}$ of $L(D)$ satisfies that, for each arc $a \in \bar{A}$, there is an arc $b \in \bar{A}$ such that the terminal endpoints of a and b are the same, then the number of (k, l)-kernels of $L(D)$ is less than or equal to the number of (k, l)-kernels of D.*

Proof. Let $1 \leq l < k$, $\mathcal{K}_1$ be the set of all (k, l)-kernels of $L(D)$, let $\mathcal{K}$ be the set of all (k, l)-kernels of D, and let $\bar{f}' : \mathcal{K}_1 \to \mathcal{K}$ be the restriction of $\bar{f}$ to $\mathcal{K}_1$. From Theorem 2.5 it suffices to prove that $\bar{f}'$ is an injective function.

(c) If $\bar{K}_1 \neq \bar{K}_2$ and $\{\bar{K}_1, \bar{K}_2\} \subseteq \mathcal{K}_1$, then $\bar{f}'(\bar{K}_1) \neq \bar{f}'(\bar{K}_2)$.

Since $\bar{K}_1 \neq \bar{K}_2$, we can assume, without loss of generality, that $\bar{K}_1 \setminus \bar{K}_2 \neq \emptyset$. Let $a = (u,x) \in \bar{K}_1 \setminus \bar{K}_2$. It follows from Definition 2.1 that $x \in \bar{f}'(\bar{K}_1)$, and we will show that $x \notin \bar{f}'(\bar{K}_2)$.

By contradiction, suppose that $x \in \bar{f}'(\bar{K}_2)$. Hence there exists $b = (v,x) \in \bar{K}_2$. Since $a = (u,x) \notin \bar{K}_2$ and $\bar{K}_2$ is a (k,l)-kernel of $L(D)$, there exists $c \in \bar{K}_2$ such that $d_{L(D)}(a,c) \leq l < k$. Let $(a = a_0, a_1, \ldots, a_n = c)$ be a shortest directed path in $L(D)$ from a to c and consider the following two possibilities:

If $b \neq c$, then it follows from Definition 2.1 that $(b, a_1, a_2, \ldots, a_n = c)$ is a directed path in $L(D)$ from b to c of length $n \leq l < k$ with $\{b,c\} \subseteq \bar{K}_2$, $b \neq c$. This contradicts part (1) of Definition 1.6, as $\bar{K}_2$ is a (k,l)-kernel of $L(D)$.

If $b = c$, we have from the hypothesis of Corollary 2.2 that there exists an arc $d \in \bar{K}_2$ such that $d \neq b$ and that d and b have the same terminal endpoint x. Then it follows from Definition 2.1 that $(d, a_1, a_2, \ldots, a_n = c = b)$ is a directed path in $L(D)$ of length $n \leq k < l$ from d to b with $d \neq b$, $\{d,b\} \subseteq \bar{K}_2$, contradicting part (1) of Definition 1.6, as $\bar{K}_2$ is a (k,l)-kernel of $L(D)$. □

THEOREM 2.6. *Every digraph has a $(k, 2k-2)$-kernel.*

Proof. We proceed by induction on $|V(D)|$.

For D with $|V(D)| = 1$ it is obvious. Suppose that if D' is a digraph with $|V(D')| < n$, then D' has a $(k, 2k-2)$-kernel, and let D be a digraph with $|V(D)| = n$.

Let $x_0 \in V(D)$ and $D^* = D[V(D) \setminus \{x \in V(D) | d_D(x, x_0) \leq k-1\}]$. Clearly $|V(D^*)| < n$, and hence D^* has a $(k, 2k-2)$-kernel, namely S^*. Consider the following two possibilities.

If there exists a directed path in D of length less than or equal to $k-1$, then S^* is a $(k, 2k-2)$-kernel of D.

If there is no directed path in D from x_0 to some point of S^* of length less than or equal to $k-1$, then $S^* \cup \{x_0\}$ is a $(k, 2k-2)$-kernel of D. □

COROLLARY 2.3. *Every digraph has a (k,l)-kernel for $l \geq 2k-2$.*

The proof is a direct consequence of Theorem 2.6 and Definition 1.6, as a (k,l)-kernel of a digraph D is also a (k,l')-kernel for every $l' \geq l$.

Remark 2.6. The hypothesis $l \geq 2k-2$ cannot be omitted in Corollary 2.3. Consider C_{2k-1} to be the directed cycle of length $2k-1$; for any $l < 2k-1$, the digraph C_{2k-1} has no (k,l)-kernel.

COROLLARY 2.4 (see [3]). *Every digraph has a quasikernel.*

The proof is a direct consequence of Theorem 2.6 by taking $k = 2$, as a quasikernel is a $(2,2)$-kernel.

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